CLIFFORD STRUCTURES ON RIEMANNIAN MANIFOLDS

ANDREI MOROIANU AND UWE SEMMELMANN

Abstract. We introduce the notion of *even Clifford structures* on Riemannian manifolds, which for rank $r = 2$ and $r = 3$ reduce to almost Hermitian and quaternion-Hermitian structures respectively. We give the complete classification of manifolds carrying *parallel* rank r even Clifford structures: Kähler, quaternion-Kähler and Riemannian products of quaternion-Kähler manifolds for $r = 2$, 3 and 4 respectively, several classes of 8-dimensional manifolds (for $5 \leq r \leq 8$), families of real, complex and quaternionic Grassmannians (for $r = 8$, 6 and 5 respectively), and Rosenfeld's elliptic projective planes $\mathbb{OP}^2$, $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$, $(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2$ and $(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$, which are symmetric spaces associated to the exceptional simple Lie groups F_4, E_6, E_7 and E_8 (for $r = 9$, 10, 12 and 16 respectively). As an application, we classify all Riemannian manifolds whose metric is bundle-like along the curvature constancy distribution, generalizing well-known results in Sasakian and 3-Sasakian geometry.

2000 *Mathematics Subject Classification*: Primary 53C26, 53C35, 53C10, 53C15.

Keywords: Clifford structure, Kähler, quaternion-Kähler, symmetric spaces, exceptional Lie groups, Rosenfeld's elliptic projective planes, curvature constancy, fat bundles.

1. Introduction

The main goal of the present paper is to introduce a new algebraic structure on Riemannian manifolds, which we refer to as *Clifford structure*, containing almost complex structures and almost quaternionic structures as special cases.

Roughly speaking, by a Clifford (resp. even Clifford) structure on a Riemannian manifold (M, g) we understand a Euclidean vector bundle (E, h) over M, called *Clifford bundle*, together with a representation of the Clifford algebra bundle $\mathrm{Cl}(E, h)$ (resp. $\mathrm{Cl}^0(E, h)$) on the tangent bundle TM. One might notice the duality between spin and Clifford structures: While in spin geometry, the spinor bundle is a representation space of the Clifford algebra bundle of TM, in the new framework, it is the tangent bundle of the manifold which becomes a representation space of the (even) Clifford algebra bundle of the Clifford bundle E.

Several approaches to the concept of Clifford structures on Riemannian manifolds can be found in the literature. We must stress from the very beginning on the somewhat misleading fact that the same terminology is used for quite different notions. Most authors have introduced Clifford structures as a family of *global* almost complex structures satisfying the Clifford relations, i.e. as a pointwise representation of the Clifford algebra Cl_n on each tangent space of the manifold. In the sequel we will refer to these structures as *flat Clifford*

Date: December 14, 2010.

This work was supported by the French-German cooperation project Procope no. 17825PG.

structures. In contrast, our definition only involves *local* almost complex structures, obtained from local orthonormal frames of the Clifford bundle E, and reduces to the previous notion when E is trivial.

Flat Clifford structures were considered by Spindel et al. in [20], motivated by the fact that in the 2-dimensional supersymmetric σ-model, a target manifold with $N-1$ independent parallel anti-commuting complex structures gives rise to N supersymmetries. They claimed that on group manifolds $N \leq 4$ but later on, Joyce showed that this restriction does not hold in the non-compact case (cf. [14]) and provided a method to construct manifolds with arbitrarily large Clifford structures. At the same time, Barberis et al. constructed in [3] flat Clifford structures on compact flat manifolds, by means of 2-step nilpotent Lie groups.

Yet another notion of Clifford structures was used in connection with the Osserman Conjecture. Following ideas of Gilkey, Nikolayevsky defined in [18] Clifford structures on Riemannian manifolds with an additional assumption on the Riemannian curvature tensor.

An author who comes close to our concept of even Clifford structure, but restricted to a particular case, is Burdujan. His *Clifford-Kähler* manifolds, introduced in [6] and [7], correspond in our terminology to manifolds with a rank 5 parallel even Clifford structure. He proves that such manifolds have to be Einstein (a special case of Proposition 2.9 below).

The core of the paper consists of the classification of manifolds carrying parallel even Clifford structures, cf. Theorem 2.13. In rank $r = 2$ and $r = 3$ this reduces to Kähler and quaternion-Kähler structures respectively. We obtain Riemannian products of quaternion-Kähler manifolds for $r = 4$, several classes of 8-dimensional manifolds (for $5 \leq r \leq 8$), families of real, complex and quaternionic Grassmannians (for $r = 8,\ 6$ and 5 respectively), and Rosenfeld's elliptic projective planes $\mathbb{OP}^2$, $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$, $(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2$ and $(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$, which are symmetric spaces associated to the exceptional simple Lie groups F_4, E_6, E_7 and E_8 (for $r = 9,\ 10,\ 12$ and 16 respectively). Using similar arguments we also classify manifolds carrying parallel Clifford structures, showing that parallel Clifford structures can only exist in low rank ($r \leq 3$), in low dimensions ($n \leq 8$) or on flat spaces (cf. Theorem 2.14).

In Section 3, we give a geometric application of our classification theorem to the theory of manifolds with *curvature constancy*, a notion introduced in the 60's by Gray [11]. Roughly speaking, a tangent vector X on a Riemannian manifold (Z, g_Z) belongs to the curvature constancy $\mathcal{V}$ if its contraction with the Riemannian curvature tensor R^Z equals its contraction with the algebraic curvature tensor of the round sphere, cf. (28) below. One reason why Gray was interested in this notion is that on the open set of Z where the dimension of the curvature constancy achieves its minimum, $\mathcal{V}$ is a totally geodesic distribution whose integral leaves are locally isomorphic to the round sphere.

Typical examples of manifolds with non-trivial curvature constancy are Sasakian and 3-Sasakian manifolds, the dimension of $\mathcal{V}$ being (generically) 1 and 3 respectively. Rather curiously, Gray seems to have overlooked these examples when he conjectured in [11] that if the curvature constancy of a Riemannian manifold (Z, g_Z) is non-trivial, then the manifold is locally isometric to the round sphere. By the above, this conjecture is clearly false, but one may wonder whether counter-examples, other than Sasakian and 3-Sasakian structures, do exist.

Using Theorem 2.13, we classify Riemannian manifolds Z admitting non-trivial curvature constancy $\mathcal{V}$ under the additional assumption that the metric is *bundle-like* along the distribution $\mathcal{V}$, i.e. such that Z is locally the total space of a Riemannian submersion $Z \to M$ whose fibres are the integral leaves of $\mathcal{V}$, cf. Theorem 3.8. Notice that Sasakian and 3-Sasakian manifolds appear in this classification, being total spaces of (locally defined) Riemannian submersion over Kähler and quaternion-Kähler manifolds respectively.

Bundle-like metrics with curvature constancy also occur as a special case of *fat bundles*, introduced by Weinstein in [21] and revisited by Ziller (cf. [22], [10]). A Riemannian submersion is called fat if the sectional curvature is positive on planes spanned by a horizontal and a vertical vector. Homogeneous fat bundles were classified by Bérard-Bergery in [4]. It turns out that all our homogeneous examples with curvature constancy (cf. Table 3) may be found in his list. It is still an open question whether examples of non-homogeneous fat bundles with fibre dimension larger than one exist (cf. [22]).

Acknowledgments. We are indebted to P. Gauduchon for several suggestions and remarks, to P.-A. Nagy who has drawn our attention on the curvature constancy problem, and to W. Ziller for having communicated us his work [22] on fat bundles and for several useful comments.

2. Clifford structures

We refer to [16] for backgrounds on Clifford algebras and Clifford bundles.

Definition 2.1. A rank r *Clifford structure* on a Riemannian manifold (M^n, g) is an oriented rank r Euclidean bundle (E, h) over M together with a non-vanishing algebra bundle morphism, called *Clifford morphism*, $\varphi : \mathrm{Cl}(E, h) \to \mathrm{End}(TM)$ which maps E into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(TM)$.

The image by φ of every unit vector $e \in E_x$ is a Hermitian structure J_e on T_xM (i.e. a complex structure compatible with the metric g):

$$J_e^2 = \varphi(e) \circ \varphi(e) = \varphi(e \cdot e) = \varphi(-h(e, e)) = -\mathrm{id}_{T_xM}.$$

Since the square norm of a Hermitian structure J is equal to the dimension n of the space on which it acts, we see that (E, h) can be identified by φ with its image $\varphi(E) \subset \mathrm{End}^-(TM)$ endowed with the Euclidean metric $\frac{1}{n}g$.

The universality property of the Clifford algebra immediately shows that a rank r Clifford structure on (M, g) is a rank r sub-bundle of $\mathrm{End}^-(TM)$, locally spanned by anti-commuting almost complex structures J_i, $i = 1, \ldots, r$. In terms of G-structures, a Clifford structure is equivalent to the reduction of the orthonormal frame bundle of M to the normalizer of $\mathrm{Pin}(r)$ in $\mathrm{SO}(n)$ (where the inclusion $\mathrm{Pin}(r) \subset \mathrm{SO}(n)$ is given by the orthogonal representation of Cl_r on $\mathbb{R}^n$ defined, up to conjugacy, by the restriction of the map φ to some fibre $\mathrm{Cl}(E, h)_x$). There are thus topological restrictions to the existence of Clifford structures on a given manifold, but we will not further develop this point of view here.

A Clifford structure (M, g, E, h) is called *parallel* if the sub-bundle $\varphi(E)$ of $\mathrm{End}^-(TM)$ is parallel with respect to the Levi-Civita connection ∇^g of (M, g).

Since every oriented rank 1 vector bundle is trivial, there is a one-to-one correspondence between rank 1 Clifford structures and almost Hermitian structures on (M, g) (up to a sign). A rank 1 Clifford structure is parallel if and only if the corresponding almost Hermitian structure is Kähler.

Every hyper-Kähler manifold (M^n, g, I, J, K) carries parallel rank 2 Clifford structures (e.g. the sub-bundle of $\mathrm{End}^-(TM)$ generated by I and J). The converse holds for $n > 4$ (cf. Theorem 2.14 below). Notice also that by the very definition, a quaternion-Kähler structure is nothing else but a parallel rank 3 Clifford structure.

The classification of n-dimensional Riemannian manifolds carrying rank r parallel Clifford structures will be given in Theorem 2.14 below. It turns out that parallel Clifford structures can only exist either in low ranks ($r \leq 3$), or in low dimensions ($n \leq 8$) or on flat spaces. Therefore, even though it provides a common framework for Kähler, quaternion-Kähler and hyper-Kähler geometries, the notion of parallel Clifford structure is in some sense too restrictive.

We will now introduce a natural extension of Definition 2.1, by requiring the Clifford morphism to be defined only on the *even* Clifford algebra bundle of E. We gather in this way much more flexibility and examples, while a complete classification in the parallel case is still possible.

Definition 2.2. A rank r *even Clifford structure* ($r \geq 2$) on a Riemannian manifold (M^n, g) is an oriented rank r Euclidean bundle (E, h) over M together with a non-trivial algebra bundle morphism, called *Clifford morphism*, $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(TM)$ which maps $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(TM)$. Recall that $\Lambda^2 E$ is viewed as a sub-bundle of $\mathrm{Cl}^0(E, h)$ by identifying $e \wedge f$ with $e \cdot f + h(e, f)$ for every $e, f \in E$.

As before, an even Clifford structure is equivalent to the reduction of the orthonormal frame bundle of M to the normalizer of the image of $\mathrm{Spin}(r)$ in $\mathrm{SO}(n)$ through some non-trivial representation of the Clifford algebra Cl_r on $\mathbb{R}^n$. In more familiar terms, an even Clifford structure can be characterized as follows:

Lemma 2.3. *Let $\{e_1, \ldots, e_r\}$ be a local orthonormal frame on E. The local endomorphisms $J_{ij} := \varphi(e_i \cdot e_j) \in \mathrm{End}(TM)$ are skew-symmetric for $i \neq j$ and satisfy*

$$(1) \qquad \begin{cases} J_{ii} = -\mathrm{id} & \textit{for all } 1 \leq i \leq r, \\ J_{ij} = -J_{ji} \textit{ and } J_{ij}^2 = -\mathrm{id} & \textit{for all } i \neq j, \\ J_{ij} \circ J_{ik} = J_{jk} & \textit{for all } i, j, k \textit{ mutually distinct}, \\ J_{ij} \circ J_{kl} = J_{kl} \circ J_{ij} & \textit{for all } i, j, k, l \textit{ mutually distinct}. \end{cases}$$

Moreover, if $r \neq 4$, then

$$(2) \qquad < J_{ij}, J_{kl} >= 0, \qquad \textit{unless } i = j,\ k = l \textit{ or } i = k \neq j = l \textit{ or } i = l \neq k = j.$$

Proof. The first statements follow directly from the usual relations in the Clifford algebra

$$\begin{cases} e_i \cdot e_j = -e_j \cdot e_i & \text{for all } \ i \neq j, \\ (e_i \cdot e_j)^2 = -\mathrm{id} & \text{for all } \ i \neq j, \\ (e_i \cdot e_j) \cdot (e_i \cdot e_k) = (e_j \cdot e_k) & \text{for all } i, j, k \text{ mutually distinct}, \\ (e_i \cdot e_j) \cdot (e_k \cdot e_l) = (e_k \cdot e_l) \cdot (e_i \cdot e_j) & \text{for all } i, j, k, l \text{ mutually distinct}. \end{cases}$$

The orthogonality of J_{ij} and J_{kl} is obvious when exactly two of the subscripts coincide (since the corresponding endomorphisms anti-commute). For $r = 3$, (2) is thus satisfied. Assume now that $r \geq 5$ and that all four subscripts are mutually distinct. We then choose s different from i, j, k, l and write, using the fact that J_{sl} and J_{ij} commute:

$$\begin{aligned} < J_{ij}, J_{kl} > &= \mathrm{tr}(J_{ij} J_{kl}) = \mathrm{tr}(J_{ij} J_{sk} J_{sl}) = \mathrm{tr}(J_{sl} J_{ij} J_{sk}) \\ &= \mathrm{tr}(J_{ij} J_{sl} J_{sk}) = \mathrm{tr}(J_{ij} J_{lk}) = - < J_{ij}, J_{kl} > . \end{aligned}$$

□

Every Clifford structure E induces an even Clifford structure of the same rank. To see this, one needs to check on a local orthonormal frame $\{e_1, \ldots, e_r\}$ of E that $\varphi(e_i \wedge e_j)$ is skew-symmetric for all $i \neq j$. This is due to the fact that $\varphi(e_i \wedge e_j) = \varphi(e_i) \circ \varphi(e_j)$ is the composition of two anti-commuting skew-symmetric endomorphisms.

The converse also holds if the rank of the Clifford bundle E is equal to 3 modulo 4. Indeed, if $r = 4k + 3$, the Hodge isomorphism $E \simeq \Lambda^{r-1} E \subset \mathrm{Cl}^0(E, h)$ extends by the universality property of the Clifford algebra to an algebra bundle morphism $h : \mathrm{Cl}(E, h) \to \mathrm{Cl}^0(E, h)$. Thus, if $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(TM)$ is the Clifford morphism defining the even Clifford structure, then $\varphi \circ h : \mathrm{Cl}(E, h) \to \mathrm{End}(TM)$ is an algebra bundle morphism mapping E into $\mathrm{End}^-(TM)$ (because the image by $\varphi \circ h$ of every element of E is a composition of $2k + 1$ mutually commuting skew-symmetric endomorphisms of TM).

If the rank of the Clifford bundle E is not equal to 3 modulo 4, the representation of $\mathrm{Cl}^0(E, h)$ on TM cannot be extended in general to a representation of the whole Clifford algebra bundle $\mathrm{Cl}(E, h)$. This can be seen on examples as follows. If $r = 1$, 2, 4 or 8 modulo 8, one can take $M = \mathbb{R}^n$ to be the representation space of an irreducible representation of Cl_r^0 and E to be the trivial vector bundle of rank r over M. Then the obvious even Clifford structure E does not extend to a Clifford structure simply for dimensional reasons (the dimension of any irreducible representation of Cl_r is twice the dimension of any irreducible representation of Cl_r^0 for r as above). For $r = 5$, an example is provided by the quaternionic projective space $\mathbb{HP}^2$ which carries an even Clifford structure of rank 5 (cf. Theorem 2.13). On the other hand, any Riemannian manifold carrying a rank 5 Clifford structure is almost Hermitian (with respect to the endomorphism induced by the volume element of the Clifford algebra bundle), and it is well-known that $\mathbb{HP}^2$ carries no almost complex structure, its first Pontryagin class being an odd multiple of the generator of $H^4(\mathbb{HP}^2, \mathbb{Z})$. Finally, for $r = 6$, an example is given by the complex projective space $\mathbb{CP}^4$, which carries a rank 6 even Clifford structure (cf. Theorem 2.13), but no rank 6 Clifford structure, since this would imply the triviality of its canonical bundle. Similar examples can be constructed for all $r = 5$ and 6 mod 8.

An even Clifford structure (M, g, E, h) is called *parallel*, if there exists a metric connection ∇^E on (E, h) such that φ is connection preserving, i.e.

$$\varphi(\nabla^E_X \sigma) = \nabla^g_X \varphi(\sigma) \tag{3}$$

for every tangent vector $X \in TM$ and section σ of $\mathrm{Cl}^0(E, h)$.

Remark 2.4. For r even, the notion of an even Clifford structure of rank r admits a slight extension to the case where E is no longer a vector bundle but a *projective* bundle, i.e. a locally defined vector bundle associated to some G-principal bundle via a projective representation $\rho : G \to \mathrm{PSO}(r) = \mathrm{SO}(r)/\{\pm I_r\}$. Since the extension of the standard representation of $\mathrm{SO}(r)$ from $\mathbb{R}^r$ to $\Lambda^2\mathbb{R}^r$ factors through $\mathrm{PSO}(r)$, we see that the second exterior power of any projective vector bundle is a well-defined vector bundle, so Definition 2.2 can be adapted to this setting and the corresponding structure will be referred to as *projective even Clifford structure* in the sequel.

The main goal of this section is to classify (cf. Theorem 2.13) complete simply connected Riemannian manifolds (M, g) which carry parallel even Clifford structures as introduced in Definition 2.2, in the extended sense of Remark 2.4. The results are listed in Tables 1 and 2 below. The classification of manifolds carrying parallel Clifford structures will then be obtained as a by-product of Theorem 2.13 by a case-by-case analysis.

We start by examining even Clifford structures of low rank.

Example 2.5. A rank 2 even Clifford structure induces an almost Hermitian structure on (M, g) (the image by φ of the volume element of $\Lambda^2 E$). Conversely, every almost Hermitian structure J on (M, g) induces a rank 2 even Clifford structure by taking (E, h) to be an arbitrary oriented rank 2 Euclidean bundle and defining φ by the fact that it maps the volume element of (E, h) onto J. An even Clifford structure is parallel if and only if the corresponding almost Hermitian structure J is a Kähler structure on (M, g).

Example 2.6. A rank 3 even Clifford structure induces a quaternionic structure on (M, g) i.e. a rank 3 sub-bundle S of $\mathrm{End}(TM)$ locally spanned by three almost Hermitian structures satisfying the quaternion relations. If $\{e_1, e_2, e_3\}$ is a local orthonormal basis of E, S is spanned by $I := \varphi(e_1 \cdot e_2)$, $J := \varphi(e_2 \cdot e_3)$ and $K := \varphi(e_3 \cdot e_1)$. Conversely, every quaternionic structure S on (M, g) induces a rank 3 even Clifford structure by taking $E = S$ with the induced Euclidean structure and defining φ as the Hodge isomorphism $\Lambda^2 E \simeq E = S$. By this correspondence, a parallel even Clifford structure is equivalent to a quaternion-Kähler structure on M.

Note that the quaternion-Kähler condition is empty in dimension 4. There are several ways to see this, e.g. by saying that $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) = \mathrm{SO}(4)$ so there is no holonomy restriction. In our setting, this corresponds to the fact that the bundle $E := \Lambda^2_+ M$ of self-dual 2-forms canonically defines a rank 3 parallel even Clifford structure on every 4-dimensional (oriented) Riemannian manifold.

We thus see that Kähler and quaternion-Kähler geometries fit naturally in the more general framework of parallel even Clifford structures.

The fact that the Lie algebra $\mathfrak{so}(4)$ is not simple reduces the case $r = 4$ to $r = 3$ (see Proposition 2.9 (i) below).

Let us now make the following:

Definition 2.7. A parallel even Clifford structure (M, E, ∇^E) is called *flat* if the connection ∇^E is flat.

Theorem 2.8. *A complete simply connected Riemannian manifold (M^n, g) carrying a flat even Clifford structure E of rank $r \geq 5$ is flat, (and thus isometric with a Cl^0_r representation space).*

Proof. One can choose a parallel global orthonormal frame $\{e_i\}, i = 1, \ldots, r$, on E, which induces global parallel complex structures $J_{ij} := \varphi(e_i \cdot e_j)$ on M for every $i < j$.

We claim that if M is irreducible, then it is flat. Since M is hyper-Kähler with respect to the triple J_{12}, J_{31}, J_{23}, it has to be Ricci-flat. According to the Berger-Simons holonomy theorem, M is either symmetric (hence flat, since a symmetric Ricci-flat manifold is flat), or has holonomy $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)$ or $\mathrm{Spin}(7)$. The last three cases actually do not occur. Indeed, the space of parallel 2-forms on M corresponds to the fixed points of the holonomy representation on $\Lambda^2\mathbb{R}^n$, or equivalently to the centralizer of the holonomy Lie algebra $\mathfrak{hol}(M)$ in $\mathfrak{so}(n)$. This centralizer is zero for $\mathrm{Hol}(M) = \mathrm{Spin}(7)$, 1-dimensional for $\mathrm{Hol}(M) = \mathrm{SU}(n/2)$ and 3-dimensional $\mathrm{Hol}(M) = \mathrm{Sp}(n/4)$. On the other hand, the space of parallel 2-forms on M has dimension at least $r - 1 \geq 4$ (any two of J_{1i}, $1 < i \leq r$ anti-commute so they are linearly independent), a contradiction which proves our claim.

Back to the general case, the de Rham decomposition theorem states that M is a Riemannian product $M = M_0 \times M_1 \times \ldots \times M_k$, where M_0 is flat, and each M_i, $i \geq 1$ is irreducible, non-flat. It is well-known that a parallel complex structure J on a Riemannian manifold (M, g) preserves the tangent bundle of every irreducible non-flat factor of M. Indeed, if M_1 is such a factor, then $J(TM_1) \cap TM_1$ is a parallel sub-bundle of TM_1, so either $J(TM_1) = TM_1$ or $g(JX, Y) = 0$ for all $X, Y \in TM_1$. But the latter case is impossible since otherwise the Bianchi identity would imply

$$R(X, Y, X, Y) \;=\; R(X, Y, JX, JY) = R(X, JX, Y, JY) + R(JX, Y, X, JY) = 0$$

for all $X, Y \in TM_1$, so M_1 would be flat.

Consequently, each non-flat irreducible factor in the de Rham decomposition of M is preserved by every J_{ij}, and thus inherits a flat even Clifford structure of rank r. The first part of the proof shows that no such factor exists, so $M = M_0$ is flat. $\square$

The next result is crucial for the classification of parallel even Clifford structures.

Proposition 2.9. *Assume that the complete simply connected Riemannian manifold (M^n, g) carries a parallel non-flat even Clifford structure (E, ∇^E) of rank $r \geq 3$. Then the following holds:*

(i) *If $r = 4$ then (M, g) is a Riemannian product of two quaternion-Kähler manifolds.*
(ii) *If $r \neq 4$ and $n \neq 8$ then*

(a) *The curvature of* ∇^E*, viewed as a map from* $\Lambda^2 M$ *to* $\mathrm{End}^-(E) \simeq \Lambda^2 E$ *is a non-zero constant times the metric adjoint of the Clifford map* φ*.*
(b) M *is Einstein with non-vanishing scalar curvature and has irreducible holonomy.*

(iii) *If* $r \neq 4$ *and* $n = 8$*, then (a) implies (b).*

Proof. Any local orthonormal frame $\{e_1, \dots, e_r\}$ on E induces local endomorphisms on M defined as before by $J_{ij} := \varphi(e_i \cdot e_j)$. We denote by ω_{ij} the curvature forms of the connection ∇^E with respect to the local frame $\{e_i\}$:

$$R^E_{X,Y} e_i = \sum_{j=1}^{r} \omega_{ji}(X,Y) e_j.$$

From (3) we immediately get $\varphi \circ R^E_{X,Y} = R_{X,Y} \circ \varphi$, where R denotes the Riemannian curvature tensor on (M, g). Consequently,

$$\begin{aligned} R_{X,Y} J_{ij} =& R_{X,Y} \varphi(e_i \cdot e_j) = \varphi[R^E_{X,Y}(e_i \cdot e_j)] \\ =& \varphi[\sum_{s=1}^{r} \omega_{si}(X,Y) e_s \cdot e_j + e_i \cdot \sum_{s=1}^{r} \omega_{sj}(X,Y) e_s] \\ =& \sum_{s=1}^{r} [\omega_{si}(X,Y) J_{sj} + \omega_{sj}(X,Y) J_{is}]. \end{aligned} \tag{4}$$

We take $i \neq j$, apply this to some vector Z and take the scalar product with $J_{ij}(W)$ to obtain

$$\begin{aligned} R(X,Y,J_{ij}(Z),J_{ij}(W)) - R(X,Y,Z,W) &= -2\omega_{ij}(X,Y) g(J_{ij}(Z), W) \\ &\quad + \sum_{s=1}^{r} [\omega_{si}(X,Y) g(J_{si}(Z), W) + \omega_{sj}(X,Y) g(J_{sj}(Z), W)]. \end{aligned} \tag{5}$$

For $i \neq j$ we define the local two-forms R^{ij} on M by

$$R^{ij}(X,Y) := \sum_{a=1}^{n} R(J_{ij} X_a, X_a, X, Y), \tag{6}$$

where $\{X_a\}$ denotes a local orthonormal frame on M. In other words, R^{ij} is twice the image of the 2-form J_{ij} via the curvature endomorphism $R : \Lambda^2 M \to \Lambda^2 M$. The first Bianchi identity easily shows that $R^{ij}(X,Y) = 2 \sum_{a=1}^{n} R(X, X_a, J_{ij} X_a, Y)$.

(i) Assume that $r = 4$. The image $v := \varphi(\omega)$ of the volume element $\omega := e_1 \cdot e_2 \cdot e_3 \cdot e_4 \in \mathrm{Cl}^0(E)$ is a parallel involution of TM commuting with the $\mathrm{Cl}^0(E)$-action, so the tangent bundle of M splits into a parallel direct sum $TM = T^+ \oplus T^-$ of the ± 1 eigen-distributions of v. By the de Rham decomposition theorem, M is a Riemannian product $M = M^+ \times M^-$. The restriction of φ to $\Lambda^2_\pm E$ is trivial on $T^\pm$ and defines a rank 3 Clifford structure on $M^\pm$. More explicitly, one can define a local orthonormal frame

$$e_1^\pm := \frac{1}{2}\Big(e_1 \wedge e_2 \pm e_3 \wedge e_4\Big), \quad e_2^\pm := \frac{1}{2}\Big(e_1 \wedge e_3 \mp e_2 \wedge e_4\Big), \quad e_3^\pm := \frac{1}{2}\Big(e_1 \wedge e_4 \pm e_2 \wedge e_3\Big) \tag{7}$$

of $\Lambda^2_\pm E$ and it is clear that the local endomorphisms $J^\pm_{ij} := \varphi(e^\pm_i)\circ\varphi(e^\pm_j)$ vanish on $M^\pm$ and satisfy the quaternionic relations on $M^\mp$. In fact it is straightforward to check the relations

$$J^\pm_{12} = \pm\frac{1}{2}\Big(J_{14} \pm J_{23}\Big), \quad J^\pm_{31} = \pm\frac{1}{2}\Big(J_{13} \mp J_{24}\Big), \quad J^\pm_{23} = \pm\frac{1}{2}\Big(J_{12} \pm J_{34}\Big) \tag{8}$$

This shows that M is a Riemannian product of two quaternion-Kähler manifolds.

For later use, we remark that the curvature forms $\omega^\pm_{ij}$, $1 \le i,j \le 3$ of the connection on $\Lambda^2_\pm E$ with respect to the local frame $\{e^\pm_i\}$ are related to the forms ω_{ij} by

$$\omega^\pm_{12} = \pm(\omega_{14} \pm \omega_{23}), \qquad \omega^\pm_{31} = \pm(\omega_{13} \mp \omega_{24}), \qquad \omega^\pm_{23} = \pm(\omega_{12} \pm \omega_{34}). \tag{9}$$

(ii) Assume now that $r \ne 4$. Let us choose some k different from i and j. Taking $Z = X_a$, $W = J_{ik}(X_a)$, summing over a in (5) and using (1) yields

$$2R^{ik} = \sum_{s=1}^{r}[\omega_{si} < J_{si}, J_{ik} > + \omega_{sj} < J_{sj}, J_{ik} >] = n\omega_{ik}. \tag{10}$$

Taking now $Y = Z = X_a$ and summing over a in (5) yields

$$\frac{1}{2}R^{ij}(X, J_{ij}W) = \mathrm{Ric}(X,W) + 2\omega_{ij}(X, J_{ij}(W)) - \sum_{s=1}^{r}[\omega_{si}(X, J_{si}(W)) + \omega_{sj}(X, J_{sj}(W))].$$

We identify 2-forms and endomorphisms on M using g. The previous relation reads

$$-\frac{1}{2}J_{ij}\circ R^{ij} = \mathrm{Ric} - 2J_{ij}\circ\omega_{ij} + \sum_{s=1}^{r}[J_{si}\circ\omega_{si} + J_{sj}\circ\omega_{sj}],$$

so taking (10) into account we get for every $i \ne j$

$$0 = \mathrm{Ric} + (n/4 - 2)J_{ij}\circ\omega_{ij} + \sum_{s=1}^{r}[J_{si}\circ\omega_{si} + J_{sj}\circ\omega_{sj}]. \tag{11}$$

It turns out that this system in the unknown endomorphisms $J_{ij}\circ\omega_{ij}$ has a unique solution for $n > 8$. Indeed, if we denote by $S_i := \sum_{s=1}^r J_{si}\circ\omega_{si}$ and sum over j in (11), we get

$$0 = r\mathrm{Ric} + (n/4 - 2)S_i + rS_i + \sum_{j=1}^{r} S_j,$$

so $S_i = S_j$ for all i,j. From (11) again we see that $J_{ij}\circ\omega_{ij}$ are all equal for $i \ne j$, and thus proportional with Ric:

$$J_{ij}\circ\omega_{ij} = \frac{1}{4 - n/4 - 2r}\mathrm{Ric}, \qquad \forall\ i \ne j. \tag{12}$$

Since the right term is symmetric, the two skew-symmetric endomorphisms from the left term commute, so J_{ij} commutes with Ric for all i,j. This, in turn, implies like in Lemma 2.3 above that

$$< \omega_{ij}, J_{kl} >= 0 \text{ unless } i = k \ne j = l \text{ or } i = l \ne k = j. \tag{13}$$

We finally choose k different from i and j, take $X = J_{ik}(X_a)$, $Y = X_a$, sum over a in (5) and use (13) to obtain

$$-J_{ij} \circ R^{ik} \circ J_{ij} - R^{ik} = - < \omega_{ki}, J_{ik} > J_{ki}.$$

By (10) this reads

$$n\omega_{ki} = - < \omega_{ki}, J_{ik} > J_{ki}$$

and (12) then implies on the first hand that M is Einstein and on the other hand that the Ricci tensor does not vanish, since otherwise ∇^E would be flat.

There exists thus a non-zero constant κ such that

$$\omega_{ij} = \kappa J_{ij} \tag{14}$$

for all $i \neq j$. This is equivalent to the statement (a).

We will now prove (iib) and (iii) simultaneously. From now on n might be equal to 8, but we assume that (a) holds. We can re-express (4) and (12) as

$$R_{X,Y} J_{ij} = \kappa \sum_{s=1}^{r} [g(J_{si}(X), Y) J_{sj} + g(J_{sj}(X), Y) J_{is}]. \tag{15}$$

and

$$\mathrm{Ric} = \kappa(n/4 + 2r - 4). \tag{16}$$

Assume that M were reducible, i.e. that TM is the direct sum of two parallel distributions T_1 and T_2. For all $X \in T_1$ and $Y \in T_2$ we have $R_{X,Y} = 0$, so (15) implies

$$0 = \kappa \sum_{s=1}^{r} [g(J_{si}(X), Y) J_{sj} + g(J_{sj}(X), Y) J_{is}].$$

Taking the scalar product with J_{ik} for some $k \neq i, j$ and using (13) yields

$$0 = g(J_{kj}(X), Y).$$

This shows that each J_{kj}, and hence the whole even Clifford structure, preserves the splitting $TM = T_1 \oplus T_2$. In other words, each integral leaf M_i of T_i $(i = 1, 2)$ carries a parallel even Clifford structure. Notice that the relations $\omega_{ij} = \kappa J_{ij}$ for all $i \neq j$ continue to hold on M_1 and M_2. Formula (11) then shows that the Ricci tensor of each factor T_i must satisfy $\mathrm{Ric}^{T_i} = \kappa(\dim(T_i)/4 + 2r - 4)$, which of course contradicts (16). This finishes the proof of (iib) and (iii). $\square$

In order to proceed we need the following algebraic interpretation:

Proposition 2.10. *Let (M^n, g) be a simply connected Riemannian manifold with holonomy group $H := \mathrm{Hol}(M)$ acting on $\mathbb{R}^n$. A parallel rank r $(3 \leq r \neq 4)$ even Clifford structure on M is equivalent to an orthogonal representation $\rho : H \to \mathrm{SO}(r)$ of H on $\mathbb{R}^r$ together with a H-equivariant algebra morphism $\phi : \mathrm{Cl}_r^0 \to \mathrm{End}(\mathbb{R}^n)$ mapping $\mathfrak{so}(r) \subset \mathrm{Cl}_r^0$ into $\mathfrak{so}(n) \subset \mathrm{End}(\mathbb{R}^n)$.*

Proof. Assume that ρ and ϕ satisfy the conditions above. Let P be the holonomy bundle of (M, g) through some orthonormal frame u_0, with structure group H. The Levi-Civita connection of M restricts to P and induces a connection on the Euclidean bundle $E := P \times_\rho \mathbb{R}^r$. The bundle morphism

$$\varphi : \mathrm{Cl}^0(E) \to \mathrm{End}(TM), \qquad [u, a] \mapsto [u, \phi(a)]$$

is well-defined since ϕ is H-equivariant and clearly induces a parallel rank r even Clifford structure on (M, g).

Conversely, if (E, ∇^E) defines a parallel even Clifford structure on M, we claim that E is associated to the holonomy bundle P through u_0 and that ∇^E corresponds to the Levi-Civita connection. Let x_0 be the base point of u_0, let Γ be the based loop space at x_0 and let Γ_0 be the kernel of the holonomy morphism $\Gamma \to H$. The parallel transport with respect to ∇^E of E_{x_0} along curves in Γ defines a group morphism $\tilde{\rho} : \Gamma \to \mathrm{SO}(E_{x_0})$. If $\gamma \in \Gamma_0$, the fact that (E, ∇^E) is a parallel even Clifford structure is equivalent to $\varphi(\Lambda^2(\tilde{\rho}(\gamma))(\omega)) = \varphi(\omega)$ for all $\omega \in \Lambda^2(E)$. Since $\mathfrak{so}(r)$ is simple for $3 \leq r \neq 4$, the map φ is injective. The relation above reduces to $\Lambda^2(\tilde{\rho}(\gamma)) = \mathrm{id}$, thus to $\tilde{\rho}(\gamma) = \mathrm{id}$. This shows that $\Gamma_0 = \mathrm{Ker}(\tilde{\rho})$, so by taking the quotient, $\tilde{\rho}$ defines a faithful orthogonal representation ρ of $H = \Gamma/\Gamma_0$ on E_{x_0}. It is easy to check that the map $P \times_\rho E_{x_0} \to E$ given by

$$[u, e] \mapsto \tau_\gamma^E(e),$$

where γ is any curve in M whose horizontal lift to P through u_0 ends at u and τ_γ^E denotes the parallel transport on E with respect to ∇^E along γ, is a well-defined bundle morphism preserving the covariant derivatives. The existence of the H-equivariant algebra morphism $\phi : \mathrm{Cl}_r^0 \to \mathrm{End}(\mathbb{R}^n)$ mapping $\mathfrak{so}(r) \subset \mathrm{Cl}_r^0$ into $\mathfrak{so}(n) \subset \mathrm{End}(\mathbb{R}^n)$ is now straightforward. $\square$

It is easy to check that this result holds *verbatim* for projective even Clifford structures, by replacing orthogonal representations with projective ones. Notice that if $\rho : H \to \mathrm{PSO}(r)$ is a projective representation, $\Lambda^2\rho$ is a linear representation, so the vector bundle $\Lambda^2 E := P \times_{\Lambda^2\rho} \Lambda^2(\mathbb{R}^r)$ is globally defined, even though $E := P \times_\rho \mathbb{R}^r$ is only locally defined.

Corollary 2.11. *Assume that (M^n, g) satisfies the hypotheses of Proposition 2.9. Then the Lie algebra $\mathfrak{h}$ of the holonomy group H (associated to some holonomy bundle P) is a direct sum of Lie sub-algebras, one of which is isomorphic to $\mathfrak{so}(r)$.*

Proof. Every orthonormal frame $u_0 \in P$ over $x_0 \in M$, defines a natural Lie algebra isomorphism from $\mathfrak{so}(n)$ to $\Lambda^2 M_{x_0}$. In this way, the holonomy algebra $\mathfrak{h}$ is naturally identified with a sub-algebra of $\Lambda^2 M_{x_0}$ and the image $\mathfrak{k}$ of $\mathfrak{so}(r)$ through the map ϕ defined in Proposition 2.10 is naturally identified with $\varphi(\Lambda^2 E_{x_0})$.

The Ambrose-Singer Theorem ([15], Thm. 8.1 Ch.II) shows that $\mathfrak{h}$ contains the image of $\Lambda^2 M_{x_0}$ through the curvature endomorphism. With the notation (6), we thus get $(R^{ij})_{x_0} \in \mathfrak{h}$ for all $1 \leq i, j \leq r$. Taking (10) and (14) into account shows that $\mathfrak{k} \subset \mathfrak{h}$.

Moreover, by Proposition 2.10, $\mathfrak{k}$ is an ideal of $\mathfrak{h}$. Since $\mathfrak{h}$ is the Lie algebra of a compact Lie group, we immediately obtain the Lie algebra decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}^\perp$, where $\mathfrak{k}^\perp$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{h}$ with respect to any $\mathrm{ad}_\mathfrak{h}$-invariant metric on $\mathfrak{h}$. $\square$

We are now ready for the first important result of this section.

Theorem 2.12. *A Riemannian manifold (M^n, g) carrying a parallel non-flat even Clifford structure (E, ∇^E) of rank $r \geq 5$ is either locally symmetric or 8-dimensional.*

Proof. Assume that M is not locally symmetric. By replacing M with its universal cover, we may assume that M is simply connected. According to Proposition 2.9, M has irreducible holonomy and non-vanishing scalar curvature. The Berger-Simons Holonomy Theorem implies that there are exactly three possibilities for the holonomy group H of M: $H = \mathrm{SO}(n)$, $H = \mathrm{U}(n/2)$ or $H = \mathrm{Sp}(n/4) \cdot \mathrm{Sp}(1)$. The second exterior power of the holonomy representation is of course irreducible in the first case and decomposes as

$$\mathfrak{so}(n) = \mathfrak{su}(n/2) \oplus \mathbb{R} \oplus \mathfrak{p}_1,$$

$$\mathfrak{so}(n) = \mathfrak{sp}(n/4) \oplus \mathfrak{sp}(1) \oplus \mathfrak{p}_2$$

in the latter two cases. A summand isomorphic to some $\mathfrak{so}(r)$ $(r \geq 5)$ occurs in the above decompositions if and only if $r = n$ in the first case, or is obtained from the low-dimensional isomorphisms

$$\mathfrak{su}(n/2) \simeq \mathfrak{so}(r) \qquad \text{for } n = 8 \text{ and } r = 6,$$

$$\mathfrak{sp}(n/4) \simeq \mathfrak{so}(r) \qquad \text{for } n = 8 \text{ and } r = 5.$$

In the latter cases one has $n = 8$, so we are left with the case when M has generic holonomy $\mathrm{SO}(n)$. By Proposition 2.10, $\mathbb{R}^n$ inherits a Cl^0_n-module structure, which for dimensional reasons may only occur when $n = 8$. $\square$

Using this result we will now obtain the classification of complete simply connected manifolds with parallel rank r even Clifford structures. From the above discussion it is enough to consider the cases when $r \geq 5$ and either $\dim(M) = 8$ or M is symmetric.

Case 1. $\dim(M) = 8$**.** Proposition 2.10 has several consequences:

(a) $\mathbb{R}^8$ is a Cl^0_r representation, thus $5 \leq r \leq 8$.
(b) The inclusion $\phi : \mathfrak{so}(r) \to \mathfrak{so}(8)$ is defined by the spin (or half-spin for $r = 8$) representation.
(c) The holonomy group H is contained in the connected component of the identity $N^0_{\mathrm{SO}(8)}\mathfrak{so}(r)$ of the normalizer of $\mathfrak{so}(r)$ in $\mathrm{SO}(8)$, acting on its Lie algebra by the adjoint representation.

Using again the low-dimensional isomorphisms $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$ and $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$ we easily get

$$N^0_{\mathrm{SO}(8)}\mathfrak{so}(5) = \mathrm{Sp}(2) \cdot \mathrm{Sp}(1), \qquad N^0_{\mathrm{SO}(8)}\mathfrak{so}(6) = \mathrm{U}(4), \qquad N^0_{\mathrm{SO}(8)}\mathfrak{so}(7) = \mathrm{Spin}(7).$$

Thus a necessary condition for a simply connected 8-dimensional manifold to carry a parallel even Clifford structure of rank r is that M is quaternion-Kähler for $r = 5$, Kähler for $r = 6$ and has holonomy contained in $\mathrm{Spin}(7)$ for $r = 7$ (no condition at all for $r = 8$). Conversely, if M satisfies one of these conditions for $r = 5, 6, 7$ or is an arbitrary manifold in the case

$r = 8$, we define E to be associated to the holonomy bundle of M with respect to the following representations of the holonomy group:

$r = 5$: $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \to \mathrm{SO}(5)$, $a \cdot b \mapsto \xi(a)$, where $\xi : \mathrm{Sp}(2) \simeq \mathrm{Spin}(5) \to \mathrm{SO}(5)$ is the spin covering.
$r = 6$: $\mathrm{U}(4) \to \mathrm{PSO}(6)$ induced by taking the $\mathbb{Z}_4$ quotient in the projection onto the first factor in $\mathrm{SU}(4) \times \mathrm{U}(1) \to \mathrm{SU}(4) \simeq \mathrm{Spin}(5)$.
$r = 7$: The spin covering $\mathrm{Spin}(7) \to \mathrm{SO}(7)$.
$r = 8$: One of the two representations $\mathrm{SO}(8) \to \mathrm{PSO}(8)$ obtained by taking the $\mathbb{Z}_2$ quotient in the half-spin representations $\mathrm{Spin}(8) \to \mathrm{SO}(\Delta_\pm)$.

Notice that for $r = 6$ and $r = 8$ the defining representation of E is projective, so E is only locally defined if M is non-spin. On the contrary, if M is spin then E is a well-defined vector bundle, associated to the spin holonomy bundle of M.

The attentive reader might have noticed the subtlety of the case $r = 8$. In all other cases the equivariant Lie algebra morphism ϕ is constructed by identifying $\mathfrak{so}(r)$ with a factor of the Lie algebra of the holonomy group acting on $\mathbb{R}^8$ by the *spin* representation (therefore extending to a representation of the even Clifford algebra). For $r = 8$ however, the holonomy representation *is not* the spin representation. What still makes things work in this case is the *triality* of the $\mathfrak{so}(8)$ representations, which is an outer automorphism of $\mathrm{Spin}(8)$ interchanging its three non-equivalent representations on $\mathbb{R}^8$. In this way, on a 8-dimensional spin manifold one has six Clifford actions: The Clifford algebra bundle of TM acts on the half spinor bundles $\Sigma_\pm M$, $\mathrm{Cl}^0(\Sigma_+ M)$ acts on TM and $\Sigma_- M$, and $\mathrm{Cl}^0(\Sigma_- M)$ acts on TM and $\Sigma_+ M$. Of course, when M is not spin, among the six Clifford actions above, only the third and the fifth ones are globally defined.

According to Proposition 2.10, the argument above can be expressed as follows: We denote by $\xi : \mathrm{Spin}(8) \to \mathrm{SO}(8)$ the spin covering and by $\delta^\pm : \mathrm{Spin}(8) \to \mathrm{SO}(8)$ the half-spin representations. If $H \subset \mathrm{SO}(8)$ is the holonomy group of M, let $\rho : H \to \mathrm{PSO}(8)$ denote the restriction to H of the $\mathbb{Z}_2$-quotient of δ^+. The isomorphism $\phi : \mathfrak{so}(8) \to \mathfrak{so}(8)$, $\phi = \xi_* \circ (\delta^+_*)^{-1}$ is tautologically equivariant with respect to the representations of H on $\mathfrak{so}(8)$ induced by ρ and ξ respectively, and it extends to a Clifford action due to triality.

Case 2. $M = G/H$ **is symmetric.** According to Proposition 2.10 and Corollary 2.11, there are two necessary conditions for M to carry a parallel even Clifford structure of rank $r \geq 5$:

(a) $\mathfrak{so}(r)$ occurs as a summand in the Lie algebra $\mathfrak{h}$ of the isotropy group H.
(b) The dimension of M has to be a multiple of the dimension $N_0(r)$ of the irreducible Cl^0_r representation.

Notice that Proposition 2.10 shows that if $M = G/H$ is a compact symmetric space solution of our problem, its non-compact dual G^*/H is a solution too, since the isotropy representations are the same. We will thus investigate only the symmetric spaces of compact type.

After a cross-check in the tables of symmetric spaces of Type I and II ([5], pp. 312-317) we are left with the following cases:

(1) $G = \mathrm{SU}(n)$, $H = \mathrm{SO}(n)$. Condition (a) is verified for $r = n$ but it is easy to check that $\dim(M) = (r-1)(r+2)/2$ can not be a multiple of $N_0(r)$.

(2) $G = \mathrm{SU}(2n)$, $H = \mathrm{Sp}(n)$. Condition (a) is verified for $n = 2$ and $r = 5$, but $\dim(M) = 5$ is not a multiple of $N_0(5) = 8$.

(3) $G = \mathrm{SU}(p+q)$, $H = S(\mathrm{U}(p) \times \mathrm{U}(q))$. Both conditions are verified for $p = 4$, $r = 6$ and arbitrary q.

(4) $G = \mathrm{SO}(p+q)$, $H = \mathrm{SO}(p) \times \mathrm{SO}(q)$. By condition (a) one can assume $r = p \geq 5$. The isotropy representation is the tensor product $\mathbb{R}^{pq}$ of the standard representations of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$. Assume that $p \neq 8$. It is well-known that the group $\mathrm{SO}(p)$ has exactly one non-trivial representation on $\mathbb{R}^p$. This is due to the fact that $\mathrm{SO}(p)$ has no outer automorphisms for p odd, while for p even the only outer automorphisms are the conjugations by matrices in $\mathrm{O}(p) \setminus \mathrm{SO}(p)$. Restricting our attention to the subgroup $\mathrm{SO}(p)$ of the holonomy group H, the map ϕ given by Proposition 2.10 defines a $\mathrm{SO}(p)$-equivariant representation of $\mathfrak{so}(p)$ on $\mathbb{R}^p \oplus \ldots \oplus \mathbb{R}^p$ (q times) and is thus defined by q^2 equivariant components $\phi_{ij} : \mathfrak{so}(p) \to \mathrm{End}(\mathbb{R}^p)$. It is easy to see that each ϕ_{ij} is then scalar: $\phi_{ij}(A) = \lambda_{ij} A$ for all $A \in \mathfrak{so}(p)$. Finally, the fact that ϕ extends to the Clifford algebra implies that $\phi(A)^2 = -\mathrm{id}$ for $A = \xi_*(e_1 \cdot e_2)$ (here ξ denotes the spin covering $\mathrm{Spin}(p) \to \mathrm{SO}(p)$), and this is impossible since

$$(\phi(A)^2)_{ij} = \sum_{k=1}^{q} \lambda_{ik} \lambda_{kj} A^2,$$

and A^2 is not a multiple of the identity. Thus $r = p = 8$ is the only admissible case.

(5) $G = \mathrm{SO}(2n)$, $H = \mathrm{U}(n)$. Condition (a) is verified for $n = 4$ and $r = 6$, but $\dim(M) = 12$ is not a multiple of $N_0(6) = 8$.

(6) $G = \mathrm{Sp}(n)$, $H = \mathrm{U}(n)$. Condition (a) is verified for $n = 4$ and $r = 6$, but $\dim(M) = 20$ is not a multiple of $N_0(6) = 8$.

(7) $G = \mathrm{Sp}(p+q)$, $H = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$. Both conditions are verified for $p = 2$, $r = 5$ and arbitrary q.

(8) If G is one of the exceptional simple Lie groups F_4, E_6, E_7, E_8, both conditions are simultaneously verified for $H = \mathrm{Spin}(9)$, $\mathrm{Spin}(10) \times \mathrm{U}(1)$, $\mathrm{Spin}(12) \times \mathrm{SU}(2)$ and $\mathrm{Spin}(16)$ respectively. The corresponding symmetric spaces are exactly Rosenfeld's elliptic projective planes $\mathbb{OP}^2$, $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$, $(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2$ and $(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$.

(9) Finally, no symmetric space of type II (i.e. $M = H \times H/H$) can occur: condition (a) is satisfied for $H = \mathrm{SU}(4)$, $r = 6$ and $H = \mathrm{SO}(n)$, $r = n$ but the dimension of M is 15 in the first case and $n(n-1)/2$ in the second case, so condition (b) does not hold.

The only candidates of symmetric spaces carrying parallel even Clifford structures of rank $r \geq 5$ are thus those from cases (3), (4), (7) and (8). Conversely, all these spaces carry a (projective) parallel even Clifford structure. This is due to the fact that the restriction of the infinitesimal isotropy representation to the $\mathfrak{so}(r)$ summand is the spin representation in all cases except for $\mathfrak{so}(8)$, where the triality argument applies. Summarizing, we have proved the following

Theorem 2.13. *The list of complete simply connected Riemannian manifolds M carrying a parallel rank r even Clifford structure is given in the tables below.*

r	M	dimension of M
2	Kähler	$2m,\ m \geq 1$
3	hyper-Kähler	$4q,\ q \geq 1$
4	reducible hyper-Kähler	$4(q_1+q_2),\ q_1 \geq 1,\ q_2 \geq 0$
arbitrary	Cl_r^0 representation space	multiple of $N_0(r)$

Table 1. Manifolds with a flat even Clifford structure

r	type of E	M	dimension of M
2		Kähler	$2m,\ m \geq 1$
3		quaternion-Kähler (QK)	$4q,\ q \geq 1$
4	projective	product of two QK manifolds	$4q,\ q \geq 1$
5		QK	8
6	projective if M non-spin	Kähler	8
7		Spin(7) holonomy	8
8	projective if M non-spin	Riemannian	8
5		$\mathrm{Sp}(k+2)/\mathrm{Sp}(k)\times\mathrm{Sp}(2)$	$8k,\ k \geq 2$
6	projective	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(4))$	$8k,\ k \geq 2$
8	projective if k odd	$\mathrm{SO}(k+8)/\mathrm{SO}(k)\times\mathrm{SO}(8)$	$8k,\ k \geq 2$
9		$\mathbb{OP}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$	16
10		$(\mathbb{C}\otimes\mathbb{O})\mathbb{P}^2 = \mathrm{E}_6/\mathrm{Spin}(10)\cdot\mathrm{U}(1)$	32
12		$(\mathbb{H}\otimes\mathbb{O})\mathbb{P}^2 = \mathrm{E}_7/\mathrm{Spin}(12)\cdot\mathrm{SU}(2)$	64
16		$(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2 = \mathrm{E}_8/\mathrm{Spin}(16)$	128

Table 2. Manifolds with a parallel non-flat even Clifford structure [1]

We end up this section with the classification of manifolds carrying parallel Clifford structures.

Theorem 2.14. *A simply connected Riemannian manifold (M^n, g) carries a parallel rank r Clifford structure if and only if one of the following (non-exclusive) cases occur:*

(1) *$r = 1$ and M is Kähler.*
(2) *$r = 2$ and either $n = 4$ and M is Kähler or $n \geq 8$ and M is hyper-Kähler.*
(3) *$r = 3$ and M is quaternion-Kähler.*
(4) *$r = 4$, $n = 8$ and M is a product of two Ricci-flat Kähler surfaces.*
(5) *$r = 5$, $n = 8$ and M is hyper-Kähler.*

[1]In this table we adopt the convention that the QK condition is empty in dimension 4. For the sake of simplicity, we have omitted in Table 2 the non-compact duals of the compact symmetric spaces. The meticulous reader should add the spaces obtained by replacing $\mathrm{Sp}(k+8)$, $\mathrm{SU}(k+4)$, $\mathrm{SO}(k+8)$, F_4, E_6, E_7 and E_8 in the last seven rows with $\mathrm{Sp}(k,8)$, $\mathrm{SU}(k,4)$, $\mathrm{SO}_0(k,8)$, F_4^{-20}, E_6^{-14}, E_7^{-5} and E_8^{8} respectively.

(6) *$r = 6$, $n = 8$ and M is Kähler Ricci-flat.*
(7) *$r = 7$ and M is an 8-dimensional manifold with* $\mathrm{Spin}(7)$ *holonomy.*
(8) *r is arbitrary and M is flat, isometric to a nontrivial representation of the Clifford algebra* Cl_r.

Proof. Assume that (M^n, g) carries a rank r parallel Clifford structure $(E, h) \subset (\Lambda^2 M, \frac{1}{n} g)$. The image by $\varphi : \mathrm{Cl}(E, h) \to \mathrm{End}(TM)$ of the volume element is a parallel endomorphism v of TM which satisfies $v \circ v = (-1)^{\frac{r(r+1)}{2}}$ and commutes (resp. anti-commutes) with every element of E for r odd (resp. even). We start by considering the cases $r \leq 4$.

• $r = 1$. It was already noticed that a parallel rank 1 Clifford structure corresponds to a Kähler structure on M.

• $r = 2$. The rank 2 Clifford structure E induces a rank 3 Clifford structure $E' := E \oplus \Lambda^2 E$ on M. Explicitly, if $\{e_1, e_2\}$ is a local orthonormal basis of E, then $e_3 := e_1 \circ e_2$ is independent of the chosen basis and $\{e_1, e_2, e_3\}$ satisfy the quaternionic relations. Moreover, $e_3 = v$ is a parallel endomorphism of TM, so (M, g) is Kähler. In the notation of Proposition 2.9 we have $\omega_{13} = \omega_{23} = 0$. Formula (11) yields

$$0 = \mathrm{Ric} + n/4 J_{ij} \circ \omega_{ij} + J_{si} \circ \omega_{si} + J_{sj} \circ \omega_{sj}$$

for every permutation $\{i, j, s\}$ of $\{1, 2, 3\}$. If $n > 4$ this system shows that $\omega_{12} = 0$, so M is hyper-Kähler. Conversely, if either $n = 4$ and (M, g, J) is Kähler, or $n > 4$ and (M, g, I, J, K) is hyper-Kähler, then $E = \Lambda^{(2,0)+(0,2)} M$ in the first case, or $E =< I, K >$ in the second case, define a rank 2 parallel Clifford structure on M.

• $r = 3$. It was already noticed that because of the isomorphism $\Lambda^2 E \cong E$, every rank 3 even Clifford structure is automatically a Clifford structure, and corresponds to a quaternion-Kähler structure (which, we recall, is an empty condition for $n = 4$).

• $r = 4$. The endomorphism v is now a parallel involution of TM anti-commuting with every element of the Clifford bundle $E \subset \Lambda^2 M$. Correspondingly, the tangent bundle of M splits in a parallel direct sum $TM = T^+ \oplus T^-$, such that $v|_{T^\pm} = \pm \mathrm{id}$. If we denote by J_i, $1 \leq i \leq 4$ a local orthonormal basis of E, each J_i maps $T^\pm$ to $T^\mp$. The de Rham decomposition theorem shows that M is a Riemannian product $M = M^+ \times M^-$ and $TM^\pm = T^\pm$. The Riemannian curvature tensor of M is the sum of the two curvature tensors of M^+ and M^-: $R = R^+ + R^-$. Let ω_{ij} denote the curvature forms (with respect to the local frame $\{J_i\}$) of the Levi-Civita connection on E:

$$R_{X,Y} J_i = \sum_{j=1}^{4} \omega_{ji}(X, Y) J_j.$$

We take $X, Y \in T^+$ and apply the previous relation to some $Z \in T^+$ and obtain

$$R^+_{X,Y} Z = \sum_{j=1}^{4} \omega_{ji}(X, Y) J_j J_i Z, \qquad \forall\ 1 \leq i \leq 4.$$

For $1 \leq j \leq 3$ we denote by $\omega_j := \omega_{j4}$ and $I_j := -J_j J_4$. Since by definition $v = J_1 J_2 J_3 J_4$, it is easy to check that I_j are anti-commuting almost complex structures on M^+ satisfying the quaternionic relations $I_1 I_2 = I_3$ etc. The previous curvature relation reads

$$R^+_{X,Y} = -\sum_{j=1}^{3} \omega_j(X,Y) I_j, \qquad \forall\ X, Y \in T^+. \tag{17}$$

The symmetry by pairs of R^+ implies that $\omega_i = \sum_{j=1}^{3} a_{ji} I_j$ for some smooth functions a_{ij} satisfying $a_{ij} = a_{ji}$. Moreover, the first Bianchi identity applied to (17) yields

$$\sum_{i,j=1}^{3} a_{ij} I_i \wedge I_j = 0. \tag{18}$$

If $\dim(M^+) > 4$, we may choose non-vanishing vectors $X, Y \in T^+$ such that Y is orthogonal to X and to $I_i X$ for $i = 1, 2, 3$. Applying (18) to $X, I_i X, Y, I_j Y$ yields $a_{ij} + a_{ji} = 0$, so $\omega_j = 0$. By (17) we get $R^+ = 0$ and similarly $R^- = 0$, so M is flat. It remains to study the case $\dim(M^+) = 4$. In this case I_1, I_2 and I_3 are a basis of the space of self-dual 2-forms $\Lambda^2_+ M^+$, so (17) is equivalent to the fact that M^+ is self-dual and has vanishing Ricci tensor (see e.g. [5], p.51). In other words, M^+ is Kähler (with respect to any parallel 2-form in $\Lambda^2_- M^+$) and Ricci-flat, and the same holds of course for M^-.

Conversely, assume that $M = M^+ \times M^-$ is a Riemannian product of two simply connected Ricci-flat Kähler surfaces. The holonomy of M is then a subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2) \simeq \mathrm{Spin}(4)$, so the frame bundle of M and the Levi-Civita connection reduce to a principal $\mathrm{SU}(2) \times \mathrm{SU}(2)$-bundle P. Let ξ denote the representation of $\mathrm{Spin}(4)$ on $\mathbb{R}^4$ coming from the spin covering $\mathrm{Spin}(4) \to \mathrm{SO}(4)$ and let ρ denote the representation of $\mathrm{Spin}(4)$ on $\mathfrak{so}(8)$ obtained by restricting the adjoint action of $\mathrm{SO}(8)$ to $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathrm{SO}(8)$). The irreducible representation of Cl_4 on $\mathbb{R}^8$ defines a $\mathrm{Spin}(4)$-equivariant map from $\mathbb{R}^4$ to $\mathfrak{so}(8)$ (with respect to the above actions of $\mathrm{Spin}(4)$). The above map defines an embedding of the rank 4 vector bundle $E := P \times_\xi \mathbb{R}^4$ into $\Lambda^2 M = P \times_\rho \mathfrak{so}(8)$, which is by construction a parallel Clifford structure on M.

For $r \geq 5$ we will use the fact that E defines tautologically a rank r parallel even Clifford structure on M, and apply Theorems 2.8 and 2.13 to reduce the study to manifolds appearing in Table 2.

• $r = 5$. The volume element v defines a Kähler structure on M in this case. The quaternionic Grassmannians $\mathrm{Sp}(k+8)/\mathrm{Sp}(k) \cdot \mathrm{Sp}(2)$ are obviously not Kähler (since the Lie algebra of the isometry group of every Kähler symmetric space has a non-trivial center), so it remains to examine the case $n = 8$, when, according to Theorem 2.13, M is quaternion-Kähler. More explicitly, if E is the rank 5 Clifford bundle, $\varphi(\Lambda^2 E)$ is a Lie sub-algebra of $\mathrm{End}^-(TM) \simeq \mathfrak{so}(8)$ isomorphic to $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$ and its centralizer is a Lie sub-algebra $\mathfrak{s}$ of $\mathrm{End}^-(TM)$ isomorphic to $\mathfrak{so}(3)$, defining a quaternion-Kähler structure. Moreover v belongs to $\mathfrak{s}$ (being the image of a central element in the Clifford algebra bundle of E), so we easily see that its orthogonal complement $v^\perp$ in $\mathfrak{s}$ defines a rank 2 parallel Clifford structure on M. By the case $r = 2$ above, M is then hyper-Kähler.

Conversely, every 8-dimensional hyper-Kähler manifold carries parallel Clifford structures of rank 5 obtained as follows. Let ξ denote the representation of $\mathrm{Spin}(5)$ on $\mathbb{R}^5$ coming from the spin covering $\mathrm{Spin}(5) \to \mathrm{SO}(5)$ and let ρ denote the representation of $\mathrm{Spin}(5)$ on $\mathfrak{so}(8)$ obtained by restricting the adjoint action of $\mathrm{SO}(8)$ to $\mathrm{Spin}(5) \simeq \mathrm{Sp}(2) \subset \mathrm{SO}(8)$. The irreducible representation of Cl_5 on $\mathbb{R}^8$ defines a $\mathrm{Spin}(5)$-equivariant map from $\mathbb{R}^5$ to $\mathfrak{so}(8)$ (with respect to the above actions of $\mathrm{Spin}(5)$). If P denotes the holonomy bundle of M with structure group $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$, the above map defines an embedding of the rank 5 vector bundle $E := P \times_\xi \mathbb{R}^5$ into $\Lambda^2 M = P \times_\rho \mathfrak{so}(8)$, which is by construction a parallel Clifford structure on M.

• $r = 6$. The volume element v is now a Kähler structure anti-commuting with every element of the Clifford bundle E. If we denote by J_i, $1 \le i \le 6$ a local orthonormal basis of E, each J_i is a 2-form of type $(2,0)+(0,2)$ with respect to v, so the curvature endomorphism vanishes on J_i:

$$0 = R(J_i)(X,Y) = \sum_{a=1}^{n} R(J_i X_a, X_a, X, Y) = 2\sum_{a=1}^{n} R(X, X_a, J_i X_a, Y). \tag{19}$$

Let ω_{ij} denote the curvature forms (with respect to the local frame $\{J_i\}$) of the Levi-Civita connection on E:

$$R_{X,Y} J_i = \sum_{j=1}^{6} \omega_{ji}(X,Y) J_j.$$

We can express this as follows:

$$R(X,Y,J_iZ,J_iW) - R(X,Y,Z,W) = \sum_{j=1}^{6} \omega_{ji}(X,Y) g(J_j Z, J_i W).$$

Taking the trace in Y and Z and using (19) yields

$$\mathrm{Ric} = -\sum_{j=1}^{6} J_j \circ J_i \circ \omega_{ji}.$$

This relation, together with (11), shows that $\mathrm{Ric} = 0$.

Conversely, every 8-dimensional Ricci-flat Kähler manifold carries parallel Clifford structures of rank 6 defined by the $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$-equivariant embedding of $\mathbb{R}^6$ into $\mathfrak{so}(8)$ coming from the irreducible representation of Cl_6 on $\mathbb{R}^8$, like in the case $r = 5$.

• $r = 7$. Theorem 2.13 shows that M has to be a 8-dimensional manifold with holonomy $\mathrm{Spin}(7)$. By an argument similar to the previous ones, every such manifold carries parallel Clifford structures of rank 7 defined by the $\mathrm{Spin}(7)$-equivariant embedding of $\mathbb{R}^7$ into $\mathfrak{so}(8)$ coming from one of the irreducible representations of Cl_7 on $\mathbb{R}^8$.

• $r = 8$. The dimension of M has to be at least equal to 16 in this case (since the dimension of the irreducible Cl_8-representation is 16). Moreover, the volume element v is a parallel involution of TM anti-commuting with every element of E, so TM splits in a parallel direct sum of the ± 1 eigen-distributions of v. This contradicts Proposition 2.9.

• Finally, for $r \ge 9$, the spaces appearing in the last four rows of Table 2 cannot carry a Clifford structure since the dimension of the irreducible representation of Cl_r for $r = 9,\ 10,\ 12,\ 16$

is 32, 64, 128, 256 respectively, which is exactly twice the dimension of the corresponding tangent spaces in each case. □

3. Bundle-like curvature constancy

As an application of Theorem 2.13, we classify in this section bundle-like metrics with curvature constancy. We first show in Subsection 3.1 that every Riemannian submersion $Z \to M$ with totally geodesic fibres is associated to a G-principal bundle $P \to M$ (where G is the isometry group of some given fibre), which carries a canonical G-invariant connection. The curvature of this connection is a 2-form ω on M with values in the adjoint bundle $\mathrm{ad}(P)$. We then compute the different components of the Riemannian curvature tensor of Z in terms of the Riemannian curvature of M and of the curvature form ω.

Most of this material can be found in the literature (cf. [12], see also [22]), but we include it here for summing up the notations, conventions and usual normalizations. Readers familiar with Riemannian geometry can pass directly to Subsection 3.2, where we interpret the curvature constancy condition (28) by the fact that ω defines a *parallel even Clifford structure* on M. The classification is obtained in Subsection 3.3 by a case-by-case analysis through the manifolds in Table 2.

3.1. **Riemannian submersions with totally geodesic fibres.** Let $\pi : Z^{k+n} \to M^n$ be a Riemannian submersion with totally geodesic fibres. Assume that Z is complete. We denote by $Z_x := \pi^{-1}(x)$ the fibre of π over $x \in M$. From Theorem 1 in [13], all fibres are isometric to some fixed Riemannian manifold (F, g_F) and π is a locally trivial fibration with structure group the Lie group $G := \mathrm{Iso}(F)$ of isometries of F.

For every tangent vector $X \in T_xM$ and $z \in Z_x$, we denote by X^* its horizontal lift at z. For every curve γ on M and $z \in Z_{\gamma(0)}$ there exists a unique curve $\tilde\gamma$ with $\tilde\gamma(0) = z$ whose tangent vector at t is the horizontal lift of $\dot{\gamma(t)}$ at $\tilde\gamma(t)$ for every t. This is called the *horizontal lift* of γ through z. Hermann's result in [13] mainly says that for every curve γ on M, the mapping $\tau_t : Z_{\gamma(0)} \to Z_{\gamma(t)}$, which maps z to the value at t of the horizontal lift of γ through z, is an isometry between the two fibres, (each endowed with the induced Riemannian metric).

We define the G-principal fibre bundle P over M as the set of isometries from F to the fibres of π:

$$P := \{u : F \to Z \mid \exists\, x \in M \text{ such that } u \text{ maps } F \text{ isometrically onto } Z_x\}.$$

We denote by $p : P \to M$ the natural projection and by P_x the fiber of p over x:

$$P_x := \{u : F \to Z_x \mid u \text{ is an isometry}\}.$$

The right action of $G = \mathrm{Iso}(F)$ on P is given by $ua := u \circ a$ for every $u \in P$ and $a \in G$.

Proposition 3.1. *The horizontal distribution on Z induces a G-invariant connection on P.*

Proof. For $X \in T_xM$ and $u \in P_x$, we define its *horizontal lift* $\tilde X \in T_uP$ as follows. Take any curve x_t in M such that $X = \dot x_0$. The isometry τ_t between Z_{x_0} and Z_{x_t} described above, defines a curve $u_t := \tau_t \circ u$ which obviously satisfies $p(u_t) = x_t$. We then set $\tilde{X} := \dot{u}_0$ and claim that this does not depend on the curve x_t. This is actually a direct consequence of the following more general result:

Lemma 3.2. *Let $p : P \to M$ be a G-principal fibre bundle and assume that G acts effectively on some manifold F. Define $Z := P \times_G F$ and for each $f \in F$, the smooth map $R_f : P \to Z$, $R_f(u) = u(f)$. Then a tangent vector $X \in T_uP$ vanishes if and only if $p_*(X) = 0$ and $(R_f)_*(X) = 0$ for every $f \in F$.*

Proof. Since the result is local, one may assume that $P = M \times G$ is trivial and $u = (x, 1)$. One can write $X = (X', X'')$, with $X' \in T_xM$ and $X'' \in \mathfrak{g}$. Since $p_*(X) = 0$, we get $X' = 0$. From $(R_f)_*(X) = 0$ we obtain $\exp(tX'')(f) = f$ for every $t \in \mathbb{R}$ and $f \in F$. If X'' were not zero, this would contradict the effectiveness of the action of G. □

Returning to our argument, we see that $p_*(\tilde{X}) = X$ and

$$(R_f)_*(\tilde{X}) = \frac{\partial}{\partial t}\Big|_{t=0}(u_t(f)) = \frac{\partial}{\partial t}\Big|_{t=0}(\tau_t(u(f)) = X^*_{u(f)}$$

only depend on X, not on x_t. The map $T_xM \to T_uP$, $X \mapsto \tilde{X}$ is thus well-defined for every $x \in M$ and $u \in p^{-1}(x)$. We denote by H_u the image of this map.

Lemma 3.2 also shows that H_u is a vector subspace of T_uP, supplementary to the tangent space to the fibre of P through u. The collection $\{H_u, u \in P\}$ is called the horizontal distribution, and it is easy to see that it is invariant under the action of G: If $a \in G$ $u \in P$ and x_t is a curve in M with $x_0 = p(u)$, then (denoting $X := \dot{x}_0$):

$$(R_a)_*(\tilde{X}_u) = (R_a)_*\frac{\partial}{\partial t}\Big|_{t=0}(\tau_t \circ u) = \frac{\partial}{\partial t}\Big|_{t=0}(\tau_t \circ ua) = \tilde{X}_{ua}.$$

This proves the proposition. □

We will now express the Riemannian curvature of Z in terms of the curvature of the connection on P defined above (we will denote this connection by θ in the sequel). In order to do this, we need to introduce some notation. The adjoint bundle $\mathrm{ad}(P)$ of P, is the vector bundle associated to P via the adjoint representation of G on its Lie algebra:

$$\mathrm{ad}(P) := P \times_{\mathrm{ad}} \mathfrak{g},$$

where for every $g \in G$, $\mathrm{ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity of $\mathrm{Ad}_g : G \to G$ defined as usually by $\mathrm{Ad}_g(h) := ghg^{-1}$. The curvature of the connection θ defined by Proposition 3.1 is a G-equivariant 2-form $\tilde{\omega}$ on P with values in $\mathfrak{g}$ or, equivalently, a 2-form ω on M with values in the vector bundle $\mathrm{ad}(P)$, i.e. a section of $\Lambda^2M \otimes \mathrm{ad}(P)$. The forms ω and $\tilde{\omega}$ are related by

$$[u, \tilde{\omega}(\tilde{X}, \tilde{Y})_u] = \omega(X, Y)_{p(u)}, \tag{20}$$

where $X, Y \in T_{p(u)}M$ are tangent vectors on M with horizontal lifts $\tilde{X}, \tilde{Y} \in T_uP$ to tangent vectors on P.

For each $x \in M$, the fibre $\mathrm{ad}(P)_x$ of $\mathrm{ad}(P)$ over x has a Lie algebra structure (it is actually naturally isomorphic to the Lie algebra of the isometry group of the fibre Z_x). Every element α of $\mathrm{ad}(P)_x$ induces a Killing vector field denoted α^* on the corresponding fibre Z_x. If α is represented by $A \in \mathfrak{g}$ in the frame $u \in P_x$ (i.e. $\alpha = [u, A]$), and $z \in Z_x$ is represented by $f \in F$ in the same frame u (i.e. $z = [u, f]$), then α^*_z is the image of A by the differential at the identity of the map $G \to Z_x$, $a \mapsto [u, af]$. By a slight abuse of notation, we denote this by $\alpha^*_z = uAf$. It is easy to check that this is independent of u: If we replace u by ug, then $\alpha = [ug, \mathrm{ad}_{g^{-1}}(A)]$, $z = [ug, g^{-1}f]$, so $\alpha^*_z = ug(g^{-1}Ag)(g^{-1}f) = uAf$.

Every section α of $\mathrm{ad}(P)$ induces in this way a vertical vector field α^* on Z.

Definition 3.3. The vertical vector fields on Z obtained in this way from sections of $\mathrm{ad}(P)$, and the horizontal lifts X^* of vector fields X on M are called *standard vertical and horizontal* vector fields on Z.

We will compute the Levi-Civita covariant derivative on Z in a local frame consisting of standard vertical and horizontal vector fields. We start by computing the Lie brackets.

Lemma 3.4. *If X, Y are vector fields on M and α is a section of* $\mathrm{ad}(P)$*, the following relations hold:*

$$[X^*, \alpha^*] = (\nabla^\theta_X \alpha)^*, \tag{21}$$

$$[X^*, Y^*] = [X, Y]^* - \omega(X, Y)^*, \tag{22}$$

where ∇^θ is the covariant derivative on $\mathrm{ad}(P)$ *induced by the connection θ on P defined in Proposition* 3.1 *and ω is the curvature of θ, viewed as a 2-form on M with values in* $\mathrm{ad}(P)$.

Proof. The first relation is more or less the definition of the covariant derivative on $\mathrm{ad}(P)$. Indeed, the flow φ_t of X^* acts on a standard vertical vector field α^* by conjugation with the parallel transport:

$$\varphi_{t*}(\alpha^*) = (\tau_t \circ \alpha \circ \tau_t^{-1})^*,$$

whence

$$(\nabla^\theta_X \alpha)^* = -(\frac{d}{dt}\Big|_{t=0} \tau_t \circ \alpha \circ \tau_t^{-1})^* = -\frac{d}{dt}\Big|_{t=0} \varphi_{t*}(\alpha^*) = [X^*, \alpha^*].$$

In order to prove the second relation, we apply (21) and compute for every standard vertical vector field α^* using the Jacobi identity

$$\begin{aligned}
[\omega(X,Y)^*, \alpha^*] &= -[\omega(X,Y), \alpha]^* = -(\nabla^\theta_X \nabla^\theta_Y \alpha - \nabla^\theta_Y \nabla^\theta_X \alpha - \nabla^\theta_{[X,Y]} \alpha)^* \\
&= -[X^*, [Y^*, \alpha^*]] + [Y^*, [X^*, \alpha^*]] + [[X^*, Y^*], \alpha^*] \\
&= [[X^*, Y^*] - [X, Y]^*, \alpha^*].
\end{aligned}$$

Note that the first equality is a consequence of the fact that for every Lie group G acting (from the left) on a manifold M, the canonical map which associates to a right-invariant vector field on G a vector field on M is a Lie algebra anti-automorphism. ∎

From (22) we see that the tensor A of the O'Neill formulas for a Riemannian submersion is given by $A(X,Y) = -\frac{1}{2}\omega(X,Y)^*$ for horizontal vector fields X, Y on M. Moreover, as an immediate corollary of (21) and (22), together with the Koszul formula describing the Levi-Civita connection we obtain:

Lemma 3.5. *If X, Y are vector fields on M and α is a section of $\mathrm{ad}(P)$, the following relations hold:*

$$\nabla^Z_{\alpha^*} X^* = \frac{1}{2}\sum_{a=1}^{n} g_Z(\alpha^*, \omega(X, X_a)^*)X_a^*, \tag{23}$$

$$\nabla^Z_{X^*}\alpha^* = (\nabla^\theta_X \alpha)^* + \frac{1}{2}\sum_{a=1}^{n} g_Z(\alpha^*, \omega(X, X_a)^*)X_a^*, \tag{24}$$

$$\nabla^Z_{X^*} Y^* = (\nabla_X Y)^* - \frac{1}{2}\omega(X,Y)^*, \tag{25}$$

where ∇ is the Levi-Civita covariant derivative on M and $\{X_a\}$ is a local orthonormal frame on M.

We will now compute the relevant component of the curvature tensor of Z. Let X, Y be vector fields on M. We may assume that X, Y and the local orthonormal frame $\{X_a\}$ are parallel at some point $x \in M$. As before we denote with X^* the horizontal lift of X to a vector field on Z and with α^* the vertical vector field on Z corresponding to the section α of the vector bundle $\mathrm{ad}(P)$. Using Lemma 3.5 we can write:

$$\begin{aligned}
\nabla^Z_{X^*}(\nabla^Z_{\alpha^*} Y^*) &= \frac{1}{2}\sum_{a=1}^{n} \nabla^Z_{X^*}(g_Z(\alpha^*, \omega(Y, X_a)^*)X_a^*) \\
&= \frac{1}{2}\sum_{a=1}^{n}\Big(g_Z(\nabla^Z_{X^*}\alpha^*, \omega(Y,X_a)^*)X_a^* + g_Z(\alpha^*, \nabla^Z_{X^*}\omega(Y,X_a)^*)X_a^* \\
&\quad + g_Z(\alpha^*, \omega(Y,X_a)^*)\nabla^Z_{X^*}X_a^*\Big) \\
&= \frac{1}{2}\sum_{a=1}^{n}\Big(g_Z((\nabla^\theta_X\alpha)^*, \omega(Y,X_a)^*)X_a^* + g_Z(\alpha^*, ((\nabla^\theta_X\omega)(Y,X_a))^*)X_a^* \\
&\quad - \frac{1}{2} g_Z(\alpha^*, \omega(Y,X_a)^*)\omega(X,X_a)^*\Big),
\end{aligned}$$

$$\nabla^Z_{\alpha^*}(\nabla^Z_{X^*}Y^*) = \nabla^Z_{\alpha^*}((\nabla_X Y)^* - \frac{1}{2}\omega(X,Y)^*) = -\frac{1}{2}\nabla^Z_{\alpha^*}\omega(X,Y)^*\ ,$$

$$\nabla^Z_{[X^*,\alpha^*]}Y^* = \nabla^Z_{(\nabla^\theta_X\alpha)^*}Y^* = \frac{1}{2}\sum_{a=1}^{n} g_Z((\nabla^\theta_X\alpha)^*, \omega(Y,X_a)^*)X_a^*.$$

Consequently,

$$\begin{aligned} R^Z_{X^*,\alpha^*}Y^* &= \frac{1}{2}\sum_{a=1}^n g_Z(\alpha^*,((\nabla^\theta_X\omega)(Y,X_a))^*)X_a^* - \frac{1}{4}\sum_{a=1}^n g_Z(\alpha^*,\omega(Y,X_a)^*)\omega(X,X_a)^* \\ &\quad + \frac{1}{2}\nabla^Z_{\alpha^*}\omega(X,Y)^*. \end{aligned}$$

Taking the scalar product in this equation with a horizontal vector T^* and a vertical vector β^*, yields

$$g_Z(R^Z_{X^*,\alpha^*}Y^*,T^*) = \frac{1}{2}g_Z(\alpha^*,((\nabla^\theta_X\omega)(Y,T))^*), \tag{26}$$

$$\begin{aligned} g_Z(R^Z_{X^*,\alpha^*}Y^*,\beta^*) = &-\frac{1}{4}\sum_{a=1}^n g_Z(\alpha^*,\omega(Y,X_a)^*)g_Z(\beta^*,\omega(X,X_a)^*) \\ &+\frac{1}{2}g_Z(\beta^*,\nabla^Z_{\alpha^*}\omega(X,Y)^*). \end{aligned} \tag{27}$$

Note that (26) and (27) are special cases of the O'Neill curvature equations for Riemannian submersions (cf. [5]).

3.2. **Curvature constancy.** Let (Z,g_Z) be a Riemannian manifold. For every $z\in Z$ we define the curvature constancy at z by (see [11]):

$$\mathcal{V}_z := \{V\in T_zZ \mid R^Z_{V,X}Y = g_Z(X,Y)V - g_Z(V,Y)X \text{ for every } X,Y\in T_zZ\}. \tag{28}$$

The function $z\mapsto \dim(\mathcal{V}_z)$ is upper semi-continuous on Z. By replacing Z with the open subset where this function attains its minimum, we may assume that $\mathcal{V}$ is a k-dimensional distribution on Z, called the *curvature constancy*. It is easy to check that $\mathcal{V}$ is totally geodesic (cf. [11]).

We will introduce the following Ansatz in order to study the curvature constancy condition: Assume that $\mathcal{V}$ is locally the vertical distribution of a Riemannian submersion $\pi : Z\to M$ (equivalently, the metric of Z is bundle-like along $\mathcal{V}$). Since $\mathcal{V}$ is totally geodesic, the fibres of the Riemannian submersion are locally isometric to the unit sphere $\mathbb{S}^k$. All computations below being local, we can assume, by restricting to a contractible neighbourhood M' of M and taking the universal cover of $\pi^{-1}(M')$, that each fibre is globally isometric to $\mathbb{S}^k$. Consider the G-principal fibre bundle P over M defined in the previous subsection, together with the connection θ given by Proposition 3.1. We set $k+1 =: r$ so $G=\mathrm{SO}(r)$, and introduce the rank r Euclidean vector bundle $E\to M$ associated to P via the standard representation of $\mathrm{SO}(r)$. Notice that $\mathrm{ad}(P)$ is naturally identified with the bundle $\mathrm{End}^-(E)$ of skew-symmetric endomorphisms of E, and Z is identified with the unit sphere bundle of E.

The curvature constancy condition (28) can be expressed in terms of standard vertical and horizontal vector fields as follows:

$$R^Z_{\alpha^*,X^*}Y^* = g_M(X,Y)\alpha^* \quad \text{for every } X,Y\in TM,\ \alpha\in\mathrm{ad}(P). \tag{29}$$

Using (26) and (27), this is equivalent to the system

$$(\nabla^\theta_X\omega)(Y,T) = 0, \quad \text{for all } X,Y,T\in TM, \tag{30}$$

$$
\begin{aligned}
g_M(X,Y)g_Z(\alpha^*,\beta^*) =& \frac{1}{4}\sum_{a=1}^{n} g_Z(\alpha^*,\omega(Y,X_a)^*)g_Z(\beta^*,\omega(X,X_a)^*) \\
& -\frac{1}{2}g_Z(\beta^*,\nabla^Z_{\alpha^*}\omega(X,Y)^*),
\end{aligned}
\tag{31}
$$

for all $X, Y \in TM$ and $\alpha, \beta \in \mathrm{ad}(P)$. In order to exploit (31), we need to express the scalar product and covariant derivative of standard vertical vector fields in terms of the corresponding objects on E.

Lemma 3.6. *For every $z \in Z \subset E$ and $\alpha, \beta, \gamma \in \mathrm{ad}(P) = \mathrm{End}^-(E)$ in the fibre over $x := \pi(z)$ we have*

$$g_Z(\alpha^*,\beta^*)_z = g_E(\alpha z, \beta z). \tag{32}$$

$$g_Z(\nabla^Z_{\alpha^*}\gamma^*,\beta^*)_z = g_E(\gamma\alpha z, \beta z). \tag{33}$$

Proof. Any frame u of P defines an isometry from (E_x, g_E) to the standard Euclidean space $\mathbb{R}^r$. Once we fix such a frame, $\mathrm{ad}(P)_x$ becomes the space of skew-symmetric matrices, Z_x is the unit sphere in $\mathbb{R}^r$, and the vertical vector field α^* associated to a skew-symmetric matrix $\alpha \in \mathfrak{so}(r)$ is the Killing vector field on $\mathbb{S}^{r-1}$ whose value at $z \in \mathbb{S}^{r-1} \subset \mathbb{R}^r$ is $\alpha z \in T_z\mathbb{S}^{r-1}$. The first formula is now clear.

The Levi-Civita covariant derivative on $\mathbb{S}^{r-1}$ is the projection of the directional derivative in $\mathbb{R}^r$. Moreover, the derivative of the vector-valued function $f(z) = z$ on $\mathbb{R}^r$ obviously satisfies $A.f = A$ for every tangent vector $A \in T\mathbb{R}^r$. We thus get at z:

$$g_Z(\nabla^Z_{\alpha^*}\gamma^*,\beta^*)_z = g_E(\alpha z.\gamma f, \beta z) = g_E(\gamma\alpha z, \beta z).$$

□

Taking Lemma 3.6 into account, (31) is equivalent to

$$
\begin{aligned}
g_M(X,Y)g_E(\alpha z,\beta z) =& \frac{1}{4}\sum_{a=1}^{n} g_E(\alpha z,\omega(Y,X_a)z)g_E(\beta z,\omega(X,X_a)z) \\
& -\frac{1}{2}g_E(\beta z,\omega(X,Y)\alpha z),
\end{aligned}
\tag{34}
$$

for all $z \in Z = S(E)$, $\alpha, \beta \in \mathrm{ad}(P) = \mathrm{End}^-(E)$ and $X, Y \in TM$.

Formula (34) can be equivalently stated as follows:

$$g_M(X,Y)g_E(v_1,v_2) = \frac{1}{4}\sum_{a=1}^{n} g_E(v_1,\omega(Y,X_a)u)g_E(v_2,\omega(X,X_a)u) - \frac{1}{2}g_E(v_2,\omega(X,Y)v_1), \tag{35}$$

for all $u, v_1, v_2 \in E_x$ with $|u|^2_E = 1$ and $v_1, v_2 \perp u$ and for all $X, Y \in T_xM$. We introduce the map $\varphi : \Lambda^2 E \to \mathrm{End}^-(TM)$, defined by

$$g_M(\varphi(u \wedge v)X, Y) := -\frac{1}{2}g_E(v,\omega(X,Y)u), \qquad \forall\ x \in M,\ u, v \in E_x,\ X, Y \in T_xM.$$

Formula (35) is then equivalent to

$$\varphi(u \wedge v) \circ \varphi(u \wedge w) = \varphi(v \wedge w) - g_E(v, w)\mathrm{id}, \tag{36}$$

for all $u, v, w \in E_x$ with $|u|_E^2 = 1$ and $v, w \perp u$ (where id denotes the identity of T_xM).

Using the universality property of the even Clifford algebra (Lemma 4.1 below), this shows that (E, φ) defines an even Clifford structure on M. We have proved the following:

Theorem 3.7. *Assume that the curvature constancy of Z is the vertical distribution of a Riemannian submersion $(Z^{k+n}, g_Z) \to (M^n, g)$. Then (M, g)*

(a) *carries a parallel even Clifford structure (E, ∇^E, φ) of rank $r = k + 1$;*
(b) *the curvature of E, viewed as an endomorphism $\omega : \Lambda^2(TM) \to \mathrm{End}^-(E)$, equals minus twice the metric adjoint of $\varphi : \Lambda^2 E \simeq \mathrm{End}^-(E) \to \mathrm{End}^-(TM) \simeq \Lambda^2(TM)$.*

Conversely, if (M, g) satisfies these conditions, then the sphere bundle Z of E, together with the Riemannian metric induced by the connection ∇^E on Z defines a Riemannian submersion onto (M, g) whose vertical distribution belongs to the curvature constancy.

3.3. **The classification.** From Theorem 3.7, every Riemannian submersion $(Z^{k+n}, g_Z) \to (M^n, g)$ whose vertical distribution belongs to the curvature constancy defines a parallel even Clifford structure $(E, \nabla^E, h, \varphi)$ of rank $r := k + 1$ on M, such that the curvature ω of ∇^E, viewed as an endomorphism $\omega : \Lambda^2(TM) \to \mathrm{End}^-(E)$, equals minus twice the metric adjoint of the Clifford morphism $\varphi : \Lambda^2 E \to \mathrm{End}^-(TM)$. In the notation of Proposition 2.9, this amounts to say that

$$\omega_{ij} = 2J_{ij}, \qquad \forall\ 1 \le i \ne j \le r. \tag{37}$$

Conversely, if $(E, \nabla^E, h, \varphi)$ is a parallel even Clifford structure of rank r on M satisfying (37), E carries a Riemannian metric defined by the metric on M, that of E, and the splitting of the tangent bundle of E given by the connection ∇^E and by Theorem 3.7, the restriction to the unit sphere bundle Z of the projection $E \to M$ is a Riemannian submersion whose vertical distribution belongs to the curvature constancy.

We will now examine under which circumstances a simply connected complete Riemannian manifold

(i) carries a parallel even Clifford structure $(E, \nabla^E, h, \varphi)$.
(ii) $(E, \nabla^E, h, \varphi)$ satisfies (37).

Notice that for every $3 \le n \ne 4$, condition (ii) together with (14) and (16) implies that the scalar curvature of M is

$$\mathrm{scal} = 2n(n/4 + 2r - 4). \tag{38}$$

• $r = 2$. In this case M is Kähler (see Example 2.5) and E is simply a rank 2 Euclidean vector bundle endowed with a metric connection ∇^E whose curvature is minus twice the Kähler form of M. By the Chern-Weil theory, this is equivalent to the cohomology class of the Kähler form being half-integer, so up to rescaling M is a Hodge manifold. It is well-known that the circle bundle Z of E carries a Sasakian structure for the corresponding rescaling of the metric on M.

• $r = 3$. By Example 2.6, condition (i) is equivalent to M being quaternion-Kähler (recall that this is an empty condition for $n = 4$) and E is either $\Lambda^2_+ M$ for $n = 4$ or the 3-dimensional sub-bundle of $\Lambda^2 M$ defining the quaternion-Kähler structure for $n > 4$. Condition (ii) is equivalent to M being anti-self-dual and Einstein with scalar curvature equal to 24 (see [5] p.51 and (38) above) for $n = 4$, and quaternion-Kähler with positive scalar curvature equal to $8q(q + 2)$ for $n = 4q > 4$. The Riemannian manifold (Z, g_Z) is the twistor space of M in the sense of Salamon [19].

• $r = 4$. Proposition 2.9 (i) shows that M is the Riemannian product of two quaternion-Kähler manifolds M^+ and M^- of dimension $4q^+$ and $4q^-$ respectively (notice that one of q^+ or q^- might vanish). Recall that the rank 4 even Clifford structure E on M induces in a natural way rank 3 even Clifford structures $\Lambda^2_\pm E$ on $M^\mp$. A local orthonormal basis e_i, $1 \le i \le 4$ of E induces local orthonormal bases $\tilde{e}_i^\pm$, $1 \le i \le 3$ of $\Lambda^2_\pm E$ by (7). Taking (8) and (9) into account, Equation (37) becomes

$$\omega^\pm_{ij} = 4J^\pm_{ij}, \qquad \text{for all } 1 \le i \ne j \le 3. \tag{39}$$

Like in the previous case, this means that $M^\pm$ is a quaternion-Kähler manifold with scalar curvature $16q^\pm(q^\pm+2)$, where now we use the usual convention that in dimension 4 quaternion-Kähler means anti-self-dual and Einstein.

In order to describe the Riemannian manifold (Z, g_Z), we need to understand in more detail the construction of the even Clifford structure of rank 4 on a product of quaternion-Kähler manifolds $M = M^+ \times M^-$. The orthonormal frame bundle of M admits a reduction to a principal bundle P with structure group $G := \mathrm{Sp}(q^+)\cdot\mathrm{Sp}(1)\times\mathrm{Sp}(q^-)\cdot\mathrm{Sp}(1) \subset \mathrm{SO}(4q^+ + 4q^-)$. The representation of the universal cover of G on $\mathbb{R}^4 \simeq \mathbb{H}$ defined by

$$(A, a, B, b)(v) = avb^{-1} \qquad \forall\ (A, a, B, b) \in \mathrm{Sp}(q^+) \times \mathrm{Sp}(1) \times \mathrm{Sp}(q^-) \times \mathrm{Sp}(1),\ \forall v \in \mathbb{H}$$

induces a projective representation $\rho : G \to \mathrm{PSO}(4)$, which in turn determines the (locally defined) bundle E and the (globally defined) manifold $Z := P \times_\rho \mathbb{RP}^3$. A Riemannian manifold obtained in this way is called *quaternion-Sasakian*. By definition, a quaternion-Sasakian manifold fibres over a product of quaternion-Kähler manifolds $M = M^+ \times M^-$, with fiber $\mathbb{RP}^3$. Notice that 3-Sasakian manifolds are special cases of quaternion-Sasakian manifolds, when one of the factors M^+ or M^- is reduced to a point.

We now examine the remaining cases in Table 2.

• $r \ge 5$ and $n = 8$. Taking (10) into account, (37) is equivalent to the fact that the restriction of the curvature endomorphism R of M to the Lie sub-algebra $\varphi(\Lambda^2 E) \subset \Lambda^2 M$ equals 4id. Moreover, we have $\kappa = 2$ in Equation (14), so (15) shows that M is Einstein with scalar curvature $2n(n/4 + 2r - 4)$.

If $r = 8$, this means that R is constant, equal to 4 on $\Lambda^2 M$, so M is the round sphere $\mathbb{S}^8(1/2)$ of radius $1/2$.

The case $r = 7$ does not occur, since a manifold with holonomy Spin(7) is Ricci-flat, contradicting Proposition 2.9 (iii).

If $r = 6$, M is Kähler and $\varphi(\Lambda^2 E)$ is just the sub-bundle $\Lambda_0^{(1,1)} M$ of primitive forms of type $(1,1)$, corresponding to the isomorphism $\mathfrak{spin}(6) \simeq \mathfrak{su}(4)$. By the above M has Einstein constant equal to 20. This shows that the curvature endomorphism of M is equal to 4 on $\Lambda_0^{(1,1)} M$, is equal to 20 on the line generated by the Kähler form (since the image of the Kähler form is the Ricci form), and vanishes on $\Lambda^{(2,0)+(0,2)} M$ (like on every Kähler manifold), so M is isometric to the complex projective space $\mathbb{CP}^4 = \mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(1) \cdot \mathrm{U}(4))$ endowed with the Fubini-Study metric with scalar curvature 160.

If $r = 5$, M is quaternion-Kähler, and by a slight abuse of notation we can write $\Lambda^2 M = \mathfrak{sp}(1) \oplus \mathfrak{sp}(2) \oplus \mathfrak{p}$. Like before, the curvature endomorphism R of M equals 4 on $\mathfrak{sp}(2) = \varphi(\Lambda^2 E)$. Moreover, on every quaternion-Kähler manifold with Einstein constant 16, R equals 4 on $\mathfrak{sp}(1)$ and vanishes on $\mathfrak{p}$. Thus M is isometric to $\mathbb{HP}^2 = \mathrm{Sp}(3)/\mathrm{Sp}(1) \times \mathrm{Sp}(2)$.

• $r \geq 5$ and $n > 8$. This case concerns the symmetric spaces M in the last seven rows of Table 2. For each of these spaces condition (ii) is automatically satisfied (by Proposition 2.9) for the specific normalization of the metric for which $\kappa = 2$ in Equation (14), which by (15) is equivalent to the scalar curvature being equal to $2n(n/4 + 2r - 4)$.

Summarizing, we have proved the following

Theorem 3.8. *There exists a Riemannian submersion from a complete Riemannian manifold (Z^{k+n}, g_Z) to a complete simply connected Riemannian manifold (M^n, g) whose vertical distribution belongs to the curvature constancy if and only if (Z, M) appears to the following list:*

Z	M	Fibre	$\dim(M)$	$\mathrm{scal}(M)$
Sasakian	Hodge	$\mathbb{S}^1$	$2m,\ m \geq 1$	
Twistor space Z	quaternion-Kähler (QK)	$\mathbb{S}^2$	$4q,\ q \geq 1$	$8q(q+2)$
Quaternion-Sasakian	product of two QK manifolds	$\mathbb{RP}^3$	$4q,\ q \geq 1$ $q = q^+ + q^-$	$16q^+(q^+ + 2)$ $+16q^-(q^- + 2)$
$\frac{\mathrm{Sp}(k+2)}{\mathrm{Sp}(k)\times\mathrm{Spin}(4)}$	$\mathrm{Sp}(k+2)/\mathrm{Sp}(k) \times \mathrm{Sp}(2)$	$\mathbb{S}^4$	$8k,\ k \geq 1$	$32k(k+3)$
$\frac{\mathrm{SU}(k+4)}{\mathrm{S}(\mathrm{U}(k)\times(\mathrm{Sp}(2)\cdot U(1)))}$	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$	$\mathbb{RP}^5$	$8k,\ k \geq 1$	$32k(k+4)$
$\frac{\mathrm{SO}(k+8)}{\mathrm{SO}(k)\times\mathrm{Spin}(7)}$	$\mathrm{SO}(k+8)/\mathrm{SO}(k) \times \mathrm{SO}(8)$	$\mathbb{RP}^7$	$8k,\ k \geq 1$	$32k(k+6)$
$F_4/\mathrm{Spin}(8)$	$F_4/\mathrm{Spin}(9)$	$\mathbb{S}^8$	16	$2^6 \cdot 3^2$
$E_6/\mathrm{Spin}(9) \cdot \mathrm{U}(1)$	$E_6/\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	$\mathbb{S}^9$	32	$2^9 \cdot 3$
$E_7/\mathrm{Spin}(11) \cdot \mathrm{SU}(2)$	$E_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$	$\mathbb{S}^{11}$	64	$2^9 \cdot 3^2$
$E_8/\mathrm{Spin}(15)$	$E_8/\mathrm{Spin}(16)$	$\mathbb{S}^{15}$	128	$2^{10} \cdot 3 \cdot 5$

Table 3. Riemannian submersions with curvature constancy.[2]

[2]We adopt in this table the usual convention for quaternion-Kähler manifolds in dimension 4 as being anti-self-dual and Einstein.

We end up this section with a short list of interesting problems related to Clifford structures and perspectives of possible further research. These are just a few examples of the numerous questions raised by our work.

- The notion of curvature constancy has a hyperbolic counterpart which leads to the notion of Lorentzian Clifford structures. This problem can be studied with methods similar to those above and could provide a new framework for theoretical physicists.
- Many notions and results from almost Hermitian geometry can be generalized to Clifford structures. One can for instance introduce the minimal connection of an (even) Clifford structure, and obtain a Gray-Hervella-type classification of Clifford structures.
- One can also address the question of the existence of global almost complex structures compatible with a parallel (even) Clifford structure, generalizing corresponding results by D. V. Alekseevsky, S. Marchiafava and M. Pontecorvo [2] obtained for quaternion-Kähler manifolds.

4. Appendix. The universality property of the even Clifford algebra

For the reader's convenience we provide here the proof of the universality property for even Clifford algebras which was needed in the proof of Theorem 3.8.

Lemma 4.1. *Let (V,h) be a Euclidean vector space and let $\mathcal{A}$ be any real algebra with unit. We make the usual identification of Λ^2V with a subspace of $\mathrm{Cl}^0(V,h)$. Then a linear map $\varphi:\Lambda^2V\to\mathcal{A}$ extends to an algebra morphism $\varphi:\mathrm{Cl}^0(V,h)\to\mathcal{A}$ if and only if it satisfies*

$$\varphi(u\wedge v)\circ\varphi(u\wedge w)=\varphi(v\wedge w)-h(v,w)1_{\mathcal{A}} \tag{40}$$

for all $u,v,w\in V$ with $|u|^2_V=1$ and $v,w\perp u$.

Proof. The "only if" part is obvious. Assume, conversely, that (40) holds and let $u,v,w\in V$ be arbitrary vectors. We apply (40) to the triple $\tilde u:=u/|u|$, $\tilde v:=v-h(u,v)u/|u|^2$ and $\tilde w:=w-h(u,w)u/|u|^2$ and obtain

$$\begin{aligned}\varphi(u\wedge v)\circ\varphi(u\wedge w)=&|u|^2\varphi(v\wedge w)-h(u,v)\varphi(u\wedge w)-h(u,w)\varphi(v\wedge u)\\&-(|u|^2h(v,w)-h(u,v)h(u,w))1_{\mathcal{A}}.\end{aligned} \tag{41}$$

By defining $\sigma:V\otimes V\to\mathcal{A}$, $u\otimes v\mapsto\sigma_{uv}:=\varphi(u\wedge v)-h(u,v)1_{\mathcal{A}}$, (41) becomes equivalent to

$$\sigma_{uv}+\sigma_{vu} = -2h(u,v)1_{\mathcal{A}}, \tag{42}$$

$$\sigma_{vu}\circ\sigma_{uw} = -h(u,u)\sigma_{vw} \tag{43}$$

for all $u,v,w\in V$. Let $T(V)$ denote the tensor algebra of V and

$$T^0(V):=\bigoplus_{k\geq 0}V^{\otimes 2k}.$$

By definition, $\mathrm{Cl}^0(V,h) = T^0(V)/\mathcal{I}$, where $\mathcal{I}$ is the intersection with $T^0(V)$ of the two-sided ideal of $T(V)$ generated by elements of the form $u\otimes u + h(u,u)$. The map σ clearly induces a unique algebra morphism $\sigma^* : T^0(V) \to \mathcal{A}$ such that $\sigma^* = \sigma$ on $V\otimes V$. We claim that $\mathcal{I} \subset \mathrm{Ker}(\sigma^*)$. Now, every element of $\mathcal{I}$ is a linear combination of elements of the form $A = a\otimes(u\otimes u + h(u,u))\otimes b$ or $B = a\otimes v\otimes(u\otimes u+h(u,u))\otimes w\otimes b$, with $a,b\in T^0(V)$ and $u,v,w\in V$. From (42) we have

$$\sigma^*(A) = \sigma^*(a)\circ(\sigma_{uu} + h(u,u)1_{\mathcal{A}})\circ\sigma^*(b) = 0,$$

and (43) yields

$$\begin{aligned}\sigma^*(B) &= \sigma^*(a)\circ\sigma^*(v\otimes u\otimes u\otimes w + h(u,u)v\otimes w)\circ\sigma^*(b)\\ &= \sigma^*(a)\circ(\sigma_{vu}\circ\sigma_{uw} + h(u,u)\sigma_{vw})\circ\sigma^*(b) = 0.\end{aligned}$$

Consequently σ^* descends to an algebra morphism $\mathrm{Cl}^0(V,h)\to\mathcal{A}$, whose restriction to $\Lambda^2 V$ is just φ. □

References

[1] B. Alexandrov, G. Grantcharov, S. Ivanov, *Curvature properties of twistor spaces of quaternionic Kähler manifolds,* J. Geom. **62** (1998), no. 1-2, 1–12.

[2] D.V. Alekseevsky, S. Marchiafava, M. Pontecorvo, *Compatible Almost Complex Structures on Quaternion Kähler Manifolds*, Ann. Global Anal. Geom. **16** (1998), no. 5, 419–444.

[3] M.L. Barberis, I.G. Dotti Miatello, R.J. Miatello, *On certain locally homogeneous Clifford manifolds*, Ann. Global Anal. Geom. **13** (1995), no. 3, 289–301.

[4] L. Bérard-Bergery, *Sur certaines fibrations d'espaces homogènes riemanniens,* Compositio Math. **30** (1975), 43–61.

[5] A. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **10** Springer-Verlag, Berlin, 1987.

[6] I. Burdujan, *Clifford-Kähler manifolds,* Balkan J. Geom. Appl. **13** (2008), no. 2, 12–23.

[7] I. Burdujan, *On almost Cliffordian manifolds,* Ital. J. Pure Appl. Math. No. **13** (2003), 129–144.

[8] S. Chern, N. Kuiper, *Some theorems on the isometric embedding of compact Riemann manifolds in euclidean space*, Ann. of Math. (2) **56**, (1952). 422–430.

[9] R. Cleyton, A. Swann, *Einstein metrics via intrinsic or parallel torsion*, Math. Z. **247** (2004), 513–528.

[10] L. Florit, W. Ziller, *Topological obstructions for fatness,* arXiv:1001.0967.

[11] A. Gray, *Spaces of constancy of curvature operators,* Proc. Amer. Math. Soc. **17** (1966), 897–902.

[12] D. Gromoll, G. Walschap, *Metric Foliations And Curvature*, Progress In Mathematics **268**, Birkhäuser Boston, 2009.

[13] R. Hermann, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle*, Proc. Amer. Math. Soc. **11** (1960), 236–242.

[14] D. Joyce, *Manifolds with many complex structures,* Quart. J. Math. Oxford Ser. (2) **46** (1995), no. 182, 169–184.

[15] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I*, New York, Interscience Publishers, 1963.

[16] B. Lawson, M.-L. Michelson, *Spin Geometry,* Princeton University Press, 1989.

[17] R. Maltz, *The nullity spaces of curvature-like tensors,* J. Differential Geometry **7** (1972), 519–523.

[18] Y. Nikolayevsky, *Osserman manifolds and Clifford structures,* Houston J. Math. **29** (2003), no. 1, 59–75.

[19] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics Series, 201. Longman Scientific & Technical, Harlow; New York, 1989

[20] Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen, *Extended supersymmetric σ-models on group manifolds. I. The complex structures*, Nuclear Phys. B **308** (1988), no. 2-3, 662–698.
[21] A. Weinstein, *Fat bundles and symplectic manifolds*, Adv. Math. **37** (1980), no. 3, 239–250.
[22] W. Ziller, *Fatness revisited*, Lecture Notes (2001).

Andrei Moroianu, CMLS, École Polytechnique, UMR 7640 du CNRS, 91128 Palaiseau, France

E-mail address: am@math.polytechnique.fr

Uwe Semmelmann, Mathematisches Institut, Universität zu Köln, Weyertal 86-90 D-50931 Köln, Germany

E-mail address: uwe.semmelmann@math.uni-koeln.de