ESTIMATION OF THE ABSCISSA OF CONVERGENCE OF THE MOMENT GENERATING FUNCTION

HANSPETER SCHMIDLI,* *Aarhus University*

Abstract

Let F be the distribution function of a positive random variable X and assume that $\bar{F}(x)^{-1}\bar{F}(x+y)$ converges to a strictly positive value as $x \to \infty$. It is shown that the right end point R of the interval where the moment generating function exists is finite and that the distribution function of e^{rX} is regularly varying with coefficient $-R/r$. Hence Hill's estimator is proposed for estimation of R.

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1. Introduction

Recently Embrechts and Schmidli (1994) considered an insurance risk model (R_t : $t \geq 0$) where interest and borrowing is present. In this model, ruin occurs if the paying for interest for borrowed money becomes larger than the premium income.

*Postal address: Department for Theoretical Statistics, Aarhus University, Ny Munkegade, 8000 Aarhus C, Denmark.

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They derived a Lundberg type inequality for the probability of ultimate ruin;

$$\lim_{u\to\infty} P[\tau < \infty \mid R_0 = u]e^{ru} = \begin{cases} 0 & \text{if } r < R \\ \infty & \text{if } r > R \end{cases}$$

where τ denotes the epoch of ruin and

$$R := \sup\{r \in \mathbb{R} : \int_0^\infty e^{rx}\, dF(x) < \infty\}$$

is the abscissa of convergence of the moment generating function of the claim size distribution F. Hence it would be interesting to estimate R.

In the classical Sparre Andersen model (1957) the Lundberg coefficient can be estimated via order statistics, see for instance Csörgőand Steinebach (1991) or Embrechts and Mikosch (1991) and references therein. The rate of convergence of the estimator to the true value is typically $1/\log n$, where n denotes the sample size. It will turn out that for estimating the abscissa of convergence order statistics also play an important role and we will also get a $1/\log n$ rate of convergence.

Another area where one wants to estimate the abscissa of convergence is in connection with empirical Laplace transforms (see Csörgőand Teugels (1990)). This was the motivation of Hall et al. (1992) for considering the problem. They showed that on a certain set of distribution functions the best possible rate of convergence is $1/\log n$ for any estimator.

A third application are stochastic approximation procedures involving Laplace-Stieltjes transforms. An interval to which the procedure is restricted is needed. This interval should lie in the domain where the Laplace-Stieltjes transform is defined. See for instance Herkenrath (1986).

In the present paper we restrict attention to the class of distribution functions F such that $\bar{F}(\log x)$ is regularly varying, where $\bar{F} := 1-F$ denotes the tail probability. It will turn out that the set of distribution functions considered in Hall et al. (1992) is contained in this class. We will show that the problem of estimating the abscissa of convergence is equivalent to the problem of estimating the corresponding exponent of regular variation.

In the sequel we consider a sequence $(X_i : i \in \mathbb{N})$ of iid. random variables with distribution function F. For any $n \in \mathbb{N}$ we denote by $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ the order statistics of the first n elements.

2. Main result

Theorem 1. *Let X be a positive random variable with distribution function F. Denote by $R := \sup\{r \in \mathbb{R} : E[e^{rX}] < \infty\}$ the abscissa of convergence of the moment generating function. Then the following conditions are equivalent:*

i) $\lim_{x\to\infty} \dfrac{\bar{F}(x+y_i)}{\bar{F}(x)} \in (0,\infty)$ *for at least two numbers y_1, $y_2 \neq 0$ such that $y_1/y_2 \notin \mathbb{Q}$.*

ii) $R < \infty$ *and* $\lim_{x\to\infty} \dfrac{\bar{F}(x+y)}{\bar{F}(x)} = e^{-Ry}$ *for all $y \in \mathbb{R}$.*

iii) *The tail of the distribution function of e^{rX} is regularly varying for some $r > 0$.*

iv) $R < \infty$ *and the tail of the distribution function of e^{rX} is regularly varying with index $-R/r$ for all $r > 0$.*

If $R \neq 0$ then each of the following two conditions is equivalent to i) – iv):

v) $E[X - x \mid X > x]$ *converges to ρ^{-1} for some $\rho > 0$.*

vi) $R < \infty$ *and $E[X - x \mid X > x]$ converges to R^{-1}.*

Remarks.

i) Consider the case

$$\lim_{x\to\infty} \frac{\bar{F}(x+y_0)}{\bar{F}(x)} = 1$$

for some $y_0 \neq 0$. We can assume that $y_0 > 0$. For any $y \in [-y_0, y_0]$

$$1 = \lim_{x\to\infty} \frac{\bar{F}(x+y)}{\bar{F}(x+y+y_0)} \geq \lim_{x\to\infty} \frac{\bar{F}(x+y)}{\bar{F}(x)} \geq \lim_{x\to\infty} \frac{\bar{F}(x+y_0)}{\bar{F}(x)} = 1$$

hence condition *i*) of the above theorem is fulfilled. Hence by *ii*) $E[e^{rX}] = \infty$ for all $r > 0$.

ii) Note that the mean residual life condition *v*) usually is used to decide whether F belongs to a heavy-tailed distribution function or not. Hence *v*) is a natural condition in the case $0 < R < \infty$.

iii) Condition (4) of Hall et al. (1992) is stronger than *vi*). Hence the class $\mathfrak{F}$ of functions considered in Hall et al. (1992) is contained in the class of distribution functions fulfilling one of the equivalent conditions of Theorem 1. ■

In order to prove the Theorem we need the following well-known

Lemma 1. *Let G be a distribution function with $G(0) = 0$ such that $\bar{G}$ is regularly varying with index $-\rho$ for some $\varrho \geq 0$.*

i) *If $\varrho > 1$ then $\int_0^\infty \bar{G}(x)\,dx < \infty$.*

ii) *If $\varrho < 1$ then $\int_0^\infty \bar{G}(x)\,dx = \infty$.*

Proof. *i*) Let $\varrho > 1$. By Bingham et al. (1987, p.23) there exists a non-increasing function ψ such that $\bar{G}(x) \sim x^{-\varrho} x^{(\varrho-1)/2}\psi(x)$. Hence there exists a constant C such that for $x \geq 1$ $\bar{G}(x) \leq Cx^{-(1+\varrho)/2}$ from which the assertion follows.

ii) We prove the converse statement. Let $\int_0^\infty \bar{G}(x)\,dx < \infty$ and denote by $\hat{G}(s) := \int_0^\infty e^{-sx}\,dG(x)$ the Laplace-Stieltjes-transform of G. By L'Hospital's rule we get

$$\lim_{s\downarrow 0} \frac{1-\hat{G}(s\lambda)}{1-\hat{G}(s)} = \lambda \lim_{s\downarrow 0} \frac{\hat{G}'(s\lambda)}{\hat{G}'(s)} = \lambda \ .$$

Hence it follows from Bingham et al. (1987, p.334) that $\varrho \geq 1$. □

Proof of Theorem 1. Let $r > 0$ and denote by $G_r(x) := F(r^{-1}\log x)$ the distribution function of e^{rX}. Set $\lambda_i := e^{y_i}$.

ii) $\Rightarrow$ *i*) and *iv*) $\Rightarrow$ *iii*) are trivial.

i) $\Rightarrow$ *iv*) The limits

$$\lim_{x\to\infty} \frac{\bar{G}_1(\lambda_i x)}{\bar{G}_1(x)} = \lim_{x\to\infty} \frac{\bar{F}(\log x + y_i)}{\bar{F}(\log x)}$$

exist in $(0,\infty)$ for $i \in \{1,2\}$. By Bingham et al. (1987, p.18) there exist a constant $\varrho \in [0,\infty)$ such that

$$\lim_{x\to\infty} \frac{\bar{G}_1(\lambda x)}{\bar{G}_1(x)} = \lambda^{-\varrho}$$

for all $\lambda \in \mathbb{R}$. Note that for $r > 0$

$$\lim_{x\to\infty} \frac{\bar{G}_r(\lambda x)}{\bar{G}_r(x)} = \lim_{x\to\infty} \frac{\bar{G}_1(\lambda^{1/r}x^{1/r})}{\bar{G}_1(x^{1/r})} = \lambda^{-\varrho/r} \ .$$

If $0 < r < R$ then by Lemma 1 $\varrho/r \geq 1$ and hence $R \leq \varrho$. If $R < r$ then again by Lemma 1 $\varrho/r \leq 1$ and hence $R \geq \varrho$. Therefore $R = \varrho < \infty$.

iii) $\Rightarrow$ *ii*) Let r be fixed. Denote by $-\varrho/r$ the coefficient of regular variation of the tail of the distribution function of e^{rX}. Then

$$\lim_{x\to\infty} \frac{\bar{F}(x+y)}{\bar{F}(x)} = \lim_{x\to\infty} \frac{\bar{G}_r(e^{rx}e^{ry})}{\bar{G}_r(e^{rx})} = (e^{ry})^{-\varrho/r} = e^{-\varrho y} \ .$$

Hence *i*) is fulfilled and it follows from *iv*) that $\varrho = R$.

Assume now $R \neq 0$.

$vi) \Rightarrow v)$ is trivial.

$v) \Rightarrow iii)$ Note that

$$\lim_{x\to\infty} \frac{\int_x^\infty \bar{G}_\rho(z)\,dz/z}{\bar{G}_\rho(x)} = \rho \lim_{x\to\infty} E[X - \rho^{-1}\log x \mid X > \rho^{-1}\log x] = 1 \; .$$

It follows from Karamata's Theorem (Bingham et al. (1987, p.30))that $\bar{G}_\rho(x)$ is regularly varying with index -1.

$iv) \Rightarrow vi)$ follows from Karamata's Theorem (Bingham et al. (1987, p.28)) and

$$E[X - R^{-1}\log x \mid X > R^{-1}\log x] = R^{-1}\frac{\int_x^\infty \bar{G}_R(z)\,dz/z}{\bar{G}_R(x)} \; .$$

□

By Theorem 1 the problem of estimating the abscissa of convergence of the moment generating function is translated to the problem of estimation the coefficient of regular variation for the distribution function of e^X. This problem is discussed a lot in the literature. A convenient estimator for R^{-1} is the Hill estimator (1975)

$$\hat{c}_n := \frac{1}{k_n}\sum_{i=1}^{k_n} X_{n-i+1:n} - X_{n-k_n:n} \; ,$$

where (k_n) is a sequence of natural numbers. As a consequence we get the following Corollary which is proved for the Hill estimator in Deheuvels et al. (1988).

Corollary 1. *Let $(X_i : i \in \mathbb{N})$ be an iid. sequence of positive random variables with distribution function F such that $R > 0$. Assume that the equivalent conditions of Theorem 1 are fulfilled and that $k_n \to \infty$ and $n^{-1}k_n \to 0$ as $n \to \infty$. Then, whenever $k_n/\log\log n \to \lambda \in [0,\infty]$ as $n \to \infty$,*

$$\overline{\lim_{n\to\infty}} \pm \hat{c}_n = \pm R^{-1}(1 + \alpha_\lambda^\pm) \text{ a.s.}$$

where $\alpha_\infty^\pm = 0$, $\alpha_0^- = -1$, $\alpha_0^+ = \infty$ and $-1 < \alpha_\lambda^- < 0 < \alpha_\lambda^+$ are the roots of the equation

$$\alpha - \log(1+\alpha) = \lambda^{-1}$$

if $\lambda \in (0,\infty)$. □

Hence $\hat{c}_n$ is strongly consistent if and only if $\lim_{n\to\infty} k_n/\log\log n = \infty$ and $\lim_{n\to\infty} k_n/n = 0$.

3. The case $R = \infty$

It may be possible that there exists a $y_0 > 0$ such that

$$\lim_{x\to\infty} \frac{\bar{F}(x+y)}{\bar{F}(x)} = 0 \tag{1}$$

for all $y \geq y_0$. Note that it is enough to assume that (1) holds for $y = y_0$. Because $\bar{F}$ tends to 0 very quickly one expects that $R = \infty$. That this holds is shown by the following

Lemma 2. *Let X be a random variable with distribution function F. Assume that there exist a $y > 0$ such that (1) holds. Then $R = \infty$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary and set $\lambda = e^y$. Assume that

$$\overline{\lim_{x\to\infty}}\, x^{n+2}\bar{F}(\log x) = \infty \ .$$

We want to show that this cannot hold. Define for $i \in \mathbb{N}$ the increasing sequence $x_i := \inf\{x > 0 : x^{n+2}\bar{F}(\log x) \geq i\}$. Because $\bar{F}$ is a non-increasing function $x_i^{n+2}\bar{F}(\log x_i) = i$. Then

$$\begin{aligned} 1 &= \frac{1}{i}x_i^{n+2}\bar{F}(\log x_i) = \frac{1}{i}\left(\frac{x_i}{\lambda}\right)^{n+2} \bar{F}(\log(x_i/\lambda))\ \lambda^{n+2}\frac{\bar{F}(\log x_i)}{\bar{F}(\log x_i - y)} \\ &< \lambda^{n+2}\frac{\bar{F}((\log x_i - y) + y)}{\bar{F}(\log x_i - y)} \ . \end{aligned}$$

But the right-hand side converges to 0 as $i \to \infty$ which is a contradiction. Hence there exists a constant K such that $x^{n+2}\bar{F}(\log x) \leq K$ and $x^n\bar{F}(\log x)$ is integrable over the interval $(0, \infty)$. Hence $E[e^{(n+1)X}] < \infty$ which proves the assertion. □

In order to prove the next Theorem we need the following

Lemma 3. *Let* $(X_i : i \in \mathbb{N})$ *be a sequence of iid. positive random variables with distribution function* F. *Let* $(k_n : n \in \mathbb{N})$ *be a sequence of natural numbers such that* $k_n/n \to 0$ *as* $n \to \infty$. *Then*

i) *if* $x_0 := \sup\{x \geq 0 : F(x) < 1\} < \infty$ *then*

$$\lim_{n\to\infty} X_{n-k_n:n} = x_0 \quad \text{a.s.},$$

ii) *if* $F(x) < 1$ *for all* $x \in \mathbb{R}$ *then*

$$\lim_{n\to\infty} X_{n-k_n:n} = \infty \quad \text{a.s..}$$

Proof. Choose $\ell \in \mathbb{R}$ such that $0 < F(\ell) < 1$. Note that the case $X_i = x_0$ a.s. is trivial. Because $k_n/n \to 0$ there exists a $n_0 \in \mathbb{N}$ such that $(1+n^{-1})F(\ell) \leq 1-n^{-1}k_n$ and hence

$$\binom{n}{i}\bar{F}(\ell)^i F(\ell)^{n-i} \leq \binom{n}{i+1}\bar{F}(\ell)^{i+1}F(\ell)^{n-i-1} \leq \binom{n}{k_n}\bar{F}(\ell)^{k_n}F(\ell)^{n-k_n}$$

for all $1 \leq i < k_n$ if $n \geq n_0$. Now

$$\begin{aligned}\sum_{n\geq n_0} P[X_{n-k_n:n} \leq \ell] &= \sum_{n\geq n_0}\sum_{i=0}^{k_n}\binom{n}{i}\bar{F}(\ell)^i F(\ell)^{n-i} \\ &\leq \sum_{n\geq n_0}(k_n+1)\binom{n}{k_n}\bar{F}(\ell)^{k_n}F(\ell)^{n-k_n} .\end{aligned}$$

Note that by Stirling's formula there exist a constant K such that

$$\begin{aligned}\binom{n}{k_n} &\leq K\frac{n^{n+\frac{1}{2}}e^{-n}}{k_n^{k_n+\frac{1}{2}}e^{-k_n}(n-k_n)^{n-k_n+\frac{1}{2}}e^{-(n-k_n)}} \\ &= Kn^{-\frac{1}{2}}\left(\frac{k_n}{n}\right)^{-(k_n+\frac{1}{2})}\left(1-\frac{k_n}{n}\right)^{-(n-k_n+\frac{1}{2})} .\end{aligned}$$

Because

$$\lim_{n\to\infty}\left((n+1)\binom{n}{k_n}\bar{F}(\ell)^{k_n}F(\ell)^{n-k_n}\right)^{1/n} = F(\ell)$$

it follows that

$$\sum_{n=1}^{\infty} P[X_{n-k_n:n} \leq \ell] < \infty .$$

By the Borel-Cantelli Lemma $\underline{\lim}_{n\to\infty} X_{n-k_n:n} \geq \ell$ a.s.. □

The question arises what is happening with the estimator $\hat{c}_n$ if $R = \infty$. We hope that it will converge to 0 as $n \to \infty$. That this holds is shown in the following

Theorem 2. *Let* (X_k) *be a sequence of iid. random variables with distribution function* F *such that (1) holds for all* $y > 0$ *or such that* $F(x_0) = 1$ *for some* $x_0 > 0$. *Then*

$$\lim_{n\to\infty} \hat{c}_n = 0 \quad a.s..$$

Remark. In i) of Theorem 1 it was enough to assume that a non-zero limit exists for two numbers y_i. If (1) holds on $[y_0, \infty)$ for a number $y_0 > 0$ we cannot conclude that it holds for all $y \in \mathbb{R}_+$. Hence we have to assume that (1) holds for all $y > 0$. ■

Proof. The case $F(x_0) = 1$ follows immediately from Lemma 3. Hence we can assume that $F(x) < 1$ for all $x > 0$. Let $\beta > 0$ be an arbitrary constant. It follows from Bingham et al. (1987) (Proposition 2.2.4 (iv) and (2.4.8)) that

$$\overline{\lim_{x\to\infty}} \sup_{y\geq 0} \frac{e^{\beta y}\bar{F}(x+y)}{\bar{F}(x)} = 1 \ .$$

Thus

$$\overline{\lim_{x\to\infty}} \int_0^\infty \frac{\bar{F}(x+y)}{\bar{F}(x)}\, dy \leq \int_0^\infty e^{-\beta y}\, dy = \beta^{-1} \ .$$

Because β was arbitrary we can conclude that

$$\lim_{x\to\infty} \int_0^\infty \frac{\bar{F}(x+y)}{\bar{F}(x)}\, dy \ = 0.$$

Analogously we can see that $\int_0^\infty y\bar{F}(x+y)/\bar{F}(x)\, dy$ is bounded uniformly in x.

We now condition on $X_{n-k_n:n}$. Let $(Y_i : 1 \leq i \leq k_n)$ be a sequence of iid. random variables with distribution function $1 - \bar{F}(\cdot + X_{n-k_n:n})/\bar{F}(X_{n-k_n:n})$. Then

$(Y_{1:k_n}, \ldots, Y_{k_n:k_n})$ and $(X_{n-k_n+1:n} - X_{n-k_n:n}, \ldots, X_{n:n} - X_{n-k_n:n})$ have the same conditional distribution given $X_{n-k_n:n}$. Hence

$$E[\hat{c}_n \mid X_{n-k_n:n}] = E\left[\frac{1}{k_n}\sum_{i=1}^{k_n} Y_i \,\middle|\, X_{n-k_n:n}\right] = \int_0^\infty \frac{\bar{F}(X_{n-k_n:n}+y)}{\bar{F}(X_{n-k_n:n})}\,dy$$

which converges to 0 a.s. as $n \to \infty$ by Lemma 3. Furthermore

$$E[(\hat{c}_n)^2 \mid X_{n-k_n:n}] \leq E[(Y_i)^2 \mid X_{n-k_n:n}] = \int_0^\infty \frac{2y\bar{F}(X_{n-k_n:n}+y)}{\bar{F}(X_{n-k_n:n})}\,dy$$

and thus $E[(\hat{c}_n)^2]$ is bounded uniformly in n. Therefore $\hat{c}_n$ is uniformly integrable and

$$E[\overline{\lim_{n\to\infty}}\, \hat{c}_n] = \overline{\lim_{n\to\infty}}\, E[\hat{c}_n] = E[\overline{\lim_{n\to\infty}}\, E[\hat{c}_n \mid X_{n-k_n:n}]] = 0\ .$$

Because $\hat{c}_n \geq 0$ the assertion follows. $\square$

4. Final remarks

In order to estimate the abscissa of convergence R of the moment generating function a lot of data are needed. The relation to the problem of estimating the coefficient of regular variation shows that an estimator converges only slowly to the true value if the sample size tends to infinity.

Discussions of the properties of estimators for R can be found in the literature. For Hill's estimator see for instance Hill (1975), Haeusler and Teugels (1985), Beirlant and Teugels (1986), Deheuvels et al. (1988) and references therein. In particular the problem how to choose the sequence (k_n) in an optimal way is discussed. For alternative methods for the estimation see for instance Hall (1982) and Smith (1987). A simulation study can be found in Keller and Klüppelberg (1991).

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