Extreme Value Asymptotics under Strong Mixing Conditions

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Dependence via Recurrence Relations

Generating a sequence of **dependent** random variables:

$$x_n = f\left(x_{n-1}, \varepsilon_n\right), \quad n = 1, 2, \ldots,$$

where x_n is the current state and $\{\varepsilon_n\}$ are independent external disturbance:

$$\varepsilon_n = \xi_n - \eta_n,$$

with random inflow $\xi_n \geq 0$ and random demand $\eta_n \geq 0$.

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Assuming a percentage loss $0 < \psi < 1$, then the level at successive days becomes

$$x_n = (1 - \psi)\, x_{n-1} + \xi_n - \eta_n, \quad n = 1, 2, \ldots$$

Introducing the backshift operator B notation:

$$\phi\,(B)\, x_n = \xi_n - \eta_n, \quad \text{where} \quad \phi\,(B) = 1B^0 + (1 - \psi)\, B^1$$

Autoregressive Moving Average Models

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The $\mathrm{ARMA}\,(p, q)$ $\{X_n,\ n \in \mathbb{Z}\}$ is defined as the stationary solution of

$$\phi\,(B)\,X_n = \theta\,(B)\,\xi_n,$$

where $\phi(x)$ and $\theta(x)$ are polynomials of degree p and $q \in \{0, 1, \dots\}$ and the constant term of both polynomials is assumed to be one.

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♣ [existence] If $E \log(1 + |\xi_1|) < \infty$ and $\phi(x)$ has no zeros of absolute value one , then there is a stationary ARMA solution. This solution has a moving average representation and is ergodic.

Strong Mixing Condition

Suppose a **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$. Let the measure of dependence between two σ-**fields** $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$ be

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} \left| \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \right|.$$

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Let $\{X_k, \ k \in \mathbb{Z}\}$ be a two-sided sequence of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$. For $-\infty \leq J < L \leq \infty$ define $\mathcal{F}_J^L = \sigma(X_k, \ J \leq k \leq L)$, i.e. the σ-**field generated by the family** $\{X_k, \ J \leq k \leq L\}$. For each $n \in \mathbb{N}$ define the dependence (mixing) coefficient $\alpha(n)$ by

$$\alpha(n) = \sup_{-\infty < J < \infty} \alpha\left(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^\infty\right).$$

The sequence $\{X_k, \ k \in \mathbb{Z}\}$ is said to be strongly mixing (α-mixing) if

$$\lim_{n \to \infty} \alpha(n) = 0.$$

Example: Strongly Mixing ARMA

If the ARMA solution admits a moving average representation

$$X_n = \sum_{k=-\infty}^{\infty} a_k \xi_{n-k},$$

the resulting weights are of geometric order, i.e.

$$|a_j| = O\left(\rho^j\right) \ (j \to \pm\infty) \quad \text{for some} \quad 0 < \rho < 1,$$

Additionally, assume that ξ_1 has a smooth density f such that

$$\int_{\mathbb{R}} |f(y+x) - f(y)| \, \lambda^1(dy) \leq C\,|x|$$

with $C > 0$ a constant, then the ARMA solution is α-mixing.

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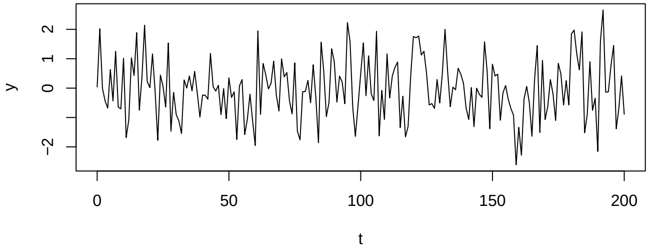
♣ `The standard normal density`

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-x^2/2\right\}$$

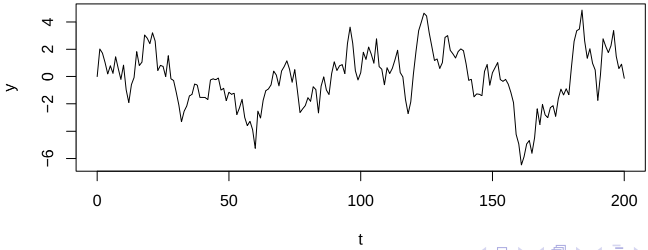
`satisfies`

$$\int_{\mathbb{R}} |\phi(y+x) - \phi(y)| \, \lambda^1(dy) \leq 2\,|x|\,.$$

$$x_k = 0x_{k-1} + \varepsilon_k$$



$$x_k = 0.85x_{k-1} + \varepsilon_k$$



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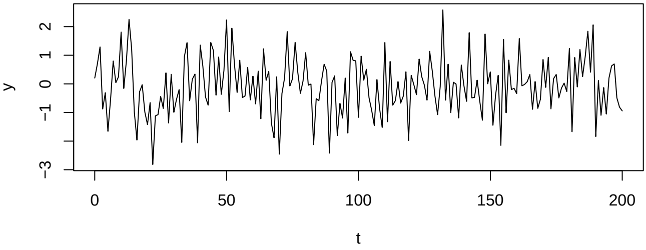

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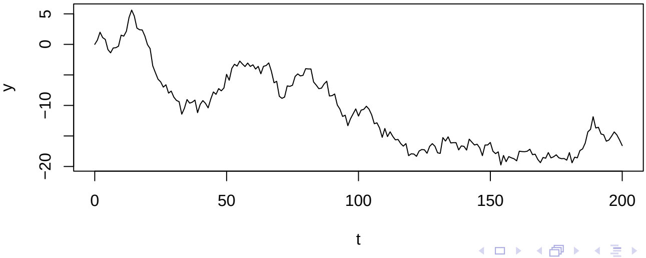

Dependence: Markov Chains

Time-homogeneous Markov process with state space $\mathcal{X}$ and stationary one-step transition probability function

$$\pi\left(x_{n-1}, \{x_n\}\right) = \pi\left(x_1, \ldots, x_{n-1}, \{x_n\}\right).$$

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Higher step transition probabilities can be generated from the one-step recursively

$$\pi^{(n+1)}\left(x, A\right) = \int_{\mathcal{X}} \pi^{(n)}\left(x, dy\right) \pi\left(y, A\right).$$

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The measure μ is called invariant with respect to the transition probability $\pi(\cdot, \cdot)$ if

$$\int_{\mathcal{X}} \mu(dx)\pi(x, A) = \mu\left(A\right).$$

Example: discrete Ornstein-Uhlenbeck I

Let $|\rho| < 1$ and consider the autoregressive model

$$X_{n+1} = \rho X_{n+1} + \xi_{n+1}, \quad n = 1, 2, \ldots,$$

where $\{\xi_k,\ k \in \mathbb{Z}\}$ is sequence of independent Gaussian random variables with mean $E\xi_1 = 0$ and variance $E\xi_1^2 = \sigma^2$. Then the transition probability distribution is given by

$$P\left(X_{n+1} \leq y | X_n = x\right) = \Phi_{0,\sigma^2}\left(y - \rho x\right).$$

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♣ `[invariant measure]` `For every Borel set` $B \in \mathcal{B}$ `the invariant measure` $\mu(B)$ `of the discrete Ornstein Uhlenbeck process is` $N\left(0, \sigma^2 / \left(1 - \rho^2\right)\right)$`,` `i.e. satisfies`

$$\mu(B) = \frac{\sqrt{1-\rho^2}}{\sqrt{2\pi}\sigma} \int_B \exp\left\{-\frac{y^2\left(1-\rho^2\right)}{2\sigma^2}\right\} \lambda^1(dy).$$

Example: discrete Ornstein-Uhlenbeck II

♣ [higher step transition] For every Borel set $B \in \mathcal{B}$ the n-step transition probability $\pi^{(n)}(x, B)$ of the discrete Ornstein-Uhlenbeck process is $N\left(\rho^n x, \sigma^2 \sum_{k=1}^n \rho^{2k}\right)$, i.e.

$$\pi^{(n)}(x, B) = \left(2\pi\sigma^2 \sum_{k=1}^{n} \rho^{2k}\right)^{-1/2} \int_B \exp\left\{-\frac{(y - \rho^n x)^2}{2\sigma^2 \sum_{k=1}^n \rho^{2k}}\right\} \lambda^1(dy).$$

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♣ [geometric ergodicity] For every x the n-step transition probability $\pi^{(n)}(x, B)$ converges to the invariant measure with geometric decay, i.e.

$$|P(X_n \in B | X_0 = x) - \mu(B)| \leq (2 + x) \exp\left\{-|\ln \rho^2| n\right\}.$$

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♣ ''Geometric Ergodicity is equivalent to β-mixing''.

β-Mixing

Suppose a **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$. Let the measure of dependence between two σ-**fields** $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$ be

$$\beta(\mathcal{A}, \mathcal{B}) = \sup \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} \left| P(A_i \cap B_j) - P(A_i) P(B_j) \right|,$$

where the supremum is taken over all pairs of finite partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of Ω such that $A_i \in \mathcal{A}$ for each i and $B_j \in \mathcal{B}$ for each j.

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$$\beta(n) = \sup_{-\infty < J < \infty} \beta\left(\mathcal{F}_{-\infty}^{J}, \mathcal{F}_{J+n}^{\infty}\right).$$

The sequence $\{X_k,\ k \in \mathbb{Z}\}$ is said to be absolutely regular (β-mixing) if

$$\lim_{n \to \infty} \beta(n) = 0.$$

Change-Point Problem I

Let $X_1, \ldots, X_n$ be (dependent) real-valued observation. Test the no-change null hypothesis

$$H_0 : EX_1 = EX_2 = \cdots = EX_n$$

against one-time shift alternative

$$H_A : EX_1 = \cdots = EX_{k^*} \neq EX_{k^*+1} = \cdots = EX_n$$

for some $1 \leq k^* < n$.

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♣ [Csörgő and Horváth (1997)] Reject the no-change null hypothesis , if

$$T_n(k) = \left(\frac{n}{k(n-k)} \right)^{1/2} \left| \sum_{i=1}^{k} X_i - \frac{k}{n} \sum_{i=1}^{n} X_i \right|$$

is large for some $1 \leq k \leq n-1$

Change-Point Problem II

♣ [along Csörgő and Horváth (1997)] Let $\{X_k,\ k \geq 1\}$ be a strictly stationary β-mixing sequence with $EX_1 = 0$. If

$$E|X_1|^{2(1+\lambda)} < \infty \quad \text{for some} \quad 0 < \lambda < 1/2$$

and

$$\beta(n) = O\left(n^{-(1+\epsilon)\left(1+\frac{1}{\lambda}\right)}\right) \quad \text{for some} \quad \epsilon > 0$$

Then

$$\lim_{n\to\infty} P\left[\max_{1\leq k\leq n-1}\left(\frac{n}{k(n-k)}\right)^{1/2}\left|\sum_{i=1}^{k} X_i - \frac{k}{n}\sum_{i=1}^{n} X_i\right| \geq t\right] = 1$$

for all $t \geq 0$.

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♣ Csörgő and Horváth (1997): Darling-Erdős type limit theorem

Asymptotics for Type-1 Error

♣ [extension of Ling(2007) to β-mixing] Let $\{X_k,\ k \geq 1\}$ be a strictly stationary β-mixing sequence with $EX_1 = 0$. If

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for some $0 < \lambda < 1/2$ and $\epsilon > 0$. Then

$$\lim_{n\to\infty} P\left[A(\log n)\frac{1}{\sigma}\max_{1\leq k\leq n-1}|T_n(k)| \geq t + D(\log n)\right] = 1 - \exp\left(-2e^{-t}\right)$$

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where

$$A(x) = (2\log x)^{1/2}$$

and

$$D(x) = 2\log x + \frac{1}{2}\log\log x - \frac{1}{2}\log \pi.$$

Moreover

$$0 < \lim_{n\to\infty} n^{-1} E\left(\sum_{k=1}^{n} X_k\right)^2 = \sigma^2 < \infty$$

Duan (1997): augmented GARCH

Let $\{\eta_k,\ k \in \mathbb{Z}\}$ be centered i.i.d. random variables. Let $\mathcal{F}_{k-1}$ denote the sigma field generated by the family $\{\ldots, \eta_{k-2}, \eta_{k-1}\}$. We consider the model

$$\varepsilon_k = \sigma_k \eta_k, \quad k \in \mathbb{Z},$$

where σ_k is measurable with respect to $\mathcal{F}_{k-1}$ for every $k \in \mathbb{Z}$ and

$$\Lambda\left(\sigma_k^2\right) = c\left(\eta_{k-1}\right)\Lambda\left(\sigma_{k-1}^2\right) + g\left(\eta_{k-1}\right) \quad k = 1, 2, \ldots,$$

where $\Lambda(\cdot), c(\cdot)$ and $g(\cdot)$ are continuous real-valued function.

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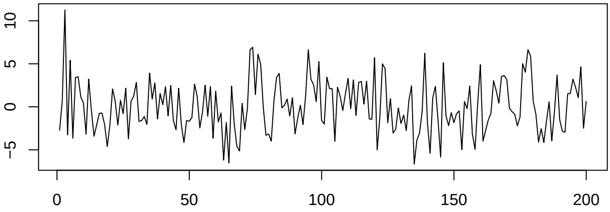
where $\Lambda(\cdot), c(\cdot)$ and $g(\cdot)$ are continuous real-valued function.

♣ [Carrasco and Chen (2002)] If η_1 has a continuous density and the density is positive on the whole real line and

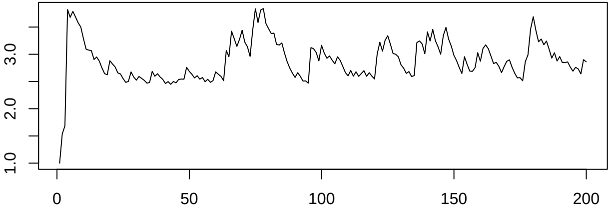
$$|c(0)| < 1, \quad E\left|c\left(\eta_1\right)\right| < 1 \quad \text{and} \quad E\left|g\left(\eta_1\right)\right| < \infty$$

then the augmented GARCH satisfies geometric ergodicity.

$$\varepsilon_k = \sigma_k \eta_k$$



$$\sigma_k^2 = 0.9 + 0.8\sigma_{k-1}^2 + 0.09\varepsilon_{k-1}^2$$



Asymptotics for Type-1 Error

♣ Let $\{X_k,\ k \geq 1\}$ be a strictly stationary augmented GARCH . If

$$E|X_1|^{2(1+\lambda)} < \infty \quad \text{and} \quad \beta(n) = O\left(\theta^n\right)$$

for some $0 < \theta < 1$. Then

$$\lim_{n\to\infty} P\left[A\left(\log n\right)\frac{1}{\hat{\sigma}_n}\max_{1\leq k\leq n-1}|T_n(k)| \leq t + D\left(\log n\right)\right] = \exp\left(-2e^{-t}\right)$$

where

$$\hat{\sigma}_n^2 = n^{-1}\sum_{k=1}^{n} X_k^2$$

and

$$\hat{\sigma}_n^2 - EX_1^2 = o_P\left(\left(\log\log\right)^{-1}\right) \quad (n\to\infty)$$

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