SPIN^c STRUCTURES AND SCALAR CURVATURE ESTIMATES

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Abstract. In this note, we look at estimates for the scalar curvature κ of a Riemannian manifold M which are related to spin^c Dirac operators: We show that one may not enlarge a Kähler metric with positive Ricci curvature without making κ smaller somewhere on M. We also give explicit upper bounds for $\min\kappa$ for arbitrary Riemannian metrics on certain submanifolds of complex projective space. In certain cases, these estimates are sharp: we give examples where equality is obtained.

0. Introduction

There is a relation between the positivity of the scalar curvature κ of a compact Riemannian manifold M and its topology: Let M be an m-dimensional, compact, orientable, spin manifold. If the generalized $\hat{A}$-genus $\alpha(M) \in KO^{-m}(\mathrm{pt})$ does not vanish, then M carries no metric of positive scalar curvature by theorems of Lichnerowicz and Hitchin. The proof uses Dirac operators, combining the Bochner-Weitzenböck-Lichnerowicz formula (short: BLW-formula) and the Atiyah-Singer index theorem. The converse is much harder to establish: If $m \geq 5$ and M is compact, simply-connected and either not spin or spin with $\alpha(M) = 0$, then by theorems of Gromov, Lawson and Stolz, there exists a metric of positive scalar curvature on M. If M is not simply connected, the problem of the existence of metrics of positive scalar curvature on M is not yet completely solved. An overview of related theorems can be found in [S2] or in [LM], sections IV.4–7. On the other hand, assume that $m \geq 3$, and let $f: M \to \mathbb{R}$ be a function which is negative somewhere on M. Then there exists a Riemannian metric on M with scalar curvature $\kappa = f$ by a theorem of Kazdan and Warner ([KW]).

Suppose that there exists a metric g of positive scalar curvature κ on M. We call such a metric *extremal* if any other metric g' on M which is "larger" in a suitable sense has smaller scalar curvature in at least one point of M. In [G], section $5\frac{4}{9}$, Gromov asked, which manifolds M possess such an extremal metric, and how such a metric may look like. He proposed to specify the word "larger" above to mean larger on two-vectors. In this case, we call g *area-extremal.* Gromov also proposed to investigate not only variations of the metric on M itself, but to consider also area-nonincreasing spin maps of non-vanishing $\hat{A}$-degree from other Riemannian manifolds to M. A metric that is area-extremal in this stronger sense will be called area-extremal *in the sense of Gromov.*

Up to now, all known examples of extremal metrics are symmetric: Llarull showed that the round metrics on spheres are area-extremal in the sense of Gromov ([L1]). He also showed that it is not enough to restrict oneself to metrics which are larger only on three-vectors ([L2]). Min-Oo proved that Hermitian symmetric spaces of compact type are area-extremal ([M]), and Kramer established extremality of complex and quaternionic projective spaces ([Kr]). A general theorem for all compact Riemannian symmetric spaces G/K with $\operatorname{rk} G = \operatorname{rk} K$ will be given in [GS].

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In this paper, we show that Kähler metrics of non-negative Ricci curvature on M are "spin^c area-extremal" according to Definition 2.2. Here, we compare the scalar curvatures of M and another Riemannian manifold N via area-nonincreasing maps $f\colon N \to M$ of non-vanishing spin^c degree, see (2.1). The following theorem follows from Corollary 2.4 and Theorem 2.10 below:

0.1. Theorem. *Let M be a compact, connected Kähler manifold with non-negative Ricci curvature and scalar curvature κ, equipped with its canonical spin^c structure. Let N be a compact, connected, orientable Riemannian manifold with scalar curvature $\bar\kappa$. If $f\colon N \to M$ is an area-nonincreasing spin map of non-vanishing spin^c degree, then*

$$\bar\kappa \not> \kappa \circ f\ .$$

If g is Ricci-positive and $\bar\kappa \geq \kappa \circ f$, then $\bar\kappa = \kappa \circ f$. Moreover, N is then isometric to $M \times F$, where F admits a non-trivial parallel untwisted spinor, and f is the projection onto the first factor.

Here $\bar\kappa \not> \kappa\circ f$ means that there exists a $p \in N$ with $\bar\kappa(p) \leq \kappa(f(p))$. For a complex manifold M, the spin^c degree of the identity id_M with respect to the natural spin^c structure is just the Todd genus of M, which can be zero. Thus, in order to conclude that a Kähler metric g of non-negative Ricci curvature is area-extremal among metrics on M, we need some extra condition, cf. Remark 2.3. On the other hand, for the estimate in Theorem 0.1 alone, we may replace the condition that f be area-nonincreasing by a weaker assumption, cf. Remark 2.6.

It would be interesting to know the class of complex manifolds whose Ricci-positive Kähler metrics are not only spin^c area-extremal, but also area-extremal in the sense of Gromov. In this paper, we establish this property for two series of Hermitian symmetric spaces. Let

$$Q^n := \mathrm{SO}_{n+2}/\mathrm{SO}_n \times \mathrm{SO}_2 \cong \left\{\, [z_0 : \ldots : z_{n+1}] \;\middle|\; z_0^2 + \cdots + z_{n+1}^2 = 0 \,\right\} \subset \mathbb{C}P^{n+1}$$

be the complex hyperquadric. The following estimate generalizes Min-Oo's theorem 7 ([M]) in the special cases of $\mathbb{C}P^n$ and Q^n:

0.2. Theorem. *Let g be a Kähler metric on the complex manifold $M = \mathbb{C}P^n$ or $M = Q^n$ with non-negative Ricci curvature and scalar curvature κ. Let N be a compact, connected orientable Riemannian manifold with scalar curvature $\bar\kappa$. If $f\colon N \to M$ is an area-nonincreasing spin map of non-vanishing $\hat A$-degree, then*

$$\bar\kappa \not> \kappa \circ f\ .$$

If g is Ricci-positive and $\bar\kappa \geq \kappa \circ f$, then $\bar\kappa = \kappa \circ f$. Moreover, N is then isometric to $M \times F$, where F admits a non-trivial parallel untwisted spinor, and f is the projection onto the first factor.

Remark. Both Theorem 0.1 and Theorem 0.2 exhibit large families of area-extremal metrics on Ricci-positive Kähler manifolds, cf. Remark 2.5 and Remark 3.7 below. This is a new phenomenon: The only examples of area-extremal metrics discovered before were the standard metrics on certain symmetric spaces, and thus unique (up to rescaling) on the underlying differentiable manifold.

Since all Kähler manifolds with positive Ricci curvature admit a holomorphic (but not necessarily isometric) embedding into complex projective space, it is natural to study the scalar curvature κ of smooth projective varieties. More generally, one may consider maps f from a manifold M into $\mathbb{C}P^N$. Recently, Bär and Bleecker compared the scalar curvature of a Riemannian manifold M with the curvature tensor of another manifold N if there exists an area-nonincreasing map $f\colon M \to N$ of a certain topological type. For the special case of a complete intersection M and a map f into $\mathbb{C}P^N$ homotopic to the inclusion, they obtained an explicit upper bound for $\min\kappa$ ([BB]). In Corollary 4.1, we present a stronger estimate for a certain class of maps into $\mathbb{C}P^N$, which in particular implies Conjecture 6.1 in [BB]. In this estimate, we obtain equality if $M = \mathbb{C}P^n$ or $M = Q^n$.

For general M and f, Corollary 4.1 is rather coarse since it does not reflect the topology of M. If we restrict our attention to complete intersections $V \subset \mathbb{C}P^N$, which form a subclass of the class of smooth projective varieties, we obtain much stronger estimates:

0.3. Theorem. *Let $V = V^n(a_1, \dots, a_r)$ be a complete intersection of total degree $|a| := a_1 + \dots + a_r$. Let V be equipped with a Riemannian metric such that there exists an area-nonincreasing map $f: V \to \mathbb{C}P^{n+r}$ which is homotopic to the inclusion. Then*

$$\min_{p \in V} \kappa(p) \leq \begin{cases} 4n(n+r+1-|a|) & \text{if } |a| \leq n+r, \\ 0 & \text{if } |a| > n+r, \ n \text{ is even, and } V \text{ is spin,} \\ 4n & \text{if } |a| > n+r, \text{ and } V \text{ is not spin, and} \\ 8n & \text{if } |a| > n+r, \ n \text{ is odd, and } V \text{ is spin.} \end{cases}$$

If V is connected, $|a| \leq n+r$ and $\kappa \geq 4n(n+r+1-|a|)$, then V is a Kähler-Einstein manifold with Einstein constant $2(n+r+1-|a|) > 0$, and f is an isometric, holomorphic embedding.

As the proofs of many of the results cited above, our arguments are based on a combination of the BLW-formula with the index theorem, applied to certain twisted Dirac operators. The estimates obtained from the BLW-formula become stronger the simpler the algebraic structure of the twisting curvature and the smaller its norm is. In this paper, we work with the Dirac operators associated to spin^c structures, i.e., we twist with a line bundle which will be of sufficiently small curvature. Since the BLW-formula for spin^c Dirac operators is particularly simple, we do not actually need the map f to be area-nonincreasing to obtain our estimates. It suffices that $\|f^* F^L\| \leq \|F^L\| \circ f$ with respect to the metrics we compare. Here F^L is the curvature of the canonical line bundle of the spin^c structure on M, and $\|F^L\|$ is defined in (1.4). We show in Lemma 2.7 that indeed $\|f^* F^L\| \leq \|F^L\| \circ f$ for any area-nonincreasing f.

We also apply the Atiyah-Singer index theorem to ensure the existence of a harmonic spinor: For example, for Kähler manifolds M of non-negative Ricci curvature which is positive somewhere on M, the Todd genus of M is 1 by Bochner's theorem. This guarantees the existence of a harmonic spinor even if the metric is no longer Kähler after deformation.

In cases where we do not know the index of a particular spin^c Dirac operator explicitly, we try to construct a family of spin^c structures indexed by a parameter $k \in \mathbb{Z}$. The indices of the corresponding spin^c Dirac operators D_k are given by a polynomial in k, called the *"Hilbert polynomial"*, which is non-zero under certain mild topological restrictions. This implies that we find a harmonic spinor for some k which is not too large. For example, when considering submanifolds of complex projective space, we construct spin^c structures whose canonical line bundle is a small power $f^* H^k$ of the pulled-back hyperplane bundle H of $\mathbb{C}P^N$. In the special case of complete intersections, we can determine explicitly the smallest k which produces a non-vanishing index, and obtain the sharper estimate of Theorem 0.3.

The rest of this paper is organized as follows: In Section 1, we recall Hitchin's scalar curvature estimate for spin^c Dirac operators with non-vanishing index (Proposition 1.7). If for some Riemannian manifold (M, g) one has equality in the estimate mentioned above, then M has special holonomy (Theorem 1.9). In Section 2, we define the notion of spin^c area-extremality and prove Theorem 0.1. We introduce the Hilbert polynomial in Section 3 and use it to derive another scalar curvature estimate (Theorem 3.1) and to prove Theorem 0.2. In the last section, we establish a general scalar curvature estimate for certain submanifolds of $\mathbb{C}P^N$ (Corollary 4.1). Using the detailed knowledge of the topology of complete intersections, we derive the stronger estimate of Theorem 0.3. This estimate is then compared with other known results.

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1. Scalar Curvature, Twisted Dirac Operators and Kähler metrics

In this section, we recall some facts on spin^c structures and parallel spinors from [Hit], [LM] and [Mo]. Let M be a compact, oriented Riemannian manifold. Each spin^c structure on M possesses a natural Dirac operator. Whenever the index of this operator is non-zero, one obtains an estimate for the scalar curvature of M. We also investigate the case where equality is obtained for some metric g on M: in this case, M has to be locally the product of a Kähler manifold with a manifold that admits an untwisted parallel spinor.

We start by recalling the construction of spin^c structures and their associated Dirac operators ([Hit], section 1.1, [LM], appendix D). Let Spin_2 be the connected double cover of $\mathrm{U}_1 = \mathrm{SO}_2$, then the group Spin^c_m is defined as

$$\mathrm{Spin}^c_m := \mathrm{Spin}_m \times \mathrm{Spin}_2 / \{(1,1),(-1,-1)\} \; .$$

1.1. Definition. Let P_{SO_m} be the frame bundle associated to the tangent bundle TM of an m-dimensional Riemannian manifold M. A *spin^c structure* on M consists of a principal U_1-bundle P_U, a principal Spin^c_m-bundle P_{Spin^c}, and a bundle map $P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}_m} \times P_\mathrm{U}$, which is equivariant with respect to the natural action of Spin^c_m on $P_{\mathrm{SO}_m} \times P_\mathrm{U}$. Let L be the complex line bundle associated to P_U, then L is called the *canonical line bundle* associated to P_{Spin^c}, and its first Chern class $c = c_1(L) \in H^2(M,\mathbb{Z})$ is called the *canonical class* of P_{Spin^c}.

A spin^c structure with canonical line bundle L exists iff the second Stiefel-Whitney classes of M and L coincide, i.e. iff

$$w_2(M) := w_2(TM) = w_2(L) \in H^2(M,\mathbb{Z}_2) \; .$$

Note that $w_2(L)$ is just the reduction modulo 2 of c.

Let S be the complex, unitary spinor representation of Spin_m. Then S is irreducible if m is odd, and splits as $S = S^+ \oplus S^-$ if m is even. The groups $\mathrm{Spin}_m \times \mathrm{U}_1$ and Spin^c_m act on S, where $\mathrm{U}_1 \subset \mathbb{C}$ acts by complex multiplication. The *complex spinor bundle* associated to a spin^c structure P_{Spin^c} is the associated Hermitian vector bundle $\mathcal{S}$ with fiber S. The tangent bundle TM acts on $\mathcal{S}$ by Clifford multiplication.

Fix a unitary connection ∇^L on the canonical line bundle L, and let $F^L \in i\,\Omega^2(M)$ be the imaginary-valued curvature form of ∇^L. Then $-\frac{1}{2\pi i}F^L$ represents the image $c_\mathbb{R}$ of c in $H^2(M,\mathbb{R})$, and each closed two-form on M which represents $c_\mathbb{R}$ arises this way. Together with the Levi-Civita connection, ∇^L induces a unitary connection $\nabla^\mathcal{S}$ on $\mathcal{S}$ which is compatible with Clifford multiplication. Thus, we can define the *spin^c Dirac operator* associated to P_{Spin^c} and ∇^L as

$$D := \sum_{i=1}^m c_i \nabla^\mathcal{S}_{e_i} : \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S}) \; .$$

Here, $e_1, \dots, e_m$ is a local orthonormal frame of M, and c_i denotes Clifford multiplication with e_i. The following BLW-formula for the square of the Dirac operator D associated to P_{Spin^c} was already known to Schrödinger ([Sch], §7):

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4} + \frac{1}{2}\sum_{i<j} F^L(e_i,e_j)\, c_i c_j \; . \tag{1.2}$$

Here, $\nabla^*\nabla$ is the Bochner-Laplacian on $\mathcal{S}$, which is a non-negative elliptic differential operator of degree 2, and κ is the scalar curvature of M. Recall that on a compact manifold, $\ker(\nabla^*\nabla)$ is precisely the space of parallel spinors with respect to $\nabla^\mathcal{S}$.

We assume from now on that the dimension of M is even and equals $2n$. Then the complex spinor bundle splits as $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, and D splits as $D^\pm\colon \Gamma(\mathcal{S}^\pm) \to \Gamma(\mathcal{S}^\mp)$. From the Atiyah-Singer index theorem for spin^c Dirac operators (Theorem D.15 in [LM]), we know that

$$\operatorname{ind}(D) := \dim\ker(D^+) - \dim\ker(D^-) = \Big(\hat{A}(M)\, e^{\frac{c}{2}}\Big)[M] \;. \tag{1.3}$$

For the applications in this paper, we have to measure curvatures of unitary line bundles. Therefore, we define a norm on $\Lambda^2(M)$ by

$$\|\alpha\|_g = \sum_{j=1}^n |\lambda_j| \;, \qquad \text{where} \quad \alpha = \sum_{j=1}^n \lambda_j\, e^{2j-1} \wedge e^{2j} \in \Lambda^2(M) \tag{1.4}$$

with respect to a suitable g-orthonormal frame $e_1, \dots, e_{2n}$ with dual frame $e^1, \dots, e^{2n}$. From the proof of Lemma 2.7 below, it will become clear that $\|\cdot\|_g$ is indeed a norm.

1.5. Example. Let (M^{2n}, g, J) be a Kähler manifold, and let $\Lambda^{0,*}M$ be the bundle of antiholomorphic differential forms on M. Then M possesses a *natural spin^c structure* $P_{\mathrm{Spin}^c}M$, such that the canonical line bundle of $P_{\mathrm{Spin}^c}M$ is precisely the canonical line bundle $K = \Lambda^{0,n}M$ of M, and the canonical class c of $P_{\mathrm{Spin}^c}M$ is the canonical class $c_1(K)$ of M. The complex spinor bundle $\mathcal{S}$ of this spin^c structure is isomorphic to $\Lambda^{0,*}M$, where Clifford multiplication is given by $c_{2j-1} + ic_{2j} = -2(\iota_{2j-1} + i\iota_{2j})$ and $c_{2j-1} - ic_{2j} = \varepsilon^{2j-1} - i\varepsilon^{2j}$. Here, ι_j and ε^j denote interior and exterior multiplication with e_j. The Levi-Civita connection on M (or equivalently, the holomorphic unitary connection on K) induces a connection on $\Lambda^{0,*}M$ such that the associated spin^c Dirac operator D coincides with the Dolbeault operator $\bar\delta + \bar\delta^*$. Thus, spin^c Dirac operators can be regarded as natural generalizations of Dolbeault operators on Kähler manifolds. For Kähler manifolds, we have $\operatorname{ind}(D) = \operatorname{Td}(M)[M]$, and the constant function 1 is always a harmonic and moreover parallel spinor. Note also that the curvature of K is related to the Ricci curvature of M by

$$F^K = i\operatorname{Ric}(\,\cdot\,, J\,\cdot\,) \;, \tag{1.6}$$

so the scalar curvature of M is given by $\kappa = \operatorname{tr}(\operatorname{Ric}) \le 2\left\|F^K\right\|_g$, with equality if the Ricci curvature is non-negative.

For a spin^c structure with canonical line bundle L on Riemannian a manifold (M, g), there is still a relation between κ and $\left\|F^L\right\|_g$ if the corresponding spin^c Dirac operator has a kernel. Combining (1.2) and (1.3) above, one obtains

1.7. Proposition ([Hit], Theorem 1.1). *Let P_{Spin^c} be a spin^c structure with canonical class $c \in H^2(M, \mathbb{Z})$ on a compact, oriented Riemannian manifold (M, g), and let $c_{\mathbb{R}} \in H^2(M, \mathbb{R})$ be represented by the closed two-form $-\frac{1}{2\pi}\alpha$. If the scalar curvature of M satisfies*

$$\kappa > 2\,\|\alpha\|_g$$

everywhere on M, then $\big(\hat{A}(M)\, e^{\frac{c}{2}}\big)[M] = 0$.

Proof. Choose a connection ∇^L on L with curvature $F^L = i\alpha$, and let D be the Dirac operator associated to P_{Spin^c} and ∇^L. Let $0 \ne \psi \in \Gamma(\mathcal{S})$ be a spinor, and let $\|\cdot\|$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the

L^2-norm and L^2-scalar product, then

$$\begin{aligned}
\|D\psi\|^2 = \langle D^2\psi,\psi\rangle &= \Big\langle \Big(\nabla^*\nabla + \frac{\kappa}{4} + \frac{1}{2}\sum_{j<k} F^L(e_j,e_k)\, c_j c_k\Big)\psi,\psi\Big\rangle \\
&= \big\|\nabla^{\mathcal{S}}\psi\big\|^2 + \Big\langle \frac{\kappa}{4}\psi,\psi\Big\rangle + \Big\langle \frac{1}{2}\sum_{j=1}^{n} \alpha(e_{2j-1},e_{2j})\, ic_{2j-1}c_{2j}\,\psi,\psi\Big\rangle \\
&\geq \big\|\nabla^{\mathcal{S}}\psi\big\|^2 + \Big\langle \Big(\frac{\kappa}{4} - \frac{\|\alpha\|_g}{2}\Big)\psi,\psi\Big\rangle > 0\,,
\end{aligned} \tag{1.8}$$

Here, we have chosen the frame $e_1, \dots, e_{2n}$ at each point such that (1.4) holds. We have also used that ic_jc_k has eigenvalues ± 1. It follows that D is invertible, and that its index vanishes. □

Now, we want to know for which M and α the estimate above can be sharp. As already observed by Hitchin, equality in Proposition 1.7 implies the existence of a parallel spinor in $\Gamma(\mathcal{S})$ ([Hit]). By [Mo], M has special holonomy in this case. If moreover, $\alpha^n \neq 0$ at some point of M, then one can conclude that M is Kähler:

1.9. Theorem. *Let (M^{2n},g) be a compact, connected, orientable Riemannian manifold, let P_{Spin^c} be a spin^c structure on M with canonical class c, and let $c_{\mathbb{R}} \in H^2(M,\mathbb{R})$ be represented by the closed two-form $-\frac{1}{2\pi}\alpha$. Assume that $\big(\hat{A}(M)\, e^{\frac{c}{2}}\big)[M] \neq 0$. If the scalar curvature of M satisfies*

$$\kappa \geq 2\,\|\alpha\|_g$$

everywhere on M, then $\kappa = 2\,\|\alpha\|_g$, and the universal cover of M is the product of a Kähler manifold with a manifold which admits an untwisted parallel spinor.

Assume moreover that $\alpha^n \neq 0$ in at least one point of M. In this case, there exists a complex structure J such that (M,g,J) is a Kähler manifold with natural spin^c structure P_{Spin^c}, and $i\alpha$ is the curvature of the associated line bundle. The Ricci curvature of M is non-negative and positive somewhere on M, and the Todd genus of M is 1.

1.10. Remark. Conversely, assume that the Ricci curvature of a connected Kähler manifold M is non-negative and positive somewhere on M. Then $\kappa = 2\,\|\rho\|_g$, where ρ is the curvature of the canonical line bundle, and $\mathrm{Td}(M)[M] = 1$ by Bochner's theorem ([LM], Corollary IV.11.12).

Proof of Theorem 1.9. Let ∇^L and D be as in the proof of Proposition 1.7 above. Since $\mathrm{ind}(D) \neq 0$, there exists a harmonic spinor $0 \neq \psi \in \ker(D) \subset \Gamma(\mathcal{S})$. From (1.8), we conclude that

$$\begin{aligned}
0 = \|D\psi\|^2 &= \big\|\nabla^{\mathcal{S}}\psi\big\|^2 + \Big\langle \frac{\kappa}{4}\psi,\psi\Big\rangle + \Big\langle \frac{1}{2}\sum_{j=1}^{n} \alpha(e_{2j-1},e_{2j})\, ic_{2j-1}c_{2j}\,\psi,\psi\Big\rangle \\
&= \big\|\nabla^{\mathcal{S}}\psi\big\|^2 + \Big\langle \Big(\frac{\kappa}{4} - \frac{\|\alpha\|_g}{2}\Big)\psi,\psi\Big\rangle\,.
\end{aligned} \tag{1.11}$$

In particular, ψ is parallel with respect to $\nabla^{\mathcal{S}}$, and $i\lambda_j c_{2j-1}c_{2j}\psi = -\left|\lambda_j\right|\psi$, where at each point, the frame $e_1, \dots, e_{2n}$ is chosen such that (1.4) holds.

The proof proceeds from now on as in [Mo], chapter 3. We will nevertheless give a detailed argument here, because we will need the explicit construction of the complex structure J again in the proof of Theorem 2.10 below. Suppose that $\alpha^n \neq 0$ at some point $p \in M$. This implies that each of the λ_j is non-zero, and moreover negative if the frame $e_1, \dots, e_{2n}$ is chosen accordingly. Thus at p,

$$ic_{2j-1}c_{2j}\psi = \psi \quad \Longrightarrow \quad \big(c_{2j-1} + ic_{2j}\big)\psi = 0\,, \tag{1.12}$$

i.e., ψ is a *pure spinor,* cf. [LM], Definition IV.9.3. Now, Theorem 1.9 is proved in the same way as Proposition IV.9.8 in [LM]: since ψ is parallel, (1.12) still holds if the frame $e_1, \dots, e_{2n}$ at p is parallelly translated to any point of M along any curve. Here, we need in particular that M is connected. Hence, there is a unique parallel almost complex structure J on M such that

$$\big(c(v) + i\, c(Jv)\big)\psi = 0 \tag{1.13}$$

for all $v \in TM$, so g is a Kähler metric. The definition of J may be rephrased by saying that Clifford multiplication with antiholomorphic vectors vanishes on ψ.

In order to prove that P_{Spin^c} is the natural spin^c structure associated to J, we show that $\mathcal{S}$ is isomorphic as a Dirac bundle to the bundle $\Lambda^{0,*}M$ of antiholomorphic forms on M. We claim that there is a natural isomorphism which maps the form 1 to ψ, and more generally,

$$\big(e^{2j_1-1} - ie^{2j_1}\big) \wedge \cdots \wedge \big(e^{2j_k-1} - ie^{2j_k}\big) \quad \longmapsto \quad \big(c_{2j_1-1} - ic_{2j_1}\big) \cdots \big(c_{2j_k-1} - ic_{2j_k}\big)\,\psi \tag{1.14}$$

for all $k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$. Using (1.13), one easily checks that this defines an isomorphism of Clifford modules which is compatible with the Levi-Civita connection ∇ on $\Lambda^{0,*}M$ and with $\nabla^{\mathcal{S}}$ on $\mathcal{S}$.

Once we have identified the canonical line bundle L of our given spin^c structure with the canonical line bundle of (M, g, J), we know that α is the Ricci form of M, and the Ricci curvature is given by (1.6). On the other hand, from

$$\operatorname{tr}(\mathrm{Ric}) = \kappa = 2\,\|\alpha\|_g = 2\,\big\|F^L\big\|_g$$

we can conclude that the Ricci curvature is non-negative, and positive wherever $\alpha^n \neq 0$. By Bochner's theorem, the Todd genus of M is 1. $\square$

One might hope that the conditions of Theorem 1.9 determine the Kähler metric g uniquely. Unfortunately, this is not true in general:

1.15. Example. Let G be a semi-simple Lie group of rank k with maximal torus T. Then G/T carries as many different homogeneous complex structures as the Weyl group of G has elements. Let us fix a homogeneous complex structure J and a positive constant C. Then there exists a $(k-1)$-parameter family of homogeneous Kähler metrics with constant scalar curvature $\kappa = C$. All these metrics have the same Ricci curvature Ric ([Be], chapter 8), which is positive. Thus, the Ricci form $\alpha = \mathrm{Ric}(\,\cdot\,, J\,\cdot\,)$ depends only on J and κ, but not on the metric g. This means that g cannot be completely determined by α and P_{Spin^c}.

In order to get a better control of the metric g, one has to introduce more constraints. For example, one could restrict oneself to metrics which are in a suitable sense "larger" than a fixed background metric. This is precisely what we do in Theorem 2.10 in the next section in a slightly more general setting.

2. Spin^c Area-Extremality of Ricci-positive Kähler Metrics

We apply the results of the previous section to smooth, area-nonincreasing maps from some Riemannian manifold $(N, \bar{g})$ to a Kähler manifold (M, g, J) of positive Ricci curvature. We show that a Kähler metric of positive Ricci curvature is "area-extremal" in a sense closely related to Gromov's definition of area-extremality in [G].

We recall that a map $f\colon N \to M$ is called a *spin map* iff

$$w_2(N) = f^* w_2(M) \ .$$

Let P_{Spin^c} be spin^c structure on M with canonical class c. Then we define the *spin^c degree* of f to be

$$\big(\hat{A}(N)\, f^* e^{\frac{c}{2}}\big)[N] \ . \tag{2.1}$$

Clearly, this is an integer if f is a spin map. If f is a covering map, then $\hat{A}(N) = f^*\hat{A}(M)$, and the spin^c index is precisely $\deg f \cdot \big(\hat{A}(M)\, e^{\frac{c}{2}}\big)[M]$. Thus, the spin^c degree generalizes the degree of a covering map whenever $\big(\hat{A}(M)\, e^{\frac{c}{2}}\big)[M] = 1$. Let g and $\bar{g}$ be Riemannian metrics on M and N. Then f is called *area-nonincreasing* iff

$$\left|f_*(v \wedge w)\right|_g \leq |v \wedge w|_{\bar{g}}$$

for all decomposable two-vectors $v \wedge w \in \Lambda^2 TN$. If f and h are two functions on M, we write

$$f \not> h$$

iff there exists a $p \in M$ with $f(p) \leq h(p)$, i.e., iff f is *not everywhere* greater than h.

2.2. Definition. Let (M, g) be compact, connected, oriented Riemannian manifold with a spin^c structure with canonical class c, and let κ be the scalar curvature of M. Then the metric g is called *spin^c area-extremal* iff

$$\bar{\kappa} \not> \kappa \circ f$$

for all compact, connected, oriented Riemannian manifolds $(N, \bar{g})$ with scalar curvature $\bar{\kappa}$ and all smooth, area-nonincreasing spin maps $f\colon N \to M$ of non-vanishing spin^c degree.

2.3. Remark. This notion of spin^c area-extremality is closely related to area-extremality in the sense of Gromov ([G], section $5\frac{4}{9}$). Recall that the *$\hat{A}$-degree* of a map $f\colon N \to M$ is defined as $(\hat{A}(N)\, f^* v)[N]$, where v is the canonical generator of $H^m(M, \mathbb{Z})$ for a compact, connected, oriented m-dimensional manifold M. Gromov calls a metric g *area-extremal* if the scalar curvatures $\bar{\kappa}$ of N is smaller or equal to $\kappa \circ f$ in at least one point of N for all area-nonincreasing spin maps of non-vanishing $\hat{A}$-degree. Because the $\hat{A}$-degree generalizes the degree of a map between compact, connected, oriented manifolds of the same dimension, the identity of M always has $\hat{A}$-degree 1, and one can compare any metric on M with an area-extremal metric using $f = \mathrm{id}_M$.

On the other hand, if we want to use spin^c area-extremality to compare different metrics on M, we need M to be connected, and id_M to have non-zero spin^c degree. By definition, this means that $\big(\hat{A}(M)\, e^{\frac{c}{2}}\big)[M] \neq 0$. If we regard the natural spin^c structure of a complex manifold, then the spin^c degree of id_M is just the Todd genus of M, which can be zero. In particular, spin^c area-extremality does not automatically imply area-extremality. However, if M carries a Kähler metric g of non-negative Ricci curvature which is positive somewhere, the spin^c degree of id_M is 1 by Bochner's theorem (see Remark 1.10), so g is indeed area-extremal.

From Proposition 1.7, we get immediately a large class of spin^c area-extremal metrics:

2.4. Corollary. *Let (M,g,J) be a compact, connected Kähler manifold of non-negative Ricci curvature. Then the metric g on M is spin^c area-extremal. If the Ricci curvature is positive somewhere on M, then g is also area-extremal.*

Proof. Let $K \to M$ be the canonical line bundle with curvature $F^K = i\alpha$, and let f^*K be the pull-back of K to N, equipped with the pull-back connection. Because f is a spin map, N carries a spin^c structure with canonical line bundle f^*K. The index of the associated Dirac operator is precisely the spin^c degree of the map f. We prove in Lemma 2.7 below that $\|f^*\alpha\|_{\bar g} \leq \|\alpha\|_g \circ f = \kappa \circ f$ if f is area-nonincreasing. Now, the first claim follows by applying Proposition 1.7 to N and α. For the second statement, one uses Bochner's theorem as in the remark above. □

2.5. Remark. Let (M,g,J) be a compact, connected Kähler manifold with non-negative Ricci curvature which is positive somewhere. Then not only is (M,g) area-extremal, but the same holds for any Kähler metric g' on M which is C^2-close to g, such that $\operatorname{supp}(g'-g)$ is contained in the region $M^+ \subset M$ where the Ricci curvature of (M,g) is positive. Such metrics can be constructed by taking a C^4-small function $h\colon M \to \mathbb{R}$ with $\operatorname{supp}(h) \subset M^+$: Let ω be the Kähler form of (M,g,J), then $\omega + i\,\partial\bar\partial h$ is the Kähler form of a metric g' as above. This shows that on all compact, connected Kähler manifolds with non-negative Ricci curvature which is positive somewhere, there exists an infinite-dimensional family of area-extremal metrics.

2.6. Remark. Clearly, the statement of the corollary remains correct if in Definition 2.2, the condition that f is area-nonincreasing is replaced by the much weaker requirement that $\|f^*\alpha\|_{\bar g} \leq \|\alpha\|_g \circ f$. The full power of the stronger condition in Definition 2.2 will only become visible in Theorem 2.10 below, where we investigate the case that $\bar\kappa \geq \kappa \circ f$.

Let us check that indeed $\|f^*\alpha\|_{\bar g} \leq \|\alpha\|_g \circ f$ if f is area-nonincreasing:

2.7. Lemma. *Let (M^{2n},g) be a Riemannian manifold, and let $\|\cdot\|_g$ be the norm on $\Lambda^2 M$ introduced in (1.4). If $(N,\bar g)$ is another Riemannian manifold and $f\colon N \to M$ is a smooth, area-nonincreasing map, then*

$$\|f^*\alpha\|_{\bar g} \leq \|\alpha\|_g \circ f$$

for all alternating two-forms α. Assume that $n \geq 2$ and that $\alpha^n \neq 0$ everywhere on M, then equality implies that f is a Riemannian submersion.

Proof. We show first that $\|\alpha\|_g$ can equivalently be defined as

$$\|\alpha\|_g = \max\left\{ \sum_{j=1}^n \alpha\bigl(e_{2j-1}, e_{2j}\bigr) \;\middle|\; e_1, \dots, e_{2n} \text{ is an orthonormal frame with respect to } g \right\} . \tag{2.8}$$

Let $\alpha = a_1 e^1 \wedge e^2 + \dots + a_n e^{2n-1} \wedge e^{2n}$ be an alternating form on $\mathbb{R}^{2n}$, equipped with the standard Euclidean metric g. Let J be the matrix of the standard complex structure on $\mathbb{R}^{2n}$, and let A be the skew-symmetric matrix that represents α. Then (2.8) can clearly be rewritten as

$$\frac{1}{2} \max\bigl\{ \operatorname{tr}\bigl(BAB^{-1}J\bigr) \bigm| B \in \mathrm{O}(2n) \bigr\} .$$

Writing $B_t = e^{tX}B$ for a skew-symmetric matrix X, we see that B is critical point of the functional $B \mapsto \operatorname{tr}(BAB^{-1}J)$ iff

$$0 = \frac{\partial}{\partial t}\Big|_{t=0} \operatorname{tr}\bigl(B_t A B_t^{-1} J\bigr) = \operatorname{tr}\bigl(XBAB^{-1}J\bigr) - \operatorname{tr}\bigl(BAB^{-1}XJ\bigr) = 2\operatorname{tr}\bigl(XBAB^{-1}J\bigr)$$

for all $X \in \mathfrak{so}(2n)$. This happens iff $BAB^{-1}J$ is a symmetric matrix, i.e. iff J and BAB^{-1} commute. In this case, the eigenvalues of $BAB^{-1}J$ are precisely $\pm a_1, \dots, \pm a_n$, each with multiplicity 2. The maximum clearly occurs if all these eigenvalues are positive. For such a B, $\frac{1}{2}\operatorname{tr}(BAB^{-1}J)$ is indeed the norm defined in (1.4). This proves (2.8). Note that from (2.8), one easily concludes that $\|\cdot\|_g$ is sub-additive, so it is indeed a norm.

Let $\alpha = \alpha_1 + \dots + \alpha_n$ with $\alpha_i = a_i e^{2i-1} \wedge e^{2i}$. Then $\|\alpha\|_g = \|\alpha_1\|_g + \dots + \|\alpha_n\|_g$. Suppose that $f\colon \mathbb{R}^l \to \mathbb{R}^{2n}$ is a linear area-nonincreasing map with respect to the standard Euclidean metrics $\bar g$ on $\mathbb{R}^l$ and g on $\mathbb{R}^{2n}$. Then $\left\|f^*(e^j \wedge e^k)\right\|_{\bar g} \le \left\|e^j \wedge e^k\right\|_g$. Thus, we have

$$\|f^*\alpha\|_{\bar g} \le \|f^*\alpha_1\|_{\bar g} + \dots + \|f^*\alpha_1\|_{\bar g} \le \|\alpha_1\|_g + \dots + \|\alpha_1\|_g = \|\alpha\|_g \ , \tag{2.9}$$

which proves the inequality in the lemma.

Let us assume that equality holds, and that $n \ge 2$ and $\alpha^n \ne 0$. Splitting $\alpha = \alpha_1 + \dots + \alpha_n$ as above, we find that by (2.9), there are orthonormal frames $\bar e_1, \dots, \bar e_l$ of $\mathbb{R}^l$ and $e_1, \dots, e_{2n}$ of $\mathbb{R}^{2n}$ such that

$$f^*\alpha = \sum_{j=1}^n \lambda_j \, \bar e^{2j-1} \wedge \bar e^{2j} \qquad \text{and} \qquad \alpha = \sum_{j=1}^n \lambda_j \, e^{2j-1} \wedge e^{2j}$$

with $\lambda_1, \dots, \lambda_n > 0$, and moreover, $f_*(\bar e_{2j-1} \wedge \bar e_{2j}) = e_{2j-1} \wedge e_{2j}$ for $1 \le j \le n$. We may choose the frames above such that $f_* \bar e_{2j-1} = \mu_j e_{2j-1}$ and $f_* \bar e_{2j} = \mu_j^{-1} e_{2j}$, with $\mu_j \ge 1$. Because f is area-nonincreasing,

$$\mu_j \mu_k = |f_*(\bar e_{2j-1} \wedge \bar e_{2k-1})|_g \le 1$$

for $1 \le j < k \le n$. If $n \ge 2$, this implies clearly that $\mu_1 = \dots = \mu_n = 1$. Using once more that f is area-nonincreasing, one proves that $f_* \bar e_k = 0$ for $k > 2n$. This implies the rigidity statement in Lemma 2.7. □

Remark. On odd-dimensional manifolds, the estimate of the lemma holds unchanged. However, in the case of equality, one gets a weaker statement because one cannot control f_* on $\ker(f^*\alpha)$.

The rigidity statement of Lemma 2.7 can be used to investigate the case that we have the inequality $\bar\kappa \ge \kappa \circ f$ in Definition 2.2:

2.10. Theorem. *Let (M, g) be a compact, connected Kähler manifold of positive Ricci curvature and complex dimension $n \ge 2$, let $(N, \bar g)$ be another compact, connected, oriented Riemannian manifold, and let $f\colon N \to M$ be a smooth spin map of non-zero spin^c degree. Suppose that f is area-nonincreasing and that $\bar\kappa \ge \kappa \circ f$. Then N is a Riemannian product $N = M \times F$, and f is the projection onto the first factor. The manifold F carries a parallel untwisted spinor and is in particular Ricci flat.*

Proof. Let K be the canonical line bundle of M with curvature $F^K = i\alpha$. As in the proof of Corollary 2.4, we construct a spin^c structure P_{Spin^c} on N with canonical line bundle f^*K which has curvature f^*F^K. If f is area-nonincreasing and $\bar\kappa \ge \kappa \circ f$, then

$$2\,\|f^*\alpha\|_{\bar g} = \bar\kappa = \kappa \circ f = 2\,\|\alpha\|_g \circ f$$

as in the proof of Theorem 1.9. Because M is Ricci-positive, we have $\alpha^n \ne 0$, and f is a Riemannian submersion by Lemma 2.7. Moreover, the complex spinor bundle associated to $P_{\mathrm{Spin}^c} \to N$ has a non-trivial parallel spinor. Then by [Mo], the universal cover $\pi\colon \tilde N \to N$ splits as a product of a Kähler manifold $\tilde N_1$ with a manifold $\tilde N_2$ which admits a parallel untwisted spinor and contains no de Rham factor that is Kähler. We will prove first that $f \circ \pi$ factors over the projection $\tilde\pi_1\colon \tilde N \to \tilde N_1$.

In a second step, we show that $\tilde{N}_1$ is isometric to $M \times F_1$, such that f is the projection onto the first factor.

To prove that $f \circ \pi$ factors over $\tilde{\pi}_1$, let ψ be the parallel spinor on $\tilde{N}$. We consider the subspace

$$E := \left\{ \bar{v} \in T\tilde{N} \;\middle|\; \text{there is a vector } \bar{w} \in T\tilde{N} \text{ such that } c_{\bar{v}+i\bar{w}}\psi = 0 \right\}$$

as in [Mo]. Because ψ is parallel, this is a parallel distribution in $T\tilde{N}$, and it carries a parallel complex structure $\bar{J}$ such that $c_{\bar{v}+i\bar{J}\bar{v}}\psi = 0$ for all $\bar{v} \in E$ as in (1.13). Then E is the tangent distribution to the Kähler factor $\tilde{N}_1$ of $\tilde{N}$. Let $\bar{e}_1, \dots, \bar{e}_{\bar{n}}$ be an orthonormal frame of $T_p\tilde{N}$ such that

$$\pi^* f^* \alpha = \sum_{k=1}^{n} \lambda_k \bar{e}^{2k-1} \wedge \bar{e}^{2k} ,$$

where $\lambda_1, \dots, \lambda_n$ are negative as in the proof of Theorem 1.9. Thus, $\bar{e}_{2n+1}, \dots, \bar{e}_{\bar{n}} \in \ker(f_*)$. Because of the analog of equation (1.11) for ψ on $\tilde{N}$, we conclude that $\bar{e}_1, \dots, \bar{e}_{2n} \in E$, and that $\bar{J}\bar{e}_{2k-1} = \bar{e}_{2k}$ for $1 \le k \le n$. This implies already that $f \circ \pi$ factors over $\tilde{\pi}_1$.

Let $i_1 \colon \tilde{N}_1 \to N$ be an embedding of $\tilde{N}_1$ as a factor of N, and write $\bar{f} := f \circ \pi \circ i_1 \colon \tilde{N}_1 \to M$. Now, $e_1 := \bar{f}_*\bar{e}_1, \dots, e_{2n} := \bar{f}_*\bar{e}_{2n}$ form an orthonormal frame of $T_{f(p)}M$ with

$$\alpha = \sum_{k=1}^{n} \lambda_k e^{2k-1} \wedge e^{2k} .$$

Because α is the Ricci form of M, it follows that $Je_{2k-1} = e_{2k}$. To be more precise, write α and J as matrices with respect to the given frame. Because $\operatorname{Ric}_M = \alpha(J\,\cdot\,,\,\cdot\,)$ is symmetric, these matrices commute. Finally, because Ric_M is positive and all the λ_j are negative, J acts as indicated.

In particular, $\bar{f}$ is a holomorphic map. Then the fibers $F_q := \bar{f}^{-1}(q)$ are complex submanifolds of $\tilde{N}_1$ for all $q \in M$, so their mean curvature vector vanishes. The O'Neill formulas for the Ricci curvature ([Be], Proposition 9.36) imply for the horizontal lift $X \in T\tilde{N}_1$ of a vector in TM:

$$\operatorname{Ric}_{\tilde{N}_1}(X,X) = \operatorname{Ric}_M(\bar{f}_*X, \bar{f}_*X) - 2\sum_{j=1}^{2n} \|A(X,\bar{e}_j)\|^2 - \sum_{j=2n+1}^{2n_1} \|T(\bar{e}_j, X)\|^2 , \tag{2.11}$$

where $A(X,Y) = \frac{1}{2}[X,Y]^\perp$ is the curvature of the fibre bundle $\bar{f}\colon \tilde{N}_1 \to M$, and $T(U,X) = (\nabla_U^{\tilde{N}_1} X)^\perp$ is the shape tensor of the fibres. Here, Y is the horizontal lift of another vector in TM, U is any vertical vector, $(\,\cdot\,)^\perp$ denotes projection onto the space of vertical vectors, and $2n_1$ is the dimension of $\tilde{N}_1$.

On the other hand, the Ricci curvatures of $\tilde{N}_1$ and M are related by the equation

$$\operatorname{Ric}_{\tilde{N}_1} = -i\big(\bar{f}^* F^K\big)\big(\bar{J}\,\cdot\,,\,\cdot\,\big) = \big(\bar{f}^*\alpha\big)(\bar{J}\,\cdot\,,\,\cdot\,) = \bar{f}^*\big(\alpha(J\,\cdot\,,\,\cdot\,)\big) = \bar{f}^* \operatorname{Ric}_M .$$

By (2.11), this implies that both tensors A and T vanish, so $\bar{f}\colon \tilde{N}_1 \to M$ is locally isometric to a trivial bundle ([Be], 9.26). By a theorem of Kobayashi ([Be], Theorem 11.26), M is simply connected, so this bundle is also globally isometric to a product, and $\tilde{N}_1 = M \times F_1$, where $\bar{f}$ is the projection onto the first factor. In the same way, N also splits as $M \times F$ with $F = F_1 \times \tilde{N}_2$, such that f is the projection onto the first factor. Finally, F_1 is a Ricci flat Kähler manifold, so it carries a parallel untwisted spinor. Since $\tilde{N}_2$ also has a parallel untwisted spinor, the same holds for F. $\square$

3. The Hilbert Polynomial and Area-Extremality à la Gromov

Suppose that M^{2n} admits a spin^c structure with canonical line bundle L. Then M also admits spin^c structures that have certain tensor powers L^k of L as their canonical line bundle. The index of the associated Dirac operators D_k is a polynomial $P_L(k)$ in k, which we will call the "Hilbert polynomial" in analogy with [LM], section IV.11. If this polynomial is not identically zero, then it has at most n zeros. In this case, there exists a small k such that D_k has non-vanishing index, and we can apply the results of the previous chapter to obtain an estimate of the scalar curvature of M.

For general M, we do not obtain the best estimate possible, because the topology of M is used only superficially. For complex projective spaces and complex hyperquadrics however, the estimate is sharp. For these manifolds, we recover estimates and rigidity statements as in Section 2, but with the spin^c degree replaced by the $\hat{A}$-degree. In particular, we prove Theorem 0.2.

3.1. Theorem. *Let (M^{2n}, g) be a compact, connected, orientable Riemannian manifold of real dimension $2n$. Suppose that $w_2(M)$ is a multiple of the reduction modulo 2 of some $c \in H^2(M, \mathbb{Z})$ with $c^n[M] \neq 0$, and let $c_{\mathbb{R}} \in H^2(M, \mathbb{R})$ be represented by the closed two-form $-\frac{1}{2\pi}\alpha$. Then*

$$\kappa \not\geq \begin{cases} 2n \, \|\alpha\|_g & \text{if } w_2(M) = nc \bmod 2, \quad \text{and} \\ 2(n+1) \, \|\alpha\|_g & \text{if } w_2(M) = (n+1)c \bmod 2. \end{cases}$$

Assume that $\kappa \geq 2n \, \|\alpha\|_g$ in the first case or $\kappa \geq 2(n+1) \, \|\alpha\|_g$ in the second case. Then equality holds, and M is Kähler and biholomorphic to the complex quadric Q^n in the first case, and to $\mathbb{C}P^n$ in the second.

Recall that $Q^n \cong \mathrm{SO}_{n+2}/\mathrm{SO}_n \times \mathrm{SO}_2 \subset \mathbb{C}P^{n+1}$ was defined by the equation $z_0^2 + \cdots + z_{n+1}^2 = 0$.

Proof of Theorem 3.1. If $w_2(M)$ is a multiple of $w_2(L) = c \bmod 2 \in H^2(M, \mathbb{Z}_2)$, we may construct spin^c structures $P^k_{\mathrm{Spin}^c}$ on M with canonical line bundle L^k for all $k \in \mathbb{Z}$ such that $w_2(M) = kc \bmod 2$. Let D_k be the Dirac operator associated to $P^k_{\mathrm{Spin}^c}$ and ∇^{L^k}. By Proposition 1.7, we know that $\mathrm{ind}(D_k) = 0$ if the scalar curvature κ is everywhere larger than $2 \left\| F^{L^k} \right\|_g = 2k \, \|\alpha\|_g$.

On the other hand, we have assumed that $c^n[M] \neq 0$. Then the *"Hilbert polynomial"*

$$P_L(k) := \mathrm{ind}\big(D_k\big) = \Big(\hat{A}(M) \, e^{\frac{kc}{2}}\Big)[M]$$

is a polynomial of degree n in k. It has the non-vanishing leading term $\frac{k^n}{2^n n!} c^n[M]$, because $\hat{A}(M)$ always starts with 1 in degree zero. In particular, $P_L(k)$ has at most n different zeros.

We distinguish two cases: if $w_2(M) = nc \bmod 2$, then for the $n+1$ different values $n, n-2, \ldots,$ $-n$ of k there exists a spin^c structure $P^k_{\mathrm{Spin}^c}$, and the operator D_k is well-defined. In particular, for one k_0 with $|k_0| \leq n$, we have $\mathrm{ind}(D_{k_0}) \neq 0$, so $\kappa \not\geq 2 \, |k_0| \, \|\alpha\|_g \leq 2n \, \|\alpha\|_g$ by Proposition 1.7.

For $w_2(M) = (n+1) \, f^* w_2(H)$, the proof is completely analogous, but we have to choose k_0 among the $n+2$ different values $n+1, \, n-1, \, \ldots, \, -n-1$, which accounts for the slightly weaker estimate.

We will now study the cases where "$\geq$" holds for the estimate in the theorem. If we have $w_2(M) = nc \bmod 2$ and $\kappa \geq 2n \, \|\alpha\|_g$, then we know not only that $\kappa = 2n \, \|\alpha\|_g$ and that M is Kähler by Theorem 1.9, but also that the canonical line bundle K of M coincides with L^n. From a result of Kobayashi and Ochiai, it follows that M is biholomorphic to Q^n ([KO]). Similarly, if $w_2(M) = (n+1)c \bmod 2$ and $\kappa \geq 2(n+1) \, \|\alpha\|_g$, then $K = L^{n+1}$, and M is biholomorphic to $\mathbb{C}P^n$ ([KO]). $\square$

Assume that $M = \mathbb{C}P^n$. In this case, we immediately recover the Hilbert-polynomial of $\mathbb{C}P^n$ from the proof of Theorem 3.1:

3.2. Remark. Let $M = \mathbb{C}P^n$ be equipped with a Kähler metric of positive Ricci curvature. Then D_k is invertible for $k = n-1, n-3, \dots, 1-n$ by Proposition 1.7. This implies that the zeros of the Hilbert polynomial

$$P_n(k) := \operatorname{ind}\bigl(D_k^{\mathbb{C}P^n}\bigr) = \Bigl(\hat{A}(\mathbb{C}P^n)\, e^{\frac{kc}{2}}\Bigr)[\mathbb{C}P^n]$$

of $H \to \mathbb{C}P^N$ are precisely $k = n-1, n-3, \dots, 1-n$ (cf. [LM], chapter IV.11). Since the Todd genus $P_n(n+1)$ of $\mathbb{C}P^n$ equals 1, this completely determines $P_n(k)$. In particular, its leading term is positive. We conclude that if $k \equiv n+1 \pmod 2$, then

$$P_n(k) = 0 \quad \text{if } |k| \le n-1, \qquad \text{and} \qquad P_n(k) > 0 \quad \text{if } k > n-1. \tag{3.3}$$

Moreover, since $\hat{A}(\mathbb{C}P^n) \in H^{4*}(\mathbb{C}P^n)$, the polynomial $P_n(k)$ is even if n is even, and odd if n is odd.

The theorem above admits a reformulation for spin maps of non-vanishing $\hat{A}$-degree. Recall that the $\hat{A}$-degree of a map $f\colon N \to M$ between compact oriented manifolds was defined by

$$\deg_{\hat{A}} f := \bigl(\hat{A}(N)\, f^* v_M\bigr)[N]\ ,$$

where v_M is the canonical generator of $H^{\dim M}(M, \mathbb{Z})$.

3.4. Corollary. *Let M^{2n} be a compact, connected, oriented manifold, let $(N, \bar{g})$ be a compact, connected, oriented Riemannian manifold, and let $f\colon N \to M$ be a smooth map of non-vanishing $\hat{A}$-degree. Suppose that $w_2(N)$ is a multiple of the reduction of f^*c modulo 2 for some $c \in H^2(M, \mathbb{Z})$ with $c^n[M] \neq 0$. Let $f^*c_{\mathbb{R}} \in H^2(N, \mathbb{R})$ be represented by a closed two-form $-\frac{1}{2\pi}\bar{\alpha}$. Then for the scalar curvature $\bar{\kappa}$ of N, we have*

$$\bar{\kappa} \not> \begin{cases} 2n\ \|\bar{\alpha}\|_{\bar{g}} & \text{if } w_2(M) = nc \bmod 2, \quad \text{and} \\ 2(n+1)\ \|\bar{\alpha}\|_{\bar{g}} & \text{if } w_2(M) = (n+1)c \bmod 2. \end{cases}$$

Proof. As in the proof of Theorem 3.1, we can construct spin^c structures $P^k_{\mathrm{Spin}^c}$ whenever $w_2(N) = kf^*c \bmod 2$. The index of the corresponding Dirac operator D_k is then given by

$$P_L(k) = \Bigl(\hat{A}(N)\, e^{\frac{k\, f^*c}{2}}\Bigr)[N]\ .$$

Because $(f^*c)^\nu = f^*(c^\nu) = 0$ for $\nu > n$, this is a polynomial of degree n with non-vanishing leading term

$$\Bigl(\hat{A}(N)\, \frac{k^n}{2^n n!}\, f^*c^n\Bigr)[N] = \frac{k^n}{2^n n!}\, \deg_{\hat{A}} f \cdot c^n[M] \neq 0\ .$$

From here, the proof proceeds as above. □

We will now concentrate on $\mathbb{C}P^n$ and Q^n and prove Theorem 0.2. We have to show that a metric on $\mathbb{C}P^n$ or on Q^n with non-negative Ricci curvature is area-extremal in the sense of Gromov ([G], section $5\frac{4}{9}$):

Proof of Theorem 0.2. We start with $M = \mathbb{C}P^n$. Let $H \to \mathbb{C}P^n$ be the hyperplane bundle, then $c := c_1(H)$ generates $H^2(M, \mathbb{Z})$, and the canonical line bundle of the complex structure on M is H^{-n-1}. We have $w_2(N) = f^*w_2(M) = (n+1)\,f^*c \bmod 2$ for a spin map $f\colon N \to M$, and $c^n[M] \neq 0$. There is a unique unitary connection on H such that the induced connection on H^{-n-1} coincides with the connection on K which is induced by the Levi-Civita connection of the given Kähler metric g. Since $\mathrm{Ric}_M \geq 0$, we have

$$\kappa = \mathrm{tr}\bigl(\mathrm{Ric}_M\bigr) = 2\left\|F^K\right\|_g = 2(n+1)\left\|\alpha\right\|_g$$

with $\alpha := -iF^H$. Thus, if f is area-nonincreasing and of non-vanishing $\hat{A}$-degree, the estimate for $\bar{\kappa}$ of Theorem 0.2 follows from Lemma 2.7 and Corollary 3.4 with $\bar{\alpha} = f^*\alpha$.

Assume that $\bar{\kappa} \geq \kappa \circ f > 0$. From the proof of Theorem 3.1, we know that $P_{f^*H}(k)$ vanishes precisely for $k = n-1,\ n-3,\ \dots,\ 1-n$. This implies that the complex spinor bundle associated to the spin^c structure with canonical bundle $f^*H^{\pm(n+1)}$ admits a parallel spinor. From here on, the argument continues as in the proof of Theorem 2.10.

For $M = Q^n$, we choose H to be the pull-back of the hyperplane bundle on $\mathbb{C}P^{n+1}$ via the canonical embedding. Then $P_{f^*H}(k)$ vanishes for $k = n-2, n-4, \dots, 2-n$. Because $P_{f^*H}(-k) = (-1)^n\,P_{f^*H}(k)$ and $P_{f^*H}(k)$ has at most n zeros, it follows that $P_{f^*H}(n) = (-1)^n P_{f^*H}(-n) \neq 0$. The rest of the proof is the same as above. □

3.5. Remark. One of the reasons that the Ricci curvature has to be strictly positive in Theorem 2.10 is to prevent the following pathological situation: Suppose that $i\colon M_0 \to M$ is the inclusion of a complex submanifold such that $\left\|i^*F^K\right\|_{i^*g} = \left\|F^K\right\|_g$, then the image of N could be contained in M_0, and N might not split as in the theorem. In the setting of the theorem above, this is prevented by the non-vanishing of the $\hat{A}$-degree of f, even if we allow M to have non-negative Ricci curvature (the non-vanishing of the spin^c degree does not allow such a conclusion). It would thus be interesting to know if the rigidity statement of Theorem 0.2 continues to hold for Kähler metrics of non-negative Ricci curvature on $\mathbb{C}P^n$ or Q^n.

3.6. Remark. The proof above relies on the existence of a k-th root of the canonical bundle of the Kähler metric on M, with $k \geq n$. Thus by the theorem of Kobayashi and Ochiai ([KO]), M has to be either $\mathbb{C}P^n$ or Q^n, and the argument above has no obvious generalization to other compact Hermitian symmetric spaces. Nevertheless, in view of Section 2 and the results of [M] and [GS], one might hope that the statement of the theorem remains true for all Ricci-positive Kähler metrics on compact Hermitian symmetric spaces.

3.7. Remark. By the same argument as in Remark 2.5, there is an infinite-dimensional family of Ricci-positive Kähler metrics on $\mathbb{C}P^n$ and Q^n with the standard complex structure. All these metrics are area-extremal in the sense of Gromov. This suggests the following question: does such an infinite-parameter family of area-extremal metrics occur whenever (M,g) is area-extremal in the sense of Gromov and the curvature of M is positive in a suitable sense (e.g. Ricci-positive in the case of Kähler metrics)?

4. Estimates for Smooth Projective Varieties

Theorem 3.1 is particularly well adapted to symplectic manifolds where the symplectic form ω can be represented as the curvature of a line bundle L. We now regard the following special case: Suppose that $M \subset \mathbb{C}P^N$ is a smooth algebraic variety, or more generally, a symplectic submanifold, equipped with an arbitrary Riemannian metric. Let $f\colon M \to \mathbb{C}P^n$ be an area-nonincreasing map that is homotopic to the inclusion. Then we get a rough estimate for the minimum of the scalar curvature κ of M.

If we know more about the topology of M, we can use the parameter k in the proof of Theorem 3.1 to "fine-tune" the estimate mentioned above. For example, if M is a complete intersection, we can determine the zeros of the Hilbert polynomial to obtain a smaller upper bound for $\min\kappa$. This estimate will be sharp for all complete intersections of sufficiently small total degree that carry a Kähler metric of constant positive scalar curvature.

4.1. Corollary. *Let (M^{2n},g) be a compact, orientable Riemannian manifold, and let $f\colon M\to\mathbb{C}P^N$ be a smooth map, such that $\|f^*\omega\|_g\leq n$, where ω is the Kähler form of the Fubini-Study metric on $\mathbb{C}P^N$ with constant holomorphic sectional curvature 4. Assume that $f^*[\omega]^n\neq 0\in H^{2n}(M,\mathbb{R})$, and that the second Stiefel-Whitney class $w_2(M)$ of M is a multiple of $f^*w_2(H)\in H^2(M,\mathbb{Z}_2)$, where H is the hyperplane bundle of $\mathbb{C}P^N$. Then*

$$\kappa\not\geq\begin{cases}4n^2 & \text{if } w_2(M)=n\,f^*w_2(H),\quad\text{and}\\ 4n(n+1) & \text{if } w_2(M)=(n+1)\,f^*w_2(H).\end{cases}$$

Assume that $\kappa\geq 4n^2$ in the first case or $\kappa\geq 4n(n+1)$ in the second case, and that M is connected. Then M is isometric to the complex quadric Q^n in the first case, and to $\mathbb{C}P^n$ in the second.

By Lemma 2.7, the estimates above are applicable if f is an area-nonincreasing map: In this case, the absolute values of the n eigenvalues of $f^*\omega$ with respect to the induced metric are all smaller than 1, so $\|f^*\omega\|_g\leq n$.

Proof of Corollary 4.1. We have equipped $\mathbb{C}P^N$ with the Fubini-Study metric of constant holomorphic sectional curvature 4, i.e. scalar curvature $4N(N+1)$. In this case, the curvature of the hyperplane bundle is given by $F^H=-2i\omega$. The image of the first Chern class $c=c_1(H)$ of H in $H^2(M,\mathbb{R})$ is represented by $\frac{1}{\pi}\omega$. The estimate in the corollary follows from Theorem 3.1 with $\bar\alpha=2f^*\omega$, because $\|f^*\omega\|_g\leq n$.

If we have $\kappa\geq 4n^2$ in the first case or $\kappa\geq 4n(n+1)$ in the second case, then M carries a metric of constant scalar curvature and is biholomorphic to Q^n or $\mathbb{C}P^n$ by Theorem 3.1. In particular, the group of complex automorphisms $\mathfrak{A}(M)$ of M acts transitively on M. From a theorem of Lichnerowicz ([Be], Proposition 2.151), it follows that a maximal compact connected subgroup of $\mathfrak{A}(M)$ acts by isometries. It is easy to see that the metric on M is Hermitian symmetric, and the rigidity statement of the corollary follows. □

Remark. The topological conditions in Corollary 4.1 may be difficult to check for arbitrary manifolds M and arbitrary maps f. There is however a large class of examples where these conditions are automatically satisfied. To begin with, regard $\mathbb{C}P^N$ as a symplectic manifold. If $f^*\omega$ defines a symplectic structure on M, then $f^*[\omega]^n$ is nonzero by definition (actually it suffices that f be homotopic to a map $\bar f$ for which $\bar f^*\omega$ defines a symplectic structure). For example, each smooth projective variety $V\subset\mathbb{C}P^N$ is a symplectic submanifold. Thus, $f^*[\omega]^n\neq 0$ whenever $f\colon V\to\mathbb{C}P^N$ is homotopic to the inclusion.

Next, recall that M is spin iff $w_2(M)$ vanishes, and that f is a spin map iff $w_2(M)=f^*w_2(\mathbb{C}P^N)$. Because $w_2(\mathbb{C}P^N)=(N+1)\,w_2(H)$, we may conclude in both cases that $w_2(M)$ is a multiple of $f^*w_2(H)$.

Now, we specialize the methods developped above to study a particular type of smooth algebraic varieties: A *complete intersection* $V=V^n(a_1,\dots,a_r)$ of complex dimension n is the intersection of r nonsingular hypersurfaces in $\mathbb{C}P^{n+r}$ in general position, defined by homogeneous polynomials of degrees $a_1,\dots,a_r$ (cf. [Hir], section 22.1). We call $|a|:=a_1+\dots+a_r$ the *total degree* of V. We will look at arbitrary Riemannian metrics on V such that there exists a smooth map $f\colon V\to\mathbb{C}P^{n+r}$ which is homotopic to the identity and area-nonincreasing (actually, $\|f^*\omega\|_g\leq n$ is sufficient). In this case, Bär and Bleecker conjectured that $\min\kappa\leq 4n(n+1)$. This bound has already been established in Corollary 4.1.

In Theorem 0.3, we have stated a stronger estimate which fits well with a calculation of the average scalar curvature of V with respect to the induced Kähler metric on V ([O]). Since we have explicit formulas for $w_2(V)$ and $\hat{A}(V)$, we can find the minimal k such that the index of the operator D_k constructed in the proof of Theorem 3.1 above does not vanish. We restate Theorem 0.3 in a slightly stronger version:

4.2. Theorem. *Let $V = V^n(a_1, \dots, a_r)$ be a complete intersection, equipped with an arbitrary Riemannian metric g, and set $|a| := a_1 + \dots + a_r$. Let $f \colon V \to \mathbb{C}P^{n+r}$ be homotopic to the inclusion, such that $\|f^*\omega\|_g \leq n$. Then*

$$\min_{p \in V} \kappa(p) \leq \begin{cases} 4n(n+r+1-|a|) & \text{if } |a| \leq n+r, \\ 0 & \text{if } |a| > n+r,\ n \text{ is even, and } V \text{ is spin,} \\ 4n & \text{if } |a| > n+r, \text{ and } V \text{ is not spin, and} \\ 8n & \text{if } |a| > n+r,\ n \text{ is odd, and } V \text{ is spin.} \end{cases}$$

If V is connected, $|a| \leq n+r$ (or $|a| = n+r+1$ and n is even) and $\kappa \geq 4n(n+r+1-|a|)$, then g is a Kähler metric of constant scalar curvature $4n(n+r+1-|a|)$. If moreover, f is area-nonincreasing, then V is a Kähler-Einstein manifold, and f is an isometric, holomorphic immersion.

Before we prove Theorem 4.2, let us look at some cases where our estimate is sharp. Clearly, no upper bound for $\min \kappa$ in a theorem as the above can be negative: let g be an arbitrary Riemannian metric. Rescaling by some very large constant $C \gg 0$, we can make the absolute value of $\min \kappa$ as small as we want without violating the assumption that $\|f^*\omega\| \leq n$. In particular, 0 is always the best estimate we can hope for.

4.3. Remark. By a computation of Ogiue, the average scalar curvature of a complete intersection, equipped with the pull-back of the Fubini-Study metric, is precisely $4n(n+r+1-|a|)$ ([O], cf. [BB]). For $|a| \leq n+r$, this value coincides with our estimate for $\min \kappa$. In this case, Theorem 4.2 gives a strong rigidity statement if $\kappa \geq 4n(n+r+1-|a|)$: It says that V carries a Kähler metric with constant scalar curvature (which is not necessarily induced from the Fubini-Study metric). According to [Ha], the only complete intersections where the induced metric has constant scalar curvature are precisely the complex projective space $\mathbb{C}P^n$ and the quadric Q^n. On the other hand, some complete intersections admit Kähler-Einstein metrics which are not induced from the Fubini-Study metric. For example, Tian showed that the Fermat hypersurfaces

$$V = \left\{ [z_0 : \dots : z_{n+1}] \in \mathbb{C}P^{n+1} \;\middle|\; z_0^a + \dots + z_{n+1}^a = 0 \right\}$$

admit such a metric if $a = n$ or $a = n+1$ ([T]). The authors do not know if there is an immersion $f \colon V \to \mathbb{C}P^N$ for some $N > n$ that is homotopic to the natural inclusion map and satisfies $\|f^*\omega\|_g \leq n$.

Finally, as pointed out in [BB], any estimate for $\min \kappa$ which is based on n and $|a| - r$ alone must be positive if V is not spin: The reason is that $V = V^n(a)$ is a simply connected hypersurface if $n \geq 3$, which is not spin if $n - a$ is odd. In this case, V carries a metric of positive scalar curvature by a theorem of Gromov and Lawson ([GL]), even if a is very large. This explains the positive estimate in the third case of Theorem 4.2. Next, choose $n = 4k+3$ and a odd, then $V^n(a)$ is spin and simply connected. The generalized $\hat{A}$-genus $\alpha(M)$ vanishes in real dimension $8k+6$, so V carries a metric of positive scalar curvature by a theorem of Stolz ([S1]), regardless of the size of a. Thus, in the last case of Theorem 4.2, the estimate again must be positive. Nevertheless, since our methods produce upper bounds for $\min \kappa$ which are multiples of $4n$, one may expect that the estimates for the last two cases are rather weak.

Before we prove Theorem 4.2, we recall a few facts about complete intersections ([Hir]):

4.4. Lemma. *Let $V = V^n(a_1, \dots, a_r)$ be a complete intersection, and let $f\colon V \to \mathbb{C}P^{n+r}$ be homotopic to the inclusion. Let $\mathbf{x} := f^*c_1(H) \in H^2(V, \mathbb{Z})$. Then*

$$w_2(V) \equiv \bigl(n + r + 1 - |a|\bigr)\,\mathbf{x} \quad (\mathrm{mod}\ 2)\ ,$$

and

$$\hat{A}(V) = \left(\frac{\mathbf{x}/2}{\sinh(\mathbf{x}/2)}\right)^{n+r+1} \prod_{j=1}^{r} \frac{\sinh(a_j\mathbf{x}/2)}{a_j\mathbf{x}/2}\ .$$

In particular, V is spin iff $|a| \equiv n + r + 1 \pmod 2$. Moreover,

$$\mathbf{x}^n[V] = a_1 \cdots a_r\ .$$

Proof. Clearly, the cohomology class $\mathbf{x}$ depends only on the homotopy class of the map f. Now, the first two equations follow from the fact that the total Chern class of TV is given by

$$c(V) = (1 + \mathbf{x})^{n+r+1} \prod_{j=1}^{r} (1 + a_j\mathbf{x})^{-1}\ ,$$

cf. [Hir], chapter 22, equation (1). The last equation is also contained in [Hir]. □

Proof of Theorem 4.2. On V, there exists a spin^c structure $P^k_{\mathrm{Spin}^c}$ with canonical line bundle f^*H^k iff $k\, f^*w_2(H) = w_2(V) \in H^2(V, \mathbb{Z}_2)$. By Lemma 4.4, this is the case if $k \equiv n + r + 1 - |a| \pmod 2$. We denote the Dirac operator associated to $P^k_{\mathrm{Spin}^c}$ by D^V_k. We will determine the smallest $k \geq 0$ such that D^V_k exists and $\mathrm{ind}(D^V_k) \neq 0$. Then our assertion will follow from Proposition 1.7 as in the proof of Theorem 3.1.

By Lemma 4.4, the index of D^V_k can be calculated as follows:

$$\begin{aligned}
\text{(4.5)}\quad \mathrm{ind}\bigl(D^V_k\bigr) = \Bigl(\hat{A}(V)\, e^{\frac{k\mathbf{x}}{2}}\Bigr)[V] &= \left(\left(\frac{\mathbf{x}/2}{\sinh(\mathbf{x}/2)}\right)^{n+r+1} e^{\frac{k\mathbf{x}}{2}} \prod_{j=1}^{r} \frac{\sinh(a_j\mathbf{x}/2)}{a_j\mathbf{x}/2}\right)[V] \\
&= 2^{-n-1}\, \mathrm{res}_{x=0} \left(\sinh\Bigl(\frac{x}{2}\Bigr)^{-n-r-1} e^{\frac{kx}{2}} \prod_{j=1}^{r} \sinh\frac{a_j x}{2}\right) \\
&= 2^{-n-1}\, \mathrm{res}_{x=0} \left(\sinh\Bigl(\frac{x}{2}\Bigr)^{-n-1} e^{\frac{kx}{2}} \prod_{j=1}^{r} \frac{e^{\frac{a_j x}{2}} - e^{-\frac{a_j x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}\right) \\
&= 2^{-n-1}\, \mathrm{res}_{x=0} \left(\sinh\Bigl(\frac{x}{2}\Bigr)^{-n-1} \sum_{l_1=\frac{1-a_1}{2}}^{\frac{a_1-1}{2}} \cdots \sum_{l_r=\frac{1-a_r}{2}}^{\frac{a_r-1}{2}} e^{\frac{(2l_1+\cdots+2l_r+k)\,x}{2}}\right) \\
&= \sum_{l_1=\frac{1-a_1}{2}}^{\frac{a_1-1}{2}} \cdots \sum_{l_r=\frac{1-a_r}{2}}^{\frac{a_r-1}{2}} P_n\bigl(2l_1 + \cdots + 2l_r + k\bigr)\ .
\end{aligned}$$

Here, $P_n(k')$ denotes the index of the operator $D^{\mathbb{C}P^n}_{k'}$ constructed on $\mathbb{C}P^n = V^n(1, \dots, 1) \subset \mathbb{C}P^{n+r}$. Because in the multiple sum above,

$$2l_1 + \cdots + 2l_r + k \equiv |a| - r + k \equiv n + 1 \pmod 2\ ,$$

the operators $D^{\mathbb{C}P^n}_{2l_1+\cdots+2l_r+k}$ exist indeed.

Let us regard the case that n is even. In this case, $P_n(k')$ is an even function of k', and $P_n(k') > 0$ whenever $|k'| > n-1$ and $k' \equiv n+1 \pmod 2$ by (3.3). The multiple sum in (4.5) contains terms of the form $P_n(k')$, where k' ranges between $r - |a| + k$ and $|a| - r + k$. If $|a| \leq n + r$, then the smallest $k \in \mathbb{Z}$ with $k \equiv n + r + 1 - |a|$ such that $P_n(|a| - r + k) > 0$ is given by

$$k_0 = n + r + 1 - |a| \ .$$

By Proposition 1.7, we have in this case

$$\min_{p\in V} \kappa(p) \leq 4n\bigl(n + r + 1 - |a|\bigr) \ .$$

On the other hand, if $|a| > n + r$, we may choose $k = 0$ if V is spin and $k = 1$ otherwise. The corresponding scalar curvature estimates are

$$\min_{p\in V} \kappa(p) \leq 0 \quad \text{if } V \text{ is spin,} \qquad \text{and} \qquad \min_{p\in V} \kappa(p) \leq 4n \quad \text{otherwise.}$$

This proves the estimates of Theorem 4.2 if n is even.

If n is odd, then $P_n(k')$ is an odd function in k'. In particular by (4.5), $\operatorname{ind}(D_0^V) = 0$ if V is spin, even if $|a|$ is very large. We calculate

$$\begin{aligned} \operatorname{ind}\bigl(D_k^V\bigr) - \operatorname{ind}\bigl(D_{k-2}^V\bigr) &= 2^{-n-1} \operatorname{res}_{x=0}\biggl(\sinh\Bigl(\frac{x}{2}\Bigr)^{-n-r-1}\Bigl(e^{\frac{kx}{2}} - e^{\frac{(k-2)x}{2}}\Bigr)\prod_{j=1}^r \sinh\frac{a_jx}{2}\biggr) \\ &= 2^{-n} \operatorname{res}_{x=0}\biggl(\sinh\Bigl(\frac{x}{2}\Bigr)^{-n-r} e^{\frac{(k-1)x}{2}} \prod_{j=1}^r \sinh\frac{a_jx}{2}\biggr) = \operatorname{ind}\bigl(D_{k-1}^W\bigr) \ , \end{aligned}$$

where $W := V^{n-1}(1, a_1, \dots, a_r)$ is the transverse intersection of V with a generic hyperplane in $\mathbb{C}P^{n+r}$ (actually, we do not need the existence of W. We only need the properties of the formal expression for $\operatorname{ind}(D_{k-1}^W)$, which we established in the previous paragraphs). Let $k_0 \geq 0$ be the smallest value of k such that $\operatorname{ind}(D_k^W) \neq 0$. It follows that $k_0 + 1$ is the smallest value of k such that $\operatorname{ind}(D_k^V) \neq 0$. This completes the proof of the estimates.

Let us now look at the case where $\kappa \geq 4n(n + r + 1 - |a|)$ and $|a| \leq n + r$ (or $|a| = n + r + 1$ and n is even), Then we have

$$\kappa \geq 4n(n + r + 1 - |a|) \geq 4\,(n + r + 1 - |a|)\ \|f^*\omega\|_g = 2\left\|f^*F^{H^{-(n+r+1-|a|)}}\right\|_g \ ,$$

and the corresponding Dirac operator has non-vanishing index by our calculations above. Then as in the proof of Theorem 1.9, there exists a parallel spinor ψ on V. Since we assumed that f is homotopic to the inclusion $V \subset \mathbb{C}P^N$, we know that $(f^*\omega^n)[V] \neq 0$, in particular $f^*\omega^n \neq 0$ in at least one point of V. This is enough to ensure that ψ is a parallel, pure spinor, thus it defines a parallel complex structure on V, and g is a Kähler metric of constant scalar curvature.

If we assume in addition that f is area-nonincreasing, then each of the n eigenvalues of $f^*\omega$ has to be 1, so f is an isometric immersion, and f_*T_pV is a complex subspace of $T_{f(p)}\mathbb{C}P^{n+r}$ for all $p \in V$. Moreover, the canonical line bundle of V has curvature

$$-2i\,(n + r + 1 - |a|)\, f^*\omega = -(n + r + 1 - |a|)\, f^*F^H \ .$$

Because V has positive Ricci curvature by Theorem 1.9, this implies that f is holomorphic if we choose the proper complex structure on V. Finally, we conclude that the metric g is Kähler-Einstein with Einstein constant $2(n + r + 1 - |a|)$. $\square$

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