Backward Invariance Principles in Change-Point Analysis

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Change-Point Problem I

Consider the model: location + "white noise"

$$X_i = \beta_i + \varepsilon_i \quad \text{and} \quad \varepsilon_i \sim (\mu, \sigma^2) \quad (i = 1, \dots, n)$$

Test the no-change-in-location null hypothesis

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_n \qquad (\beta\text{'s unknown !})$$

against one-time-shift alternative

$$H_A : \text{ for some } k \quad \beta_1 = \cdots = \beta_k \neq \beta_{k+1} = \cdots = \beta_n$$

Based on comparison of the sub-sample means

$$\max_{1 \le k < n} \left| \frac{1}{k} \sum_{i=1}^{k} X_i - \frac{1}{n-k} \sum_{i=k+1}^{n} X_i \right|$$

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Therefore $H_0 : EX_i = 0 \quad (i = 1, \dots, n)$ and normalized version:

$$\max_{1 \le k < n} \frac{\left| \frac{1}{k} \sum_{i=1}^{k} X_i - \frac{1}{n-k} \sum_{i=k+1}^{n} X_i \right|}{\sigma \sqrt{\frac{1}{k} + \frac{1}{n-k}}}$$

Change-Point Problem II

Reject the no-change null hypothesis , if

$$\max_{1\le k\le n-1} \frac{n^{-1/2}\left|\sum_{i=1}^{k} X_i - \frac{k}{n}\sum_{i=1}^{n} X_i\right|}{\sigma\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}} = \max_{1\le k\le n-1} \frac{n^{-1/2}\left|T_n(k)\right|}{\sigma\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}}$$

is large.

♣ First approach via FCLT : constrained versions

$$\sup_{\varepsilon\le t\le 1-\varepsilon} \frac{n^{-1/2}\left|T_n(\lfloor (n+1)t\rfloor)\right|}{\sigma\sqrt{t(1-t)}} \xrightarrow{\mathcal{D}} \sup_{\varepsilon\le t\le 1-\varepsilon} \frac{|W(t)-tW(1)|}{\sqrt{t(1-t)}}$$

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♣ If $\varepsilon = 0$ and for instance i.i.d.r.v.'s with finite 2nd-moments then

$$(2\log\log n)^{-1/2} \sup_{\frac{1}{n+1}\le t\le \frac{n}{n+1}} \frac{n^{-1/2}\,|T_n(\lfloor (n+1)t\rfloor)|}{\sigma\sqrt{t\,(1-t)}} \xrightarrow{P} 1$$

weighted-sup-norm Donsker I

♣ [following Szyszkowicz(1991) and Horváth(1997)]

Second approach: different weight functions

Let $0 \leq \tau < 1/2$. Then

$$\sup_{0<t<1} \frac{n^{-1/2}\,|T_n(\lfloor (n+1)t \rfloor)|}{\sigma(t\,(1-t))^{\tau}} \xrightarrow{\mathcal{D}} \sup_{0<t<1} \frac{|W(t)-tW(1)|}{(t\,(1-t))^{\tau}}$$

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- "Decomposition" of the tied-down partial sum:

$$T_n(k) = \left(1 - \frac{k}{n}\right)\sum_{i=1}^{k} X_i - \frac{k}{n}\sum_{i=k+1}^{n} X_i \quad (1 \le k < n)$$

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- Reversed process is still strongly mixing:
  [Bradley's SIP (1983)]: strict. stat., $(2+\delta)$-moment and α-mixing, e.g. $\alpha(n) \ll (\log n)^{-\lambda}$ $(n \to \infty)$ $(\lambda > 1 + 3/\delta)$

$$\sum_{i=1}^{k} X_{-i} - \sigma W(k) = o(\sqrt{k/\log\log k}) \quad a.s. \quad (k \to \infty)$$

weighted-sup-norm Donsker II

♣ [convergence-in-probability version]

Let $0 \leq \tau < 1/2$. Then there exists a sequence of Brownian bridges on the extended probability space such that:

$$\sup_{0<t<1} \frac{\left| n^{-1/2} T_n(\lfloor (n+1)t \rfloor) - \sigma B_n(t) \right|}{(t(1-t))^{\tau}} \xrightarrow{P} 0.$$

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- $\{W(t) - tW(1), 0 \leq t \leq 1\} \stackrel{\mathcal{D}}{=} \{B(t), 0 \leq t \leq 1\}$
- via [Skorokhod's representation of converging laws (1956)] and [Berbee's Extension Lemma (1979)]:

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$$\sup_{0<t<1} \left| n^{-1/2} T_n(\lfloor (n+1)t \rfloor) - \sigma B_n(t) \right| \xrightarrow{P} 0$$

- via [Bradley's SIP (1983)] and [Berbee's Extension Lemma (1979)]:

$$\max_{1 \leq k \leq n-1} \left| \sum_{i=1}^{k} X_i - \sigma W_{1n}(k) \right| \Big/ \sqrt{k} = O_P(1) \quad (n \to \infty)$$

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Darling-Erdős Limit Theorem

♣ [following Davis et al.(1986, 1995), Horváth(1993)]

$$A(\log n) \max_{1 \le k \le n-1} \frac{n^{-1/2} |T_n(k)|}{\sigma \sqrt{\frac{k}{n}\left(1 - \frac{k}{n}\right)}} - D(\log n) \xrightarrow{\mathcal{D}} E \vee E' \qquad (\spadesuit)$$

where E and E' are independent i.d.r.v so that

$$P[E \le y] = P[E' \le y] = \exp\{\exp(-y)\}$$

and $A(x) = (2\log x)^{1/2}$ and $D(x) = 2\log x + \frac{1}{2}\log\log x - \frac{1}{2}\log \pi.$

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- [Kuelbs & Philipp's SIP (1980)] : strict. stat., $2+\delta$-moment and α-mixing, e.g. $\alpha(n) \ll n^{-(1+\epsilon)(1+2/\delta)}$ $(n\to\infty)$
- asymptotic independence:

$$\max_{1\le k\le \frac{n}{\log n}} \frac{\left|\sum_{i=1}^k X_i\right|}{\sqrt{k}} \quad \vee \quad \max_{n-\frac{n}{\log n}\le k\le n-1} \frac{\left|\sum_{i=k+1}^n X_i\right|}{\sqrt{n-k}}$$

♣ `[Schmitz(2011)]` If $\alpha(n) \ll (\log n)^{-\lambda}$ and $\beta(n)\downarrow 0$ $(n\to\infty)$ then via [Berbee's Coupling (1979)] we still have $(\spadesuit)$.

Autoregressive Moving Average Models

Given a weight sequence $\{a_k, \; k \in \mathbb{Z}\}$ we define a moving average on i.i.d. noise by

$$X_n = \sum_{k=-\infty}^{\infty} a_k \varepsilon_{n-k}.$$

The $\mathrm{ARMA}\,(p,q)$ $\{X_n, \; n \in \mathbb{Z}\}$ is defined as the stationary solution of

$$\phi\,(B)\,X_n = \theta\,(B)\,\varepsilon_n,$$

where $\phi(x)$ and $\theta(x)$ are polynomials of degree p and $q \in \{0, 1, \dots\}$ and the constant term of both polynomials is assumed to be one.

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♣ [existence] If $E \log_+ |\varepsilon_1| < \infty$ and $\phi(x)$ has no zeros of absolute value one, then there is a stationary ARMA solution. This solution has a moving average representation and is ergodic.

augmented GARCH

♣ [Duan(1997), Carrasco and Chen (2002)] Let $\{\eta_k,\ k \in \mathbb{Z}\}$ be centered i.i.d. random variables. Consider

$$\begin{cases} \varepsilon_k = \sigma_k \eta_k \\ h\left(\sigma_k\right) = a\left(\eta_{k-1}\right) h\left(\sigma_{k-1}\right) + \omega\left(\eta_{k-1}\right) \end{cases}$$

where $h(\cdot), h^{-1}(\cdot)$ and $a(\cdot)$ and $\omega(\cdot)$ are nonnegative functions.

$$\sigma_k^2 = (a\eta_{k-1}^2 + b)\sigma_{k-1}^2 + \omega = a\varepsilon_{k-1}^2 + b\sigma_{k-1}^2 + \omega$$

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♣ [Aue et al.(2006)] $E \log_+ a(\eta_1) < 0 \Longrightarrow P\left[h(\sigma_k) < \infty\right] = 1$

$$h(\sigma_k) = \omega(\eta_{k-1}) + \sum_{i=1}^{\infty} a(\eta_{k-1}) \ldots a(\eta_{k-i})\, \omega(\eta_{k-i-1})$$

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♣ [Francq & Zakoïan(2006)] augm. GARCH is geometric ergodic if

$$dP_\eta = \sum_{i=1}^{N} p_i \delta_i + (1-p) f d\lambda \quad \text{and} \quad E \log_+ a(\eta_1) < 0.$$

Linear Process with Dependent Errors

♣ [Francq & Zakoïan(2006)]: log returns behave like AR models with augm. GARCH errors, i.e.

$$\begin{cases} y_k = \phi y_{k-1} + \varepsilon_k, & |\phi| < 1 \\ \varepsilon_k = \sigma_k \eta_k \end{cases}$$

Question: Change-in-the-mean, i.e. $y_1, \ldots, y_k, \Delta + y_{k+1}, \ldots, \Delta + y_n$?

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♣ [Schmitz(2011) strong approximation results]

- $\{\varepsilon_k\}$ strict. stat., $(2+\delta)$-moment, geometric. α-mixing
- $\sigma_k = f(\ldots, \eta_{k-2}, \eta_{k-1})$ and $\{\eta_k\}$ centered i.i.d.,

$$\sum_{i=1}^{n} y_i - \sqrt{\Gamma} W(n) \ll n^{\frac{1}{2}-\kappa} \quad a.s. \quad (n \to \infty)$$

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- Mixingale approximation via truncation $\sum_{i=1}^n y_i \to \sum_{i=1}^n \tilde{y}_i$
- Big-Block-Small-Block approach from [Aue et al.(2006)]
- Block de-coupling via [Bradley's coupling (1983)]
- [Einmahl's SIP (1989)] for independent not identic. distributed r.v.

"Backward" Invariance Principle

- $\{\varepsilon_k\}$ strict. stat., $(2+\delta)$-moment, geometric. α-mixing
- $\sigma_k = f(\ldots, \eta_{k-2}, \eta_{k-1})$ and $\{\eta_k\}$ centered i.i.d.,
- $\|\varepsilon_k - E[\varepsilon_k | \eta_{k-m}, \ldots, \eta_k]\|_{2+\delta} \leq c_k \psi_m, \quad \sum \psi_m < \infty$

$$\sum_{i=1}^{n} y_{-i} - \sqrt{\Gamma} W(n) \ll n^{\frac{1}{2}-\kappa} \quad a.s. \quad (n \to \infty)$$

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$$\sum_{i=1}^{n} y_{-i} - \sqrt{\Gamma} W(n) \ll n^{\frac{1}{2}-\kappa} \quad a.s. \quad (n \to \infty)$$

- reversed "Mixingale" approximation via truncation
  $\sum_{i=1}^{n} y_{-i} \to \sum_{i=1}^{n} \tilde{y}_{-i}$

$$\tilde{y}_{-i} = \sum_{\ell=0}^{i^\rho} \phi^\ell \varepsilon_{-i-\ell} \quad \text{for some fixed } \rho, \ (0 < \rho < 1).$$

- reversed maximal inequality

$$\left\| \max_{m \leq j \leq n} \left| \sum_{i=m}^{j} \tilde{y}_{-i} \right| \right\|_{2+\delta} \leq C_1 \sum_{k=0}^{\infty} \psi_k \left(\sum_{i=1}^{n-m+1} \left(\sum_{\ell=0}^{k \wedge (i+m-1)^\rho} \phi^\ell c_{k-\ell} \right)^2 \right)^{1/2}$$

Weight Functions I

♣ [Aue et al.(2006)] established SIP for squared augmented GARCH and if $0 \leq \tau < 1/2$ then

$$(\Gamma n)^{-1/2} \max_{1 \leq k \leq n} \frac{\left| \sum_{i=1}^{k} \varepsilon_i^2 - \frac{k}{n} \sum_{i=1}^{n} \varepsilon_i^2 \right|}{\left(\frac{k}{n}\right)^{\tau}} \xrightarrow{\mathcal{D}} \sup_{0<t<1} \frac{|B_t|}{t^{\tau}}$$

Question: Late change-point, i.e. weight function $(1-t)^{1/2}$?

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Question: Late change-point, i.e. weight function $(1-t)^{1/2}$?

♣ Answer: Darling-Erdös limit theorem via backward SIP

$$A(\log n)\,\Gamma^{-1/2} \max_{1 \le k \le n-1} \frac{n^{-1/2} \left| \sum_{i=1}^{k} y_i - \frac{k}{n} \sum_{i=1}^{n} y_i \right|}{\sqrt{\left(1 - \frac{k}{n}\right)}} - D^*(\log n) \xrightarrow{\mathcal{D}} E$$

where

$$D^*(x) = 2\log x + \frac{1}{2}\log\log x - \frac{1}{2}\log(4\pi)$$

Question: Early or late change-point, i.e. weight function $(t(1-t))^{1/2}$?

Weight Functions II

♣ Answer:

$$A(\log n)\,\Gamma^{-1/2}\max_{1\le k\le n-1}\frac{n^{-1/2}\left|\sum_{i=1}^{k}y_i-\frac{k}{n}\sum_{i=1}^{n}y_i\right|}{\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}}-D(\log n)\xrightarrow{\mathcal{D}}E\vee E'$$

Weight Functions II

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Asymptotic independence via [Berbee's Extension Lemma (1979)]:

$$\left| \max_{1\le k\le \frac{n}{\log n}} \frac{\left|\sum_{i=1}^{k} y_i\right|}{\sqrt{k}} - \max_{1\le k\le \frac{n}{\log n}} \frac{\left|\Gamma^{1/2} W_{1n}(k)\right|}{\sqrt{k}} \right| = o_P\left((\log\log n)^{-1/2}\right)$$

the Wiener process W_{1n} is a function of $\left\{\epsilon_1, \ldots, \epsilon_{n/\log n}, V\right\}$.

$$\left| \max_{n-\frac{n}{\log n}\le k\le n-1} \frac{\left|\sum_{i=k+1}^{n} y_i\right|}{\sqrt{n-k}} - \max_{n-\frac{n}{\log n}\le k\le n-1} \frac{\left|\Gamma^{1/2} W_{2n}(n-k)\right|}{\sqrt{n-k}} \right| = o_P\left((\log\log n)^{-1/2}\right)$$

and W_{2n} is a function of $\left\{\epsilon_{n-(n/\log n)+1-(n/\log n)^{\rho}}, \ldots, \epsilon_n, U\right\}$.

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