G. Concave and Convex Functions

In this appendix we let $I$ be an interval, finite or infinite, but not a singleton.

**Definition G.1.** A function $u : I \to \mathbb{R}$ is called (strictly) concave if for all $x, z \in I$, $x \neq z$, and all $\alpha \in (0, 1)$ one has

$$u((1 - \alpha)x + \alpha z) \geq (>) (1 - \alpha)u(x) + \alpha u(z).$$

$u$ is called (strictly) convex if $-u$ is (strictly) concave.

Because results on concave functions can easily translated for convex functions we will only consider concave functions in the sequel.

Concave functions have nice properties.

**Lemma G.2.** A concave function $u(y)$ is continuous, differentiable from the left and from the right. The derivative is decreasing, i.e. for $x < y$ we have $u'(x-) \geq u'(x+) \geq u'(y-) \geq u'(y+)$. If $u(y)$ is strictly concave then $u'(x+) > u'(y-)$. 

**Remark.** The theorem implies that $u(y)$ is differentiable almost everywhere. ■

**Proof.** Let $x < y < z$. Then

$$u(y) = u\left(\frac{z - y}{z - x}x + \frac{y - x}{z - x}z\right) \geq \frac{z - y}{z - x}u(x) + \frac{y - x}{z - x}u(z)$$

or equivalently

$$(z - x)u(y) \geq (z - y)u(x) + (y - x)u(z). \tag{G.1}$$

This implies immediately

$$\frac{u(y) - u(x)}{y - x} \geq \frac{u(z) - u(x)}{z - x} \geq \frac{u(z) - u(y)}{z - y}. \tag{G.2}$$

Thus the function $h \mapsto h^{-1}(u(y) - u(y - h))$ is increasing in $h$ and bounded from below by $(z - y)^{-1}(u(z) - u(y))$. Thus the derivative $u'(y-)$ from the left exists. Analogously, the derivative from the right $u'(y+)$ exists. The assertion in the concave case follows now from (G.2). The strict inequality in the strictly concave case follows analogously. □

Concave functions have also the following property.
Lemma G.3. Let $u(y)$ be a concave function. There exists a function $k : I \rightarrow \mathbb{R}$ such that for any $y, x \in I$

$$u(x) \leq u(y) + k(y)(x - y).$$

Moreover, the function $k(y)$ is decreasing. If $u(y)$ is strictly concave then the above inequality is strict for $x \neq y$ and $k(y)$ is strictly decreasing. Conversely, if a function $k(y)$ exists such that (G.3) is fulfilled, then $u(y)$ is concave, strictly concave if the strict inequality holds for $x \neq y$.

**Proof.** Left as an exercise. □

Corollary G.4. Let $u(y)$ be a twice differentiable function. Then $u(y)$ is concave if and only if its second derivative is negative. It is strictly concave if and only if its second derivative is strictly negative almost everywhere.

**Proof.** This follows readily from Theorem G.2 and Lemma G.3. □

The following result is very useful.

**Theorem G.5. (Jensen’s inequality)** The function $u(y)$ is (strictly) concave if and only if

$$\mathbb{E}[u(Y)] \leq (<) u(\mathbb{E}[Y])$$

for all $I$-valued integrable random variables $Y$ with $\mathbb{P}[Y \neq \mathbb{E}[Y]] > 0$.

**Proof.** Assume (G.4) for all random variables $Y$. Let $\alpha \in (0, 1)$. Let $\mathbb{P}[Y = z] = 1 - \mathbb{P}[Y = x] = \alpha$. Then the (strict) concavity follows. Assume $u(y)$ is strictly concave. Then it follows from Lemma G.3 that

$$u(Y) \leq u(\mathbb{E}[Y]) + k(\mathbb{E}[Y])(Y - \mathbb{E}[Y]).$$

The strict inequality holds if $u(y)$ is strictly concave and $Y \neq \mathbb{E}[Y]$. Taking expected values gives (G.4).

Also the following result is often useful.
Theorem G.6. (Ohlin’s lemma) Let $F_i(y), i = 1, 2$ be two distribution functions defined on $I$. Assume

$$\int_I y \, dF_1(y) = \int_I y \, dF_2(y) < \infty$$

and that there exists $y_0 \in I$ such that

$$F_1(y) \leq F_2(y), \quad y < y_0,$$
$$F_1(y) \geq F_2(y), \quad y > y_0.$$

Then for any concave function $u(y)$

$$\int_I u(y) \, dF_1(y) \geq \int_I u(y) \, dF_2(y)$$

provided the integrals are well defined. If $u(y)$ is strictly concave and $F_1 \neq F_2$ then the inequality holds strictly.

**Proof.** Recall the formulae $\int_0^\infty y \, dF_i(y) = \int_0^\infty (1 - F_i(y)) \, dy$ and $\int_{-\infty}^0 y \, dF_i(y) = -\int_{-\infty}^0 F_i(y) \, dy$, which can be proved for example by Fubini’s theorem. Thus it follows that $\int_I (F_2(y) - F_1(y)) \, dy = 0$. We know that $u(y)$ is differentiable almost everywhere and continuous. Thus $u(y) = u(y_0) + \int_{y_0}^y u'(z) \, dz$, where we can define $u'(y)$ as the right derivative. This yields

$$\int_{-\infty}^{y_0} u(y) \, dF_i(y) = u(y_0)F_i(y_0) - \int_{-\infty}^{y_0} \int_{y}^{y_0} u'(z) \, dz \, dF_i(y)$$
$$= u(y_0)F_i(y_0) - \int_{-\infty}^{y_0} F_i(z)u'(z) \, dz.$$

Analogously

$$\int_{y_0}^{\infty} u(y) \, dF_i(y) = u(y_0)(1 - F_i(y_0)) + \int_{y_0}^{\infty} (1 - F_i(z))u'(z) \, dz.$$

Putting the results together we find

$$\int_{-\infty}^{\infty} u(y) \, dF_1(y) - \int_{-\infty}^{\infty} u(y) \, dF_2(y) = \int_{-\infty}^{\infty} (F_2(y) - F_1(y))u'(y) \, dy.$$

If $y < y_0$ then $F_2(y) - F_1(y) \geq 0$ and $u'(y) \geq u'(y_0)$. If $y > y_0$ then $F_2(y) - F_1(y) \leq 0$ and $u'(y) \leq u'(y_0)$. Thus

$$\int_{-\infty}^{\infty} u(y) \, dF_1(y) - \int_{-\infty}^{\infty} u(y) \, dF_2(y) \geq \int_{-\infty}^{\infty} (F_2(y) - F_1(y))u'(y_0) \, dy = 0.$$

The strictly concave case follows analogously. □
Corollary G.7. Let $X$ be a real random variable taking values in some interval $I_1$, and let $g_i : I_1 \to I_2$, $i = 1, 2$ be increasing functions with values on some interval $I_2$. Suppose
\[ \mathbb{E}[g_1(X)] = \mathbb{E}[g_2(X)] < \infty. \]

Let $u : I_2 \to \mathbb{R}$ be a concave function such that $\mathbb{E}[u(g_i(X))]$ is well-defined. If there exists $x_0$ such that
\[ g_1(x) \geq g_2(x), \quad x < x_0, \quad g_1(x) \leq g_2(x), \quad x > x_0, \]
then
\[ \mathbb{E}[u(g_1(X))] \geq \mathbb{E}[u(g_2(X))]. \]

Moreover, if $u(y)$ is strictly concave and $\mathbb{P}[g_1(X) \neq g_2(X)] > 0$ then the inequality is strict.

Proof. Choose $F_i(y) = \mathbb{P}[g_i(X) \leq y]$. Let $y_0 = g_1(x_0)$. If $y < y_0$ then
\[ F_1(y) = \mathbb{P}[g_1(X) \leq y] = \mathbb{P}[g_1(X) \leq y, X < x_0] \leq \mathbb{P}[g_2(X) \leq y, X < x_0] \leq F_2(y). \]

If $y > y_0$ then
\[ 1 - F_1(y) = \mathbb{P}[g_1(X) > y, X > x_0] \leq \mathbb{P}[g_2(X) > y, X > x_0] \leq 1 - F_2(y). \]

The result follows now from Theorem G.6. \qed