On the Monitoring of Structural Changes in Linear Models with Dependent Errors

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The Sequential Change-Point Problem I

Consider a regression model with MDF errors:

$$y_i = \beta_{i1}x_{i1} + \ldots + \beta_{ip}x_{ip} + \varepsilon_i, \quad i = 1, 2, \ldots.$$

Chu, Stinchcombe & White (1996), historical period:

$$\boldsymbol{\beta}_i = \boldsymbol{\beta}_0, \quad 1 \leq i \leq m.$$

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Check parameter stability each time new data arrive:

$$H_0 : \boldsymbol{\beta}_{m+k} = \boldsymbol{\beta}_0 \quad \text{versus} \quad H_A : \boldsymbol{\beta}_{m+k} = \boldsymbol{\beta}_* \neq \boldsymbol{\beta}_0 \quad \forall k \geq k^*.$$

The Sequential Change-Point Problem II

Horváth, Hušková, Kokoszka & Steinebach (2004) considered linear models with independent errors:

$$y_i = \beta_{i1} 1 + \beta_{i2} x_{i2} + \ldots + \beta_{ip} x_{ip} + \varepsilon_i, \quad i = 1, 2, \ldots .$$

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Stopping time of the monitoring procedure:

$$\tau_m = \inf \left\{ k \geq 1 : \ |Q_m(k)| > \ c(\alpha)\, g_m(k) \right\},$$

such that

$$\lim_{m \to \infty} \mathbb{P}_{H_0} \left(\tau_m < \infty \right) = \alpha$$

and

$$\lim_{m \to \infty} \mathbb{P}_{H_A} \left(\tau_m < \infty \right) = 1.$$

Detector and Weight Function

Now, we consider a simple linear regression with MDF errors:

$$y_i = \beta_i x_i + \varepsilon_i, \quad i = 1, 2, \ldots .$$

Residual-based CUSUM detector: $\hat{\varepsilon}_i = y_i - x_i \, \hat{\beta}_m$ and

$$Q_m(k) = \sum_{i=1}^{m+k} \left(\hat{\varepsilon}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{\varepsilon}_i \right), \quad k = 1, 2, \ldots .$$

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Horváth et al. (2004) introduced "early-change" weight function:

$$g_m(k) = m^{1/2} \left(1 + \frac{k}{m}\right)^{1-\gamma} \left(\frac{k}{m}\right)^{\gamma}, \quad k = 1, 2, \ldots,$$

where $0 \le \gamma < 1/2$. Here, we use a "moderate" version:

$$g_m^*(k) = m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m}\right)^{\gamma}, \quad k = 1, 2, \ldots.$$

Assumptions on Regressors

There is a constant $0 \leq \tau < 1/2$, such that:

$$\frac{1}{n}\sum_{i=1}^{n} x_i = o\left(n^{-\tau}\right) \quad \text{almost surely}$$

and another constant $\mathbf{d} > 0$, such that:

$$\frac{1}{n}\sum_{i=1}^{n} \left(x_i^2 - \mathbf{d}\right) = o\left(1\right) \quad \text{almost surely.}$$

Assumptions on Errors

1. MDF error sequence:

$$\{(\varepsilon_i, \mathcal{F}_i)\,,\; i \in \mathbb{N}\} \quad \text{with} \quad \mathbb{E}\left(\varepsilon_i | \mathcal{F}_{i-1}\right) = 0.$$

2. A moment condition:

$$\sup_{1 \le i < \infty} \mathbb{E}|\varepsilon_i|^{2+\delta} < \infty \quad \text{for some} \quad 0 < \delta \le 2.$$

3. Assumption on the conditional variances:

$$\sum_{i=1}^{n} \left\{\mathbb{E}\left(\varepsilon_i^2 | \mathcal{F}_{i-1}\right) - \mathbb{E}\varepsilon_i^2\right\} = o(a_n) \quad \text{almost surely,}$$

$$\text{where} \quad a_n := n^{2/(2+\delta)} (\log n)^{2(1+\delta)/(2+\delta)}.$$

4. Variance of partial sums satisfy

$$\mathbb{E}\left(\sum_{i=1}^{n} \varepsilon_i\right)^2 = n\sigma^2 + O\left(a_n\right) \quad \text{for some} \quad \sigma^2 > 0.$$

Asymptotics for Type-1 Error

♣ If $0 \le \gamma < \min\left\{\tau, \frac{1}{2} - \frac{1}{2+\delta}\right\}$, under H_0, we have

$$\lim_{m\to\infty} \mathbb{P}\left(\frac{1}{\sigma} \sup_{1\le k<\infty} \frac{|Q_m(k)|}{g_m^*(k)} > c\right) = \mathbb{P}\left(\sup_{0\le t\le 1} (1-t)^{\gamma} |W(t)| > c\right)$$

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♣ Moreover,

$$\sigma_m^2 := \frac{1}{m}\sum_{i=1}^{m} \hat{\varepsilon}_i^2 \xrightarrow{\mathbb{P}} \sigma^2 \quad \text{or let} \quad s_m^2(k) := \frac{1}{m+k}\sum_{i=1}^{m+k} \hat{\varepsilon}_i^2$$

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Example I : NED Regressors

Given sequences $\{x_i,\ i \in \mathbb{Z}\}$ and $\{\eta_i,\ i \in \mathbb{Z}\}$, and x_i is $\mathcal{F}^i_{-\infty}$ - measurable, where $\mathcal{F}^i_{i-k} = \sigma\{\eta_j,\ i-k \leq j \leq i\}$. Then $\{x_i,\ i \in \mathbb{Z}\}$ is called $\mathcal{L}_p(\nu)$ - near epoch-dependent (NED) on $\{\eta_i,\ i \in \mathbb{Z}\}$, if $\sup_{i\in\mathbb{Z}} \|x_i\|_p < \infty$ and

$$\sup_{-\infty<i<\infty} \left\| x_i - \mathbb{E}\left(x_i | \mathcal{F}^i_{i-k}\right) \right\|_p = O\left(k^{-\nu}\right) \quad (k \to \infty),$$

where $p \geq 1$ and $\nu > 0$.

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where $p \ge 1$ and $\nu > 0$.

♣ [improving Ling (2007)] Let $\{x_i,\ i \in \mathbb{N}\}$ be a centered $\mathcal{L}_p(\nu)$-NED on an independent noise. If $1 < p < 2$ and $p/(p-1) \le q < p(\nu+1)/(p-1)$, then

$$\frac{1}{n^{\frac{1}{p}+\frac{1}{q}}} \sum_{i=1}^{n} x_i \to 0 \quad \text{almost surely.}$$

Example II : Monitoring AR(1)

Now, consider an autoregressive model, i.e.

$$x_i = \phi\, x_{i-1} + \eta_i, \quad i = 1, 2, \ldots,$$

where $\{\eta_i,\ -\infty < i < \infty\}$ is a centered i.i.d.-sequence with:

$$\sigma^2 := \mathbb{E}\eta_1^2, \quad \mathbb{E}|\eta_1|^{2+\delta} < \infty \quad \text{and} \quad -1 < \phi < 1.$$

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Causal representation yields NED-property:

$$\sup_{-\infty<i<\infty} \left\| x_i - \mathbb{E}\left(x_i | \mathcal{F}_{i-k}^i\right) \right\|_{2+\delta} \ll \phi^k \quad (k \to \infty).$$

Moreover, $\left\{x_i^2 - \sigma^2 / \left(1 - \phi^2\right),\ 1 \le i < \infty\right\}$ is $\mathcal{L}_{1+\delta/2}$-NED.

Regressor-weighted CUSUM I

Remember the simple linear regression with MDF errors:

$$y_i = \beta_i x_i + \varepsilon_i, \quad i = 1, 2, \dots .$$

Following Hušková & Koubková (2005) for regressor weights:

$$Q_m(k) = \sum_{i=1}^{m+k} \left(x_i \, \hat{\varepsilon}_i - \frac{1}{m} \sum_{i=1}^{m} x_i \, \hat{\varepsilon}_i \right), \quad k = 1, 2, \dots .$$

Now, assume bounded regressors and stationary MDF errors:

$$\sup_{i \in \mathbb{N}} |x_i| \le M < \infty, \quad \mathbb{E}\varepsilon_i^2 = \sigma^2 \quad \text{for some} \quad \sigma^2 > 0.$$

Regressor-weighted CUSUM II

Assume constants $0 \leq \tau < 1/2$ and $\mathbf{d} > 0$, such that

$$\frac{1}{n}\sum_{i=1}^{n} x_i = o\left(1\right) \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}\left(x_i^2 - \mathbf{d}\right) = o\left(n^{-\tau}\right).$$

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Assume $\mathbb{E}|\varepsilon_1|^{2+\delta} < \infty$, for some $0 < \delta \le 2$, and

$$\sum_{i=1}^{n}\left\{\mathbb{E}\left(x_i^2\,\varepsilon_i^2|\mathcal{F}_{i-1}\right) - x_i^2\,\sigma^2\right\} = o(a_n) \quad \text{almost surely.}$$

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♣ `If` $0 \le \gamma < \delta/(2+\delta) \le \tau$, `under` H_0, `we have`

$$\lim_{m\to\infty}\mathbb{P}\left(\frac{1}{\sqrt{\mathbf{d}}\sigma}\sup_{1\le k<\infty}\frac{|Q_m(k)|}{g_m^*(k)} > c\right) = \mathbb{P}\left(\sup_{0\le t\le 1}(1-t)^{\gamma}|W(t)| > c\right)$$

Regressor-weighted CUSUM III

Consider one-time parameter shift alternative, i.e.

$$H_A : \boldsymbol{\beta}_{m+k} = \boldsymbol{\beta}_* \neq \boldsymbol{\beta}_0 \quad \forall k \geq k^*.$$

♣ If $0 \leq \gamma < \delta/(2+\delta) \leq \tau$, under H_A, we have

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♣ Moreover, if $0 \leq \theta < \delta/(2+\delta)$, we have

$$\frac{1}{m}\sum_{i=1}^{m} x_i^2 \hat{\varepsilon}_i^2 - \mathbf{d}\sigma^2 = o_{\mathbb{P}}\left(m^{-\theta}\right) \quad (m \to \infty).$$

Example III : Structural Change in AR(1)

Again, the autoregressive model, i.e.

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where $\{\eta_i, \; -\infty < i < \infty\}$ is a centered i.i.d.-sequence with:

$$\mathbb{E}|\eta_1|^{2+\delta} < \infty \quad \text{and} \quad (H_0) \; -1 < \phi_i = \phi < 1.$$

Hušková & Koubková (2006), one-time shift $\Delta \neq 0$, satisfying:

$$-1 < \phi + \Delta < 1 \quad \text{and} \quad \phi + \Delta \neq 0.$$

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The sequential test presented here is consistent against:

$$H_A : \phi_{m+k} = \phi_* = 0 \neq \phi \quad \forall k \geq k^*.$$

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