Budapest - Theta worksheet

1. **Stone - von Neumann over a finite field**

   Consider the finite field $\mathbb{F}_q$ with $q = p^m$ odd. Fix a non-trivial character ψ once and for all. Note that then all characters of $\mathbb{F}_q$ are given by $\psi_a(x) = \psi(ax)$ with $a \in \mathbb{F}_q$.

   Recall the Heisenberg group $H = X \oplus Y \oplus \mathbb{F}_q$ over $\mathbb{F}_q$. So the multiplication is given by

   $$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + x \cdot y' - x' \cdot y).$$

   For example, $(-x, 0, 0) \cdot (0, y, t) \cdot (x, 0, 0) = (0, y, t - 2x \cdot y)$.

   (a) Look up Mackey's irreducibility criterion for induced representations from a *normal* subgroup.

   (b) Let $H(Y) = \{(0, y, t)\}$, a normal subgroup of H of index q^n. We extend ψ to $H(Y)$ by $\psi((0, y, t)) := \psi(t)$. Use (a) to show that the induced representation $\mathrm{Ind}_{H(Y)}^{H} \psi$ is irreducible.

   (c) Let (ρ, V) be an arbitrary representation of H. Assume that the restriction $\mathrm{Res}_{H(Y)} V$ to $H(Y)$ contains ψ. Show $\langle \mathrm{Ind}_{H(Y)}^{H} \psi, V\rangle_H \geq 1$.

   (d) Now assume that (ρ, V) is irreducible with central character ψ, i.e., $\rho(0, 0, t)v = \psi(t)v$ for all $v \in V$.

   Note that V must contain an eigenvector w under the action of $H(Y)$: More precisely, there exists a $k \in \mathbb{F}_q^n$ such that

   $$\rho(0, y, t)w = \psi(k \cdot y)\psi(t)w \qquad \text{for all } (0, y, t) \in H(Y).$$

   Compute the action of $(0, y, t)$ on $w_x := \rho(x, 0, 0)w$ and show that there exists an $x \in \mathbb{F}_q^n$ such that

   $$\rho(0, y, t)w_x = \psi(t)w_x \qquad \text{for all } (0, y, t) \in H(Y).$$

   In particular, $\mathrm{Res}_{H(Y)} \rho$ contains ψ. Use (c) to conclude $\mathrm{Ind}_{H(Y)}^{H} \psi \simeq \rho$.

   In conclusion, every irreducible representation of H with non-trivial central character ψ is isomorphic to $\mathrm{Ind}_{H(Y)}^{H} \psi$.

2. **The Fock model of the Weil representation**

   We define elements of $\mathfrak{sl}_2(\mathbb{C})$,

   $$H := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad X_+ := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad X_- := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

   Note/check $[X_+, X_-] = H$, $[H, X_\pm] = \pm 2X_{-\pm}$.

   (a) Show by an explicit calculation that if in a representation (π, V) of $\mathfrak{sl}_2(\mathbb{C})$ a vector v is an eigenvector for H ("H-weight vector") with eigenvalue ("H-weight") λ, then $\pi(X_\pm)v$ is either zero or also an H-weight vector with H-weight $\lambda \pm 2$.

   (b) Let $\mathcal{F} := \mathbb{C}[z]$ be the (infinite-dimensional) $\mathbb{C}$-vector space of polynomials in one variable. We define operators on $\mathcal{F}$ by

   $$\omega(H) := z\frac{d}{dz} + \frac{1}{2}, \qquad \omega(X_+) := \frac{1}{8\pi}z^2, \qquad \omega(X_-) := -2\pi\frac{d^2}{dz^2}.$$

   Show that these operators preserve the bracket relations and hence define a Lie algebra representation ω of $\mathfrak{sl}_2(\mathbb{C})$. (The operator identities $\frac{d}{dz}z = 1 + z\frac{d}{dz}$, $\frac{d^2}{dz^2}z^2 = 2 + 4z\frac{d}{dz} + z^2\frac{d^2}{dz^2}$ should help).

(c) Find two linear independent lowest weight vectors ($\omega(X_-)p = 0$) in $\mathcal{F}$ and their weights. Show that these vectors each generate a subrepresentation, say $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$. In particular, describe $\mathcal{F}_1$ and $\mathcal{F}_2$ and give their weight structure and the action of $X_\pm$.

Finally, show that $\mathcal{F}_1$ and $\mathcal{F}_2$ are irreducible. In fact, these are exactly the holomorphic discrete series representation $D_{1/2}$ and $D_{3/2}$.

(d) Show that the intertwiner ι from the Schrödinger model $S(\mathbb{R})$ is given by

$$\iota(e^{-\pi x^2}) = 1 \qquad \text{and} \qquad \iota\left((x - \tfrac{1}{2\pi}\frac{\partial}{\partial x})^n (e^{-\pi x^2}) \right) = \left(\frac{-i}{2\pi}\right)^n z^n.$$

3. Let V be a (non-degenerate) quadratic space over $\mathbb{Q}$ of dimension m and let $\varphi \in S(V_\mathbb{R})$ be a Schwartz function of weight k under the action of (the inverse image in the metaplectic group of) $\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$. Recall that for $\tau = u + iv$ we define

$$\varphi_k(x, \tau) := v^{-k/2+m/4}\varphi(\sqrt{v}x)e^{\pi i (x,x)u}.$$

Let $L_k = -2iv^2 \frac{\partial}{\partial \bar\tau}$ and $R_k = 2i\frac{\partial}{\partial \tau} + kv^{-1}$. Show

$$(X_-\varphi)_{k-2}(x,\tau) = L_k\varphi_k(x,\tau) \qquad \text{and} \qquad (X_+\varphi)_{k+2}(x,\tau) = R_k\varphi_k(x,\tau).$$

4. **Orbit calculation for signature $(2,1)$: class numbers!**

Consider the even binary quadratic form $Q(x,y) = ax^2 + bxy + cy^2 = \frac{1}{2}\begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} 2a & b \\ b & 2c\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$ of discriminant $D = b^2 - 4ac < 0$, so in particular, $a, b, c \in \mathbb{Z}$ and $a > 0$. Note that $Q(x,y) = a(x - z_Q y)(x - \bar z_Q y)$, where $z_Q = \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}$ is the root of $az^2 + bz + c = 0$ in $\mathbb{H}$, a CM point of discriminant D.

Recall that we have an equivalence relation on these forms by the action of $\mathrm{SL}_2(\mathbb{Z})$, that is,

$$Q\left(\gamma^{-1}\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right)\right) = \frac{1}{2}\begin{pmatrix} x & y\end{pmatrix}\left[{}^t\gamma^{-1}\begin{pmatrix} 2a & b \\ b & 2c\end{pmatrix}\gamma^{-1}\right]\begin{pmatrix} x \\ y\end{pmatrix}.$$

(We restrict here to the action of $\mathrm{SL}_2(\mathbb{Z})$, that is determinant 1, and don't consider $\mathrm{GL}_2(\mathbb{Z})$)

(i) Show
$$Q\left(\gamma^{-1}\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right)\right) = a|j(\gamma, z_Q)|^2(x - (\gamma.z_Q)y)(x - (\gamma.\bar z_Q)y).$$

(ii) Use (i) to show that all Q are equivalent to one such that its associated CM point lies in $\mathcal{F}$, the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. We call such forms reduced. Show that for such a form, we have
$$|b| \le a \qquad \text{and} \qquad \sqrt{3}a \le \sqrt{|D|},$$
and conclude that there are only finitely many reduced even positive definite binary quadratic forms of fixed discriminant D.

(This finite number is called the class number and for D the discriminant of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$ is equal to its class number. Yes, one can make this correspondence explicit).

(iii) Show that for $D = -3, -4, -7, -8, -11, -19$, we have class number 1 (also for $D = -43, -67, -163$ - have fun!) For $D = -20$ class number 2 (representatives are $x^2 + 5y^2$, $2x^2 + 2xy + 3y^2$). For $D = -23$, class number 3 (rep's. are $x^2 + xy + 6y^2$,$2x^2 + xy + 3y^2$, $2x^2 - xy + 3y^2$ - yes, the last two forms are not $\mathrm{SL}_2(\mathbb{Z})$-equivalent, but have of course the same representation numbers.).

5. **Sums of two squares!**

For $D = -4$, we have $\theta(\tau, Q) = \theta^2(\tau) = \sum_{n=0}^\infty r_2(n)q^n$ lies in $M_1(4, \chi_{-4})$, where $\chi_{-4} : \mathbb{Z} \to \{\pm 1\}$ is given by $\chi_{-4}(n) = (-1)^{(n-1)/2}$ if n is odd and zero otherwise. The space is one-dimensional. Recall the Hecke operator T_p on the level of the Fourier coefficients is given by

$a_n \mapsto a_{pn} + \chi_{-4}(p)a_{n/p}$. Show that

$$\frac{1}{4}\theta^2(\tau) = E_1(\tau) := \frac{1}{4} + \sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{-4}(d)\right)q^n.$$

(Only use Hecke operators for this and do not assume that E_1 is a modular form).
Also show

$$L(\theta^2, s) = \zeta(s)L(\chi_{-4}, s) \qquad \text{with} \qquad L(\chi_{-4}, s) := \sum_{n=1}^{\infty}\chi_{-4}(n)n^{-s} = \prod_p \frac{1}{1-\chi_{-4}(p)p^{-s}}.$$

In particular, $L(\theta^2, s)$ comes with a nice Euler product! Wha is the relationship to the zeta function of the field $\mathbb{Q}(i)$? Explain!

6. **The simplest(?) case of the Artin conjecture**

   Assume $D = -23$. Then the space $M_1(23, \chi_{-23})$ is two-dimensional and the Hecke operator T_p is given by $a_n \mapsto a_{pn} + \chi_{-23}(p)a_{n/p}$.

   As seen above, we have three class of forms of discriminant -23, represented by $Q_1(x,y) = x^2 + xy + 6y^2 = (x + y/2)^2 + \frac{23}{4}y^2$, $Q_2(x,y) = 2x^2 + xy + 3y^3 = 2(x + y/4)^2 + \frac{23}{8}y^2$, $Q_3(x,y) = 2x^2 - xy + 3y^2$. Since $Q_2(x,y) = Q_3(x,-y)$, we have $\theta(\tau, Q_2) = \theta(\tau, Q_3)$.

   (i) Compute the Fourier expansion of $\theta(\tau, Q_1)$ and $\theta(\tau, Q_2)$ up to index 7.

   (ii) Show that

   $$E(\tau) := \frac{1}{2}(\theta(\tau, Q_1) + 2\theta(\tau, Q_2)) \qquad \text{and} \qquad F(\tau) := \frac{1}{2}(\theta(\tau, Q_1) - \theta(\tau, Q_2))$$

   are eigenfunctions for T_2 and hence for all T_p and therefore are normalized Hecke eigenforms! (For F you could also argue that F is a cusp form and $\dim S_1(23, \chi_{-23}) = 1$.)

   (iii) Show that $E(\tau) = \frac{3}{2} + \sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{-23}(d)\right)q^n$. Conclude:

   (a) A prime $p \neq 23$ is represented by either Q_1 or Q_2 $\iff$ $\left(\frac{p}{23}\right) = \left(\frac{-23}{p}\right) = 1.$

   (b) If a prime $p \neq 23$ is represented by either Q_1 or Q_2, then only by one but not both.

   (c) If Q_1 represents p, then $r_{Q_1}(p) = 4$; if Q_2 represents p, then $r_{Q_2}(p) = 2$.

   (d) $p = 23$ is represented only by Q_1 with $r_{Q_1} = 2$.

   Show

   $$L(E, s) = \zeta(s)L(\chi_{-23}, s) \qquad \text{with} \qquad L(\chi_{-23}, s) := \sum_{n=1}^{\infty}\chi_{-23}(n)n^{-s} = \prod_p \frac{1}{1-\chi_{-23}(p)p^{-s}}.$$

   Note that $\left(\frac{p}{23}\right) = \pm 1$ is exactly the condition whether the ideal (p) in the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{-23})$ stays prime or splits into the product of two prime ideals. If $p = Q_1(x,y) = (x + y\frac{1+\sqrt{-23}}{2})(x - y\frac{1+\sqrt{-23}}{2})$ then (p) splits into the product of two principal prime ideals whereas if $p = Q_2(x,y)$, then p is the product of two non-principal prime ideals.

   (iv) Write $F(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_1(23, \chi_{-23})$. Show

   $$L(F, s) = \prod_p \frac{1}{1 - a_p p^{-s} + \chi_{-23}(p)p^{-2s}}.$$

   Using (iii) show

   $$a_p = \begin{cases} 1 & \text{if } p = 23 \\ 0 & \text{if } \left(\frac{p}{23}\right) = -1 \\ 2 & \text{if } \left(\frac{p}{23}\right) = 1 \text{ and } p \text{ is represented by } Q_1 \\ -1 & \text{if } \left(\frac{p}{23}\right) = 1 \text{ and } p \text{ is represented by } Q_2 \end{cases}$$

This is a "reciprocity law"! The Fourier coefficients of a cusp form of weight 1 detect the "refined" splitting question in number fields! This example is a very special case of the modularity of Galois representations and is known as the Artin Conjecture and is one case of the Langlands Program!

(v) Interpret you answer in terms of ideal classes in $\mathbb{Q}(\sqrt{-23})$.

(vi) Actually $\eta(\tau)\eta(23\tau) \in S_1(23, \chi_{-23})$. Hence $F(\tau) = \eta(\tau)\eta(23\tau)$! Show that $a_n \equiv \tau(n) \pmod{23}$. Hence we obtain a congruence for the Ramanujan τ-function of a rather different kind.

7. **The easiest(?) indefinite theta integral**

Let $F = \sqrt{D}$ be real quadratic field which we view by the norm form as a rational quadratic space of signature $(1,1)$. We let $\mathcal{O}(= L)$ be the ring of integers and $\mathcal{O}_+^*(= \Gamma)$ be the positive units. Note that $\mathcal{O}_+^*$ acts on $\mathcal{O}$ as isometries. Further note that $G := \mathrm{SO}_0(1,1) \simeq \mathbb{R}$ realized respect to an orthogonal basis of $F \otimes \mathbb{R}$ by $r(t) := \left(\begin{smallmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{smallmatrix}\right)$. We embed $\mathcal{O}_+^*$ into G in the natural way.

We let φ_0 be the standard Gaussian for V of weight 0 and let

$$\theta(\tau, t, \varphi_0) = \sum_{\lambda \in \mathcal{O}} \varphi(r(t)^{-1}\lambda, \tau).$$

Compute the Fourier expansion of

$$\Theta(\tau) = \int_{\mathcal{O}_+^* \backslash G} \theta(\tau, t, \varphi_0) dt.$$

(The K-Bessel function will appear.)