DEFORMATIONS OF NEARLY KÄHLER STRUCTURES

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Abstract. We study the space of nearly Kähler structures on compact 6-dimensional manifolds. In particular, we prove that the space of infinitesimal deformations of a strictly nearly Kähler structure (with scalar curvature scal) modulo the group of diffeomorphisms, is isomorphic to the space of primitive co-closed $(1,1)$-eigenforms of the Laplace operator for the eigenvalue $2\mathrm{scal}/5$.

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1. Introduction

A nearly Kähler manifold is an almost Hermitian manifold (M, g, J) with the property that $(\nabla_X J)X = 0$ for all tangent vectors X, where ∇ denotes the Levi-Civita connection of g. A nearly Kähler manifold is called *strictly* nearly Kähler if $(\nabla_X J)$ is non-zero for every tangent vector X. Besides Kähler manifolds, there are two main families of examples of compact nearly Kähler manifolds: Naturally reductive 3-symmetric spaces, which are classified by A. Gray and J. Wolf [12] and twistor spaces over compact quaternion-Kähler manifolds with positive scalar curvature, endowed with the non-integrable canonical almost complex structure (cf. [7]).

A nearly Kähler manifold of dimension 4 is automatically a Kähler surface, and the only known examples of non-Kähler compact nearly Kähler manifolds in dimension 6 are the 3-symmetric spaces G_2/SU_3, $\mathrm{SU}_3/S^1\times S^1$, $\mathrm{Sp}_2/S^1\times\mathrm{Sp}_1$ and $\mathrm{Sp}_1\times\mathrm{Sp}_1\times\mathrm{Sp}_1/\mathrm{Sp}_1$. Moreover, J.-B. Butruille has recently shown in [4] that there are no other homogeneous examples in dimension 6.

On the other hand, using previous results of R. Cleyton and A. Swann on G-structures with skew-symmetric intrinsic torsion, the second-named author has proved in [7] that every compact simply connected nearly Kähler manifold M is isometric to a Riemannian product $M_1 \times \ldots \times M_k$, such that for each i, M_i is a nearly Kähler manifold belonging to the following list: Kähler manifolds, naturally reductive 3-symmetric spaces, twistor spaces over compact quaternion-Kähler manifolds with positive scalar curvature, and 6-dimensional nearly Kähler manifolds.

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It is thus natural to concentrate on the 6-dimensional case, all the more that in this dimension, non-Kähler nearly Kähler manifolds have several interesting features: they carry real Killing spinors (and thus are automatically Einstein with positive scalar curvature) and they are defined by a SU_3 structure whose intrinsic torsion is skew-symmetric. These manifolds were intensively studied by A. Gray in the 70's, thus motivating the following

Definition 1.1. *A compact strictly nearly Kähler manifold of dimension* 6 *is called a* Gray manifold.

The main goal of this paper is to study the deformation problem for Gray manifolds. Notice that we consider simultaneous deformations of the metric and of the almost complex structure. Indeed M. Verbitsky proved in [10] that on a 6-dimensional almost complex manifold there is up to constant rescaling at most one strictly nearly Kähler metric. Conversely it is well known (c.f. [1] or Section 4 below) that on a manifold (M^6, g) which is not locally isometric to the standard sphere, there is at most one compatible almost complex structure J such that (M, g, J) is nearly Kähler.

We start by studying deformations of SU_3 structures, then use the characterization of Gray manifolds as SU_3 structures satisfying a certain exterior differential system in order to compute the space of infinitesimal deformations of a given Gray structure modulo diffeomorphisms. In particular, we prove that this space is isomorphic to some eigenspace of the Laplace operator acting on 2-forms (see Theorem 4.2 for a precise statement).

2. Algebraic preliminaries

Let V denote the standard 6-dimensional SU_3 representation space, which comes equipped with the Euclidean product $g \in \mathrm{Sym}V^*$, the complex structure $J \in \mathrm{End}(V)$, the fundamental 2-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot) \in \Lambda^2 V^*$, and the complex volume element $\psi^+ + i\psi^- \in \Lambda^{(3,0)}V^*$.

These objects satisfy the compatibility relations

$$\omega \wedge \psi^\pm = 0, \qquad \psi^+ \wedge \psi^- = \frac{2}{3}\omega^3 = 4dv, \tag{1}$$

where dv denotes the volume form of the metric g. It is easy to check that ψ^+ and ψ^- are related by

$$\psi^-(X, Y, Z) := -\psi^+(JX, Y, Z). \tag{2}$$

We identify elements of V and V^* using the isomorphism induced by g. For any orthonormal basis $\{e_i\}$ of V adapted to J (*i.e.* $J(e_{2i-1}) = e_{2i}$) we have:

$$\omega = e^{12} + e^{34} + e^{56},$$

$$\psi^+ = e^{135} - e^{146} - e^{236} - e^{245}, \qquad \psi^- = e^{136} + e^{145} + e^{235} - e^{246}.$$

The following formulas are straightforward (it is enough to check them for $X = e_1$ and use the transitivity of the SU_3 action on spheres):

$$\psi^+ \wedge (X \lrcorner \psi^+) = X \wedge \omega^2, \qquad \psi^+ \wedge (X \lrcorner \psi^-) = -JX \wedge \omega^2, \tag{3}$$

$$\psi^- \wedge (X \lrcorner \psi^+) = JX \wedge \omega^2, \qquad \psi^- \wedge (X \lrcorner \psi^-) = X \wedge \omega^2. \tag{4}$$

Let $\Lambda : \Lambda^p V \to \Lambda^{p-2} V$ denote the metric adjoint of the wedge product with ω, $\Lambda = \frac{1}{2}\sum_i Je_i \lrcorner e_i \lrcorner$. It is easy to check that

$$\Lambda(X \lrcorner \psi^\pm) = 0, \quad \Lambda(X \wedge \psi^\pm) = JX \lrcorner \psi^\pm, \quad \forall X \in V \tag{5}$$

and

$$\Lambda(\tau \wedge \omega) = \omega \wedge \Lambda\tau + (3-p)\tau, \quad \forall \tau \in \Lambda^p V. \tag{6}$$

We next describe the decomposition into irreducible summands of $\Lambda^2 V$ and $\Lambda^3 V$. We use the notation $\Lambda^{(p,q)+(q,p)}V$ for the projection of $\Lambda^{(p,q)}V$ onto the real space $\Lambda^{p+q}V$.

$$\Lambda^2 V = (\Lambda_0^{(1,1)} V \oplus \mathbb{R}\omega) \oplus \Lambda^{(2,0)+(0,2)} V\ , \tag{7}$$

where the first two summands consist of J-invariant and the last of J-anti-invariant forms. Here $\Lambda_0^{(1,1)}V$ is the space of primitive $(1,1)$-forms, *i.e.* the kernel of the contraction map Λ. The map

$$\xi \mapsto \xi \lrcorner \psi^+ \tag{8}$$

defines an isomorphism of the second summand $\Lambda^{(2,0)+(0,2)}V$ with V. For 3-forms we have the irreducible decomposition

$$\Lambda^3 V = (\Lambda^1 V \wedge \omega) \oplus \Lambda^{(3,0)+(0,3)} V \oplus \Lambda_0^{(2,1)+(1,2)} V. \tag{9}$$

The second summand $\Lambda^{(3,0)+(0,3)}V$ is 2-dimensional and spanned by the forms $\psi^\pm$. The third summand $\Lambda_0^{(2,1)+(1,2)}V$ is 12-dimensional and can be identified with the space of symmetric endomorphisms of V anti-commuting with J. We note that because of the Schur lemma, the map given by taking the wedge product with ω vanishes on the last two summands.

An endomorphism A of V (not necessarily skew-symmetric) acts on p-forms by the formula

$$(A_\star u)(X_1, \ldots, X_p) := -\sum_{i=1}^{p} u(X_1, \ldots, A(X_i), \ldots, X_p). \tag{10}$$

A more convenient way of writing this action is

$$A_\star u = -\sum_{i=1}^{p} A^*(e_i) \wedge e_i \lrcorner u, \tag{11}$$

where A^* denotes the metric adjoint of A. Taking $A = J$ we obtain the form spaces $\Lambda^{(p,q)+(q,p)}V$ as eigenspaces of the J-action for the eigenvalues $-(p-q)^2$. In particular, we see that $J_\star\varphi = 0$ for any J-invariant 2-form $\varphi \in \Lambda^{(1,1)}V$.

Let $\mathrm{Sym}^- V$ denote the space of symmetric endomorphisms anti-commuting with J. This space is clearly invariant by composition with J. The map $S \mapsto S_\star\psi^+$, with $S \in \mathrm{Sym}^- V$, defines an isomorphism of SU(3)-representations

$$\mathrm{Sym}^- V \cong \Lambda_0^{(2,1)+(1,2)} V,$$

showing in particular that $\mathrm{Sym}^- V$ is irreducible. Taking (2) into account, we remark that for $S \in \mathrm{Sym}^- V$ we have

$$S_\star\psi^+ = (JS)_\star\psi^-. \tag{12}$$

Notice that $\mathrm{tr}(S) = 0$ for all $S \in \mathrm{Sym}^- V$.

Let h be any skew-symmetric endomorphism anti-commuting with J. Then the map $h \mapsto g(h\cdot, \cdot)$ identifies the space of skew-symmetric endomorphism anti-commuting with J with $\Lambda^{(2,0)+(0,2)} V$. Using the isomorphism (8) we can state this as

Lemma 2.1. *An endomorphism F of V anti-commuting with J can be written in a unique way*

$$F = S + \psi_\xi^+$$

for some $S \in \mathrm{Sym}^- V$ and $\xi \in V$, where ψ_ξ^+ denotes the skew-symmetric endomorphism of V defined by $g(\psi_\xi^+\cdot, \cdot) = \psi^+(\xi, \cdot, \cdot)$.

Corresponding to the decomposition of $\Lambda^3 V$ given in (9), we have

Lemma 2.2. *An exterior 3-form $u \in \Lambda^3 V$ can be written in a unique way*

$$u = \alpha \wedge \omega + \lambda\psi^+ + \mu\psi^- + S_\star\psi^+,$$

for some $\alpha \in V$, $\lambda, \mu \in \mathbb{R}$ and $S \in \mathrm{Sym}^- V$. Its contraction with ω satisfies

$$\Lambda u = 2\alpha. \tag{13}$$

Proof. The contraction map Λ obviously vanishes on $\Lambda^{(3,0)+(0,3)} V \oplus \Lambda_0^{(2,1)+(1,2)} V$, so by (6) we have $\Lambda u = \Lambda(\alpha \wedge \omega) = 2\alpha$. □

The space of symmetric endomorphisms commuting with J is identified with $\Lambda^{(1,1)} V$ via the map $h \mapsto \varphi(\cdot, \cdot) := g(hJ\cdot, \cdot)$, which in particular maps the identity of V to the fundamental form ω. If φ is a $(1,1)$-form with corresponding symmetric endomorphism h, then

$$h = h_0 + \frac{1}{6}\mathrm{tr}(h)\mathrm{id}, \tag{14}$$

where h_0 denotes the trace-free part of h. As a consequence of this formula and of the Schur Lemma we find that

$$h_\star\psi^+ = \frac{1}{6}\mathrm{tr}(h)\mathrm{id}_\star\psi^+ = -\frac{1}{2}\mathrm{tr}(h)\psi^+ \tag{15}$$

for all symmetric endomorphisms h commuting with J.

In the remaining part of this section we want to recall several properties and formulas related to the Hodge $*$-operator, which we will use in later computations. We consider the scalar product $\langle\cdot,\cdot\rangle$ on $\Lambda^k V$ characterized by the fact that the basis

$$\{e_{i_1}\wedge\ldots\wedge e_{i_k}\mid 1\le i_1<\ldots<i_k\le 6\}$$

is orthonormal. With respect to this scalar product, the interior and exterior products are adjoint operators:

$$\langle X\lrcorner\,\omega,\tau\rangle=\langle\omega,X\wedge\tau\rangle,\qquad \forall\ X\in V,\ \omega\in\Lambda^kV,\ \tau\in\Lambda^{k-1}V. \tag{16}$$

We define the Hodge $*$-operator $*:\Lambda^kV\to\Lambda^{6-k}V$ by

$$\omega\wedge*\tau:=\langle\omega,\tau\rangle dv,\qquad \forall\ \omega,\tau\in\Lambda^kV,$$

where $dv=1/6\omega^3$ denotes the volume form ($dv=e^{123456}$ in our notations). It is well-known that the following relations are satisfied:

$$*\omega=\frac{1}{2}\omega^2,\quad \langle*\omega,*\tau\rangle=\langle\omega,\tau\rangle,\quad *^2=(-1)^k\quad\text{on }\Lambda^kV \tag{17}$$

From the expression of ψ^+ and ψ^- in any orthonormal basis $\{e_i\}$ we see that $*\psi^+=\psi^-$ and $*\psi^-=-\psi^+$. For later use we compute the Hodge operator on $\Lambda_0^{(2,1)+(1,2)}V$, too. Let $S\in\mathrm{Sym}^-V$ and $\alpha\in\Lambda_0^{(2,1)+(1,2)}V$. We have $\alpha\wedge\psi^-=0$, whence

$$\begin{aligned}\langle\alpha,S_*\psi^+\rangle dv &\overset{(11)}{=} -\sum_i\langle\alpha,S(e_i)\wedge e_i\lrcorner\,\psi^+\rangle dv=-\sum_i\langle e_i\wedge S(e_i)\lrcorner\,\alpha,\psi^+\rangle dv\\ &= (S_*\alpha)\wedge*\psi^+=(S_*\alpha)\wedge\psi^-=S_*(\alpha\wedge\psi^-)-\alpha\wedge(S_*\psi^-)\\ &= -\alpha\wedge(S_*\psi^-)=\langle\alpha,*(S_*\psi^-)\rangle dv\end{aligned}$$

(we used the fact that $*^2=-1$ on 3-forms in order to get the last equality). This shows that

$$*(S_*\psi^-)=S_*\psi^+\qquad\text{and}\qquad *(S_*\psi^+)=-S_*\psi^-. \tag{18}$$

There are two other formulas which we will use later. Let φ_0 be a primitive $(1,1)$-form, $\xi\in V$ and $\alpha\in\Lambda^pV$ then

$$*(\varphi_0\wedge\omega)=-\varphi_0,\qquad *(\xi\wedge\alpha)=(-1)^p\xi\lrcorner*\alpha\ . \tag{19}$$

3. Deformations of SU_3 structures

Let M be a smooth 6-dimensional manifold.

Definition 3.1. *A SU_3 structure on M is a reduction of the frame bundle of M to SU_3. It consists of a 5-tuple $(g,J,\omega,\psi^+,\psi^-)$ where g is a Riemannian metric, J is a compatible almost complex structure, ω is the corresponding fundamental 2-form $\omega(\cdot,\cdot)=g(J\cdot,\cdot)$ and $\psi^++i\psi^-$ is a complex volume form of type $(3,0)$.*

If M carries a SU_3 structure, each tangent space T_xM has a SU_3 representation isomorphic to the standard one, so all algebraic results of the previous section transpose *verbatim* to global results on M. In the remaining part of this article we will most of the time identify tangent vectors and 1-forms on M using the isomorphism induced by the metric g.

Let $(g_t, J_t, \omega_t, \psi_t^+, \psi_t^-)$ be a smooth family of SU_3 structures on M. We omit the index t when the above tensors are evaluated at $t = 0$, and we use the dot to denote the derivative at $t = 0$ in the direction of t.

We start with the study of the 1-jet at $t = 0$ of the family of U_3 structures (g_t, J_t, ω_t).

Lemma 3.2. *There exist a vector field ξ, a section S of* Sym^-M *and a section h of* Sym^+M (i.e. *a symmetric endomorphism commuting with J), such that*

$$\dot{J} = JS + \psi_\xi^+, \tag{20}$$

$$\dot{g} = g((h+S)\cdot, \cdot), \tag{21}$$

$$\dot{\omega} = \varphi + \xi \lrcorner \psi^+, \tag{22}$$

where φ is the $(1,1)$-form defined by $\varphi(\cdot,\cdot) = g(hJ\cdot,\cdot)$.

Proof. Differentiating the identity $J_t^2 = -\mathrm{id}_{TM}$ yields that $\dot{J}$ anti-commutes with J. The first formula thus follows directly from Lemma 2.1.

We write $g_t(\cdot,\cdot) = g(f_t\cdot,\cdot)$ so $\dot{g}(\cdot,\cdot) = g(\dot{f}\cdot,\cdot)$. Let $\dot{f} = h + h'$ be the decomposition of $\dot{f}$ in J-invariant and J-anti-invariant parts. We have to check that $h' = S$. Differentiating the identity $g_t(\cdot,\cdot) = g_t(J_t\cdot, J_t\cdot)$ at $t = 0$ yields

$$\begin{aligned}
g(hX,Y) + g(h'X,Y) &= \dot{g}(X,Y) = \dot{g}(JX,JY) + g(\dot{J}X,JY) + g(JX,\dot{J}Y)\\
&= g(hJX,JY) + g(h'JX,JY) + g(JSX,JY)\\
&\quad + g(\psi_\xi^+X,JY) + g(JX,JSY) + g(JX,\psi_\xi^+Y)\\
&= g(hX,Y) - g(h'X,Y) + 2g(SX,Y),
\end{aligned}$$

showing that $h' = S$. The last formula is a direct consequence of (20) and (21):

$$\begin{aligned}
\dot{\omega}(X,Y) &= \dot{g}(JX,Y) + g(\dot{J}X,Y) = g((h+S)JX,Y) + g((JS+\psi_\xi^+)X,Y)\\
&= g(hJX,Y) + (\xi \lrcorner \psi^+)(X,Y).
\end{aligned}$$

$\square$

This result actually says that the tangent space to the set of all U_3 structures on M at (g,J,ω) is parametrized by a section (ξ, S, φ) of the bundle $TM \oplus \mathrm{Sym}^-M \oplus \Lambda^{(1,1)}M$. We now go forward and describe the 1-jet of a family of SU_3 structures. Since the reduction from a U_3 structure to a SU_3 structure is given by a section in some S^1-bundle, it is not very surprising that the extra freedom in the tangent space is measured by a real function (μ in the notation below):

Lemma 3.3. *The derivatives at $t=0$ of ψ_t^+ and ψ_t^- are given by*

$$\dot\psi^+ = -\xi\wedge\omega + \lambda\psi^+ + \mu\psi^- - \frac{1}{2}S_\star\psi^+, \tag{23}$$

$$\dot\psi^- = -J\xi\wedge\omega - \mu\psi^+ + \lambda\psi^- - \frac{1}{2}S_\star\psi^-, \tag{24}$$

where $\lambda = \frac{1}{4}\mathrm{tr}(h)$ and μ is some smooth function on M.

Proof. By Lemma 2.2, we can write

$$\dot\psi^+ = \alpha\wedge\omega + \lambda\psi^+ + \mu\psi^- + Q_\star\psi^+, \tag{25}$$

for some functions λ, μ, 1-form α and section Q of Sym^-M.

The fact that ψ_t^+ defines – in addition to the U_3 structure (g_t, J_t) – a SU_3 structure is characterized by the two equations

$$g_t(\psi_t^+,\psi_t^+) = 4 \qquad \text{and} \qquad \psi_t^+(J_tX,Y,Z) = \psi_t^+(X,J_tY,Z). \tag{26}$$

We consider the symmetric endomorphism f_t introduced above, which corresponds to g_t in the ground metric g. Since the identity acts on 3-forms by $-3\mathrm{id}$, the first part of (26) reads $g(f_{t\star}\psi_t^+,\psi_t^+) = -12$. Differentiating this at $t=0$ and using the fact that ψ^+ and $S_\star\psi^+$ live in orthogonal components of Λ^3M, we obtain

$$\begin{aligned} 0 &= g(\dot f_\star\psi^+,\psi^+) - 6g(\dot\psi^+,\psi^+) = g((h+S)_\star\psi^+,\psi^+) - 24\lambda \\ &\overset{(15)}{=} 6\mathrm{tr}(h) - 24\lambda. \end{aligned}$$

This determines the function λ. We next differentiate the identity $\psi_t^+\wedge\omega_t = 0$ at $t=0$. Since the wedge product with ω vanishes on ψ^+, ψ^- and on $Q_\star\psi^+\in\Lambda_0^{(2,1)+(1,2)}M$, we get

$$0 = \dot\psi^+\wedge\omega + \psi^+\wedge\dot\omega = \alpha\wedge\omega^2 + \psi^+\wedge(\varphi + \xi\lrcorner\psi^+) \overset{(3)}{=} (\alpha+\xi)\wedge\omega^2,$$

showing that $\alpha = -\xi$. Finally, we differentiate the second part of (26) at $t=0$:

$$\dot\psi^+(JX,Y,Z) + \psi^+(\dot JX,Y,Z) = \dot\psi^+(X,JY,Z) + \psi^+(X,\dot JY,Z).$$

Using (20) and (25), this is equivalent to the fact that the expression

$$-(\xi\wedge\omega)(JX,Y,Z) + (Q_\star\psi^+)(JX,Y,Z) + \psi^+(JSX,Y,Z) + \psi^+(\psi_\xi^+X,Y,Z)$$

is skew-symmetric in X and Y. It is easy to check that

$$-(\xi\wedge\omega)(JX,Y,Z) + \psi^+(\psi_\xi^+X,Y,Z) = (J\xi\wedge\omega)(X,Y,Z),$$

therefore the above condition reduces to

$$(Q_\star\psi^+)(JX,Y,Z) + \psi^+(JSX,Y,Z) = (Q_\star\psi^+)(X,JY,Z) + \psi^+(X,JSY,Z).$$

Using (10), this last relation becomes

$$\psi^+((2Q+S)JX,Y,Z) = \psi^+(X,(2Q+S)JY,Z), \qquad \forall X,Y,Z\in TM. \tag{27}$$

The set of all elements of the form $2Q+S$ satisfying the above relation is a SU_3-invariant subspace of Sym^-M. But Sym^-M is irreducible, and not every element of

$\mathrm{Sym}^- M$ satisfies (27) (to see this, just pick up any element in $\mathrm{Sym}^- M$ and make a direct check). This shows that $2Q + S = 0$.

Finally, the relation (24) is a straightforward consequence of (23). We simply differentiate the formula $J_{t*}\psi_t^+ = 3\psi_t^-$ (obtained from (2)) at $t = 0$ and compute. □

Summarizing, we have shown that the tangent space to the set of all SU_3 structures on M at $(g, J, \omega, \psi^+, \psi^-)$ is parametrized by a section (ξ, S, φ, μ) of the vector bundle $TM \oplus \mathrm{Sym}^- M \oplus \Lambda^{(1,1)} M \oplus \mathbb{R}$.

Let $\alpha : \Lambda^2 M \to TM$ denote the metric adjoint of the linear map

$$X \in TM \mapsto X \lrcorner \psi^+ \in \Lambda^2 M.$$

A simple check shows that

$$\alpha(X \lrcorner \psi^+) = 2X, \qquad \alpha(X \lrcorner \psi^-) = -2JX, \qquad \alpha(\tau) = 0 \quad \forall\, \tau \in \Lambda^{(1,1)} M. \tag{28}$$

Using the map α, we derive a useful relation between the components $\dot\psi^+$ and $\dot\omega$ of any infinitesimal SU_3 deformation:

$$\Lambda \dot\psi^+ \overset{(13)}{=} -2\xi \overset{(28)}{=} -\alpha(\dot\omega). \tag{29}$$

4. Deformations of Gray manifolds

Definition 4.1. *A* Gray structure *on a 6-dimensional manifold M is a SU_3 structure $G := (g, J, \omega, \psi^+, \psi^-)$ which satisfies the exterior differential system*

$$\begin{cases} d\omega = 3\psi^+ \\ d\psi^- = -2\omega \wedge \omega \end{cases} \tag{30}$$

A Gray manifold *is a compact manifold endowed with a Gray structure.*

It follows from the work of Reyes-Carrión [9] that a Gray manifold is a strictly nearly Kähler 6-dimensional compact manifold with scalar curvature $\mathrm{scal} = 30$. We refer to [6] for an introduction to nearly Kähler geometry. We will use later on the relations

$$\nabla_X \omega = X \lrcorner \psi^+, \qquad \nabla_X \psi^+ = -X \wedge \omega,$$

which show that $\nabla\omega$ and $\nabla\psi^+$ are SU_3-invariant tensor fields on M. Moreover the second equation immediately implies that ψ^+ and ω are both eigenforms of the Laplace operator for the eigenvalue 12.

Let now M be a compact 6-dimensional manifold with some Gray structure G on it. We denote by $\mathfrak{M}$ the connected component of G in the space of Gray structures on M. Let $\mathfrak{D}$ be the group of diffeomorphisms of M isotopic to the identity. This group acts on $\mathfrak{M}$ by pull-back and the orbits of this action form the *moduli space of deformations* of G.

At the infinitesimal level, the 1-jet of a curve of Gray structures $(g_t, J_t, \omega_t, \psi_t^+, \psi_t^-)$ at G is a tuple $\gamma := (\dot g, \dot J, \dot\omega, \dot\psi^+, \dot\psi^-)$ determined by a section (ξ, S, φ, μ) of the bundle $TM \oplus \mathrm{Sym}^- M \oplus \Lambda^{(1,1)}M \oplus \mathbb{R}$ *via* (20)-(24), which satisfies the linearized system of (30)

$$\begin{cases} d\dot\omega = 3\dot\psi^+, \\ d\dot\psi^- = -4\dot\omega\wedge\omega. \end{cases} \tag{31}$$

The space of all tuples γ is called the *virtual tangent space* of $\mathfrak{M}$ at G and is denoted by $T_G\mathfrak{M}$. The Lie algebra $\chi(M)$ of $\mathfrak{D}$ maps to $T_G\mathfrak{M}$ by $X \mapsto \mathcal{L}_X G$. Its image, denoted by $\chi^G(M)$, is a vector space isomorphic to $\chi(M)/\mathfrak{K}(g)$, where $\mathfrak{K}(g)$ denotes the set of Killing vector fields on M with respect to g. The *space of infinitesimal Gray deformations* of G is, by definition, the vector space quotient $T_G\mathfrak{M}/\chi^G(M)$.

The main purpose of this section is to give a precise description of this space.

Theorem 4.2. *Let $G := (g, J, \omega, \psi^+, \psi^-)$ be a Gray structure on a manifold M such that (M, g) is not the round sphere (S^6, can). Then the space of infinitesimal deformations of G is isomorphic to the eigenspace for the eigenvalue* 12 *of the restriction of the Laplace operator Δ to the space of* co-closed *primitive $(1,1)$-forms $\Lambda_0^{(1,1)}M$.*

Proof. A simple but very useful remark is that (except on the round sphere S^6), a Gray structure is completely determined by its underlying Riemannian metric. The reason for that is the fact that the metric defines (locally) a unique line of Killing spinors with positive Killing constant, which, in turn, defines the almost complex structure, and, together with the exterior derivative of the Kähler form, one recovers the whole SU_3 structure.

By the Ebin Slice Theorem [2], each infinitesimal deformation of G has a unique representative $\gamma = (\dot g, \dot J, \dot\omega, \dot\psi^+, \dot\psi^-) \in T_G\mathfrak{M}$ such that

$$\delta\dot g = 0 \qquad \text{and} \qquad \mathrm{tr}_g \dot g = 0. \tag{32}$$

Let (ξ, S, φ, μ) be the section of the bundle $TM \oplus \mathrm{Sym}^- M \oplus \Lambda^{(1,1)}M \oplus \mathbb{R}$ determined by γ *via* the equations (20)-(24). We have to interpret the system (31)-(32) in terms of (ξ, S, φ, μ). The main idea is to show that μ and $J\xi$ are eigenvectors of the Laplace operator on functions and 1-forms corresponding to eigenvalues which do not belong to its spectrum, thus forcing them to vanish.

We start by taking the exterior product with ψ^+ in the first equation of (31) and use (23) to get

$$d\dot\omega\wedge\psi^+ = 3\dot\psi^+\wedge\psi^+ = 3\mu\psi^-\wedge\psi^+ = -12\mu dv.$$

On the other hand, using (22) and taking (4) into account yields

$$d\dot\omega\wedge\psi^+ = d(\varphi + \xi \lrcorner \psi^+)\wedge\psi^+ = d((\varphi + \xi\lrcorner\psi^+)\wedge\psi^+) = d(\xi\wedge\omega^2),$$

whence

$$-12\mu dv = d(\xi\wedge\omega^2). \tag{33}$$

We apply the contraction Λ to the first equation of (31) and use (13), (22) and (23):

$$-6\xi = 3\Lambda\dot{\psi}^+ = \Lambda d\dot{\omega} = \Lambda d\varphi + \Lambda d(\xi \lrcorner \psi^+). \tag{34}$$

In order to compute the last term, we apply the general formula (29) to the particular deformation of the SU_3 structure defined by the flow of ξ:

$$\Lambda(\mathcal{L}_\xi \psi^+) = -\alpha(\mathcal{L}_\xi \omega).$$

Since $d\omega = 3\psi^+$ and $d\psi^+ = 0$, we get

$$\Lambda d(\xi \lrcorner \psi^+) = \Lambda(\mathcal{L}_\xi \psi^+) = -\alpha(\mathcal{L}_\xi \omega) = -\alpha(d(\xi \lrcorner \omega)) - \alpha(\xi \lrcorner d\omega) \overset{(28)}{=} -\alpha(dJ\xi) - 6\xi,$$

which, together with (34), yields

$$\Lambda(d\varphi) = \alpha(dJ\xi). \tag{35}$$

We now examine the second equation of the system (31). From (24) we get

$$d\dot{\psi}^- = -dJ\xi \wedge \omega + 3J\xi \wedge \psi^+ - d\mu \wedge \psi^+ + d\lambda \wedge \psi^- - 2\lambda\omega^2 - \frac{1}{2}d(S_\star \psi^-).$$

We apply the contraction Λ to this formula and use the second equation of (31) together with (5) and (6):

$$\begin{aligned} -4\dot{\omega} &= \Lambda(-4\dot{\omega} \wedge \omega) + 4(\Lambda\dot{\omega})\omega = \Lambda d\dot{\psi}^- + 4(\Lambda\dot{\omega})\omega & (36)\\ &= -dJ\xi - \Lambda(dJ\xi)\omega - 3\xi \lrcorner \psi^+ - Jd\mu \lrcorner \psi^+ + Jd\lambda \lrcorner \psi^- & (37)\\ &\quad -8\lambda\omega - \frac{1}{2}\Lambda d(S_\star \psi^-) + 4(\Lambda\dot{\omega})\omega. & (38) \end{aligned}$$

Applying α to this equality and using (2), (22) and (29) yields

$$-8\xi = -\alpha(dJ\xi) - 6\xi - 2Jd\mu + 2d\lambda - \frac{1}{2}\alpha\Lambda d(S_\star \psi^-).$$

From (35) we then get

$$\xi = Jd\mu + \frac{1}{2}\Lambda d\varphi - d\lambda + \frac{1}{4}\alpha\Lambda d(S_\star \psi^-). \tag{39}$$

Lemma 4.3. *Let $(g, J, \omega, \psi^+, \psi^-)$ be a Gray structure on a manifold M. For every section S of* $\mathrm{Sym}^- M$ *and $(1,1)$-form φ, the following relations hold:*

$$\Lambda d\varphi = \delta h + 2d\lambda \tag{40}$$

$$\delta h = -J\delta\varphi \tag{41}$$

$$\delta(S_\star \psi^+) = -\Lambda d(S_\star \psi^-) - 2\delta S \lrcorner \psi^+ \tag{42}$$

$$\alpha\Lambda d(S_\star \psi^-) = -2\delta S \tag{43}$$

$$\Lambda\delta(S_\star \psi^+) = 0 \tag{44}$$

where h is the endomorphism defined by $\varphi(\cdot,\cdot) = g(hJ\cdot,\cdot)$ and $\lambda = \frac{1}{4}\mathrm{tr}(h) = \frac{1}{2}\Lambda\varphi$. In the above formulas, δ stands for the usual co-differential when applied to an exterior form, and for the divergence operator (cf. [3], 1.59) *when acting on symmetric tensors.*

Since the proof is rather technical, we postpone it to the end of this section.

Using (21), (40), (41) and (43), the relation (39) becomes

$$\xi = Jd\mu + \frac{1}{2}\delta h - \frac{1}{2}\delta S = Jd\mu - J\delta\varphi - \frac{1}{2}\delta\dot{g}. \tag{45}$$

From (32) and (45) we obtain

$$\xi = Jd\mu - J\delta\varphi. \tag{46}$$

On the other hand, (46) shows that μ is an eigenfunction of Δ for the eigenvalue 6:

$$\Delta\mu = \delta d\mu = \delta(J\xi) = - * d * (J\xi) = -\frac{1}{2} * d(\xi \wedge \omega^2) \overset{(33)}{=} 6\mu. \tag{47}$$

Now, the Obata theorem (cf. [8, Theorem 3]) says that on a compact n-dimensional Einstein manifold of positive scalar curvature scal, every eigenvalue of the Laplace operator is greater than or equal to $\mathrm{scal}/(n-1)$, and equality can only occur on the standard sphere. Since (M, g) is not isometric to the standard sphere and is Einstein with scalar curvature $\mathrm{scal} = 30$, the Obata theorem thus implies that $\mu = 0$. Since $\lambda = \frac{1}{4}\mathrm{tr}(h) = \frac{1}{4}\mathrm{tr}(\dot{g})$, the second part of (32) also shows that $\lambda = 0$. Taking (32), (41), (42) and (46) into account, the equation (36) now becomes

$$\begin{aligned} -4\varphi - 4\xi \lrcorner \psi^+ &= -4\dot{\omega} = -dJ\xi - \Lambda(dJ\xi)\omega - 3\xi \lrcorner \psi^+ - \frac{1}{2}\Lambda d(S_\star\psi^-) \\ &= -dJ\xi - \Lambda(dJ\xi)\omega - 3\xi \lrcorner \psi^+ + \frac{1}{2}\delta(S_\star\psi^+) - \xi \lrcorner \psi^+, \end{aligned}$$

whence

$$dJ\xi = -\Lambda(dJ\xi)\omega + 4\varphi + \frac{1}{2}\delta(S_\star\psi^+).$$

Applying Λ to this relation and using (44) yields $\Lambda(dJ\xi) = 3\Lambda(dJ\xi)$, so $dJ\xi = 4\varphi + \frac{1}{2}\delta(S_\star\psi^+)$. By (46) we have $J\xi = \delta\varphi$, so

$$\Delta(J\xi) = \delta dJ\xi = 4\delta\varphi = 4J\xi,$$

i.e. $J\xi$ is an eigenform of the Laplace operator with eigenvalue 4. On the other hand, the Bochner formula on 1-forms,

$$\Delta = \nabla^*\nabla + \mathrm{Ric} = \nabla^*\nabla + 5\mathrm{id}$$

shows, by integration over M, that 4 cannot be an eigenvalue of Δ , so ξ has to vanish identically. From Lemmas 3.2 and 3.3, we get

$$\dot{\psi}^+ = -\frac{1}{2}S_\star\psi^+, \qquad \dot{\psi}^- = -\frac{1}{2}S_\star\psi^-, \qquad \dot{\omega} = \varphi \in \Lambda_0^{(1,1)}M \ .$$

Plugging these equations into (31) yields

$$\begin{cases} d\varphi = -\frac{3}{2}S_\star\psi^+, \\ \delta(S_\star\psi^+) = - * d * (S_\star\psi^+) \overset{(18)}{=} *d(S_\star\psi^-) = -2 * d\dot{\psi}^- \overset{(31)}{=} 8 * (\dot{\omega} \wedge \omega) \overset{(19)}{=} -8\varphi. \end{cases} \tag{48}$$

Thus φ is a co-closed eigenform of the Laplace operator for the eigenvalue 12.

Conversely let us assume that $\varphi \in \Omega_0^{(1,1)}M$ is co-closed and satisfies $\Delta\varphi = 12\varphi$. We have to show that φ defines an infinitesimal deformation of the given Gray structure. The main point is to remark that $d\varphi$ is a form in $\Lambda_0^{(2,1)+(1,2)}M$.

Lemma 4.4. *If φ is a co-closed form in $\Omega_0^{(1,1)}M$ then $d\varphi \in \Omega_0^{(2,1)+(1,2)}M$.*

Proof. Using Lemma 2.2, this amounts to check that $d\varphi$ satisfies the system

$$\begin{cases} d\varphi \wedge \psi^+ = 0, \\ d\varphi \wedge \psi^- = 0, \\ \langle d\varphi, X \wedge \omega\rangle = 0, \qquad \forall\, X \in TM. \end{cases} \tag{49}$$

Each of these equations follows easily:

$$d\varphi \wedge \psi^+ = d(\varphi \wedge \psi^+) = 0,$$
$$d\varphi \wedge \psi^- = d(\varphi \wedge \psi^-) - \varphi \wedge d\psi^- \overset{(30)}{=} 2\varphi \wedge \omega^2 \overset{(17)}{=} 4\langle \varphi, \omega\rangle = 0,$$
$$\langle d\varphi, X \wedge \omega\rangle = \langle \Lambda d\varphi, X\rangle \overset{(40)}{=} -\langle J\delta\varphi, X\rangle dv = 0.$$

□

Consequently, there exists a unique section S of $\mathrm{Sym}^- M$ with $d\varphi = -\frac{3}{2}S_\star\psi^+$. Taking $\xi = 0$, $\lambda = 0$ and $\mu = 0$, the equations (20)-(24) define an infinitesimal SU_3-deformation by $\dot\omega := \varphi$, $\dot\psi^+ := -\frac{1}{2}S_\star\psi^+$ and $\dot\psi^- := -\frac{1}{2}S_\star\psi^-$. It remains to show that $(\dot\omega, \dot\psi^+, \dot\psi^-)$ satisfy the linearized system (31). The first equation is clear by definition and the second is equivalent to $d(S_\star\psi^-) = 8\varphi \wedge \omega$. Using again (18) and (19), this last equation is equivalent to $\delta(S_\star\psi^+) = -8\varphi$, which follows directly from the hypothesis on φ and the definition of S. □

Proof of Lemma 4.3. By (8), $\nabla_X J = \psi^+_X \in \Lambda^{(0,2)+(2,0)}M$ so $\langle \varphi, \nabla_X J\rangle = 0$ for all vectors X. Identifying φ with the corresponding skew-symmetric endomorphism of TM, we compute in a local orthonormal basis $\{e_i\}$ parallel at some point:

$$\begin{aligned} \Lambda d\varphi &= \frac{1}{2}Je_i \lrcorner\, e_i \lrcorner\, (e_k \wedge \nabla_{e_k}\varphi) = \frac{1}{2}Je_i \lrcorner\, \nabla_{e_i}\varphi - \frac{1}{2}Je_i \lrcorner\, (e_k \wedge (\nabla_{e_k}\varphi)e_i) \\ &= \frac{1}{2}(\nabla_{e_i}\varphi)Je_i - \frac{1}{2}(\nabla_{Je_i}\varphi)e_i + \frac{1}{2}e_k(\nabla_{e_k}\varphi)(e_i, Je_i) \\ &= (\nabla_{e_i}\varphi)Je_i + d\langle\varphi, \omega\rangle - e_k\varphi(e_i, (\nabla_{e_k}J)e_i) \\ &= (\nabla_{e_i}(\varphi J))e_i - \varphi((\nabla_{e_i}J)e_i) + 2d\lambda - 2e_k\langle \varphi, \nabla_{e_k}J\rangle \\ &= -(\nabla_{e_i}h)(e_i) + 2d\lambda = \delta h + 2d\lambda. \end{aligned}$$

This proves (40). To prove the second relation, we notice that

$$g((\nabla_{e_i}J)(\varphi e_i), X) = \psi^+(e_i, \varphi e_i, X) = 2\langle \psi^+_X, \varphi\rangle = 0,$$

so

$$\delta h = -(\nabla_{e_i}h)e_i = (\nabla_{e_i}(J\varphi))e_i = -J\delta\varphi + (\nabla_{e_i}J)\varphi e_i = -J\delta\varphi.$$

We will now use several times (and signal this by a star above the equality sign in the calculations below) the fact that any SU_3-equivariant map $\mathrm{Sym}^- M \to \Lambda^2 M$ is automatically zero, by the Schur Lemma. For every 3-form τ we can write

$$\begin{aligned} \Lambda d(\tau) &= \frac{1}{2} J e_i \lrcorner e_i \lrcorner (e_k \wedge \nabla_{e_k}\tau) = e_k \wedge (\frac{1}{2} J e_i \lrcorner e_i \lrcorner \nabla_{e_k}\tau) + J e_k \lrcorner \nabla_{e_k}\tau \\ &= d\Lambda\tau - \frac{1}{2} e_k \wedge (\nabla_{e_k} J) e_i \lrcorner e_i \lrcorner \tau + J e_k \lrcorner \nabla_{e_k}\tau. \end{aligned}$$

In particular, for $\tau = S_\star\psi^+$ we have $\Lambda\tau = 0$ and $e_k \wedge (\nabla_{e_k} J) e_i \lrcorner e_i \lrcorner \tau = 0$, so from (11) and the remark above we get

$$\begin{aligned} \Lambda d(S_\star\psi^-) &= J e_k \lrcorner \nabla_{e_k}(S_\star\psi^-) = J e_k \lrcorner ((\nabla_{e_k} S)_\star \psi^-) + J e_k \lrcorner (S_\star(\nabla_{e_k}\psi^-)) \\ &\overset{*}{=} -J e_k \lrcorner ((\nabla_{e_k} S) e_j \wedge e_j \lrcorner \psi^-) \\ &= -\langle (\nabla_{e_k} S) e_j, J e_k \rangle e_j \lrcorner \psi^- + (\nabla_{e_k} S) e_j \wedge (J e_k \lrcorner e_j \lrcorner \psi^-) \\ &\overset{*}{=} \langle (\nabla_{e_k} S) J e_j, e_k \rangle J e_j \lrcorner \psi^+ + (\nabla_{e_k} S) e_j \wedge (J e_k \lrcorner e_j \lrcorner \psi^-) \\ &= -\delta S \lrcorner \psi^+ + (\nabla_{e_k} S) e_j \wedge (J e_k \lrcorner e_j \lrcorner \psi^-), \end{aligned}$$

so by (2)

$$\Lambda d(S_\star\psi^-) = -\delta S \lrcorner \psi^+ + (\nabla_{e_k} S) e_j \wedge (e_k \lrcorner e_j \lrcorner \psi^+). \tag{50}$$

On the other hand, we have

$$\begin{aligned} \delta(S_\star\psi^+) &= e_k \lrcorner \nabla_{e_k}(S e_j \wedge e_j \lrcorner \psi^+) \overset{*}{=} e_k \lrcorner ((\nabla_{e_k} S) e_j \wedge e_j \lrcorner \psi^+) \\ &= -\delta S \lrcorner \psi^+ - (\nabla_{e_k} S) e_j \wedge e_k \lrcorner e_j \lrcorner \psi^+ \\ &\overset{(50)}{=} -2\delta S \lrcorner \psi^+ - \Lambda d(S_\star\psi^-), \end{aligned}$$

thus proving (42). Let $\sigma := (\nabla_{e_k} S) e_j \wedge (e_k \lrcorner e_j \lrcorner \psi^+)$ denote the last summand in (50). A similar calculation easily shows that $J_\star\sigma = 0$, so σ belongs to $\Lambda^{(1,1)} M$. From (28) we thus get

$$\alpha\Lambda d(S_\star\psi^-) = -\alpha(\delta S \lrcorner \psi^+) = -2\delta S.$$

Finally, the relation (44) can be checked in the same way:

$$\begin{aligned} \Lambda\delta(S_\star\psi^+) &= -\frac{1}{2} J e_k \lrcorner e_k \lrcorner (e_j \lrcorner \nabla_{e_j}(S_\star\psi^+)) \\ &= -\frac{1}{2}\Big(e_j \lrcorner \nabla_{e_j}(J e_k \lrcorner e_k \lrcorner (S_\star\psi^+)) - e_j \lrcorner (\nabla_{e_j} J) e_k \lrcorner e_k \lrcorner (S_\star\psi^+)) \Big) \\ &\overset{*}{=} \delta\Lambda(S_\star\psi^+) = 0. \end{aligned}$$

This proves the lemma. □

5. Concluding Remarks

So far we have identified the space of infinitesimal deformations of a given Gray structure with the space of co-closed primitive (1,1)-forms which are eigenforms of the Laplace operator for the eigenvalue 12. In order to proceed further there are two immediate options. One could try to compute the second derivative of a curve of Gray structures and obtain additional equations. However, this leads to quadratic expressions, which for the moment seem to be rather difficult to handle.

A second natural task is to consider the known homogeneous examples and to study the question whether or not there exist at least infinitesimal deformations. This amounts to study the Laplace operator on 2-forms on certain homogeneous spaces, which should reduce to a tractable algebraic problem.

Since a nearly Kähler deformation always gives rise to an Einstein deformation, we could equally well ask for the existence of infinitesimal Einstein deformations on the known nearly Kähler manifolds. Except for the standard sphere, where no deformations exist, there seems to be nothing known in this direction.

The situation for the standard sphere is quite different. Th. Friedrich has shown in [5] that the action of the isometry group SO_7 on the set of nearly Kähler structures on the round sphere (S^6, can) is transitive. The isotropy group of this action at the standard nearly Kähler structure is easily seen to be the group G_2 (the stabilizer in SO_7 of a vector cross product). The space of nearly Kähler structures on the round sphere is thus isomorphic to $\mathrm{SO}_7/G_2 \simeq \mathbb{R}\mathrm{P}^7$ (see [4, Prop. 7.2]). More geometrically, the set of nearly Kähler structures compatible with the round metric on S^6 can be identified with the set of non-zero real Killing spinors (modulo constant rescalings), and it is well-known that the space of real Killing spinors on (S^6, can) is isomorphic to $\mathbb{R}^8$.

The counterpart of Theorem 4.2 on the round sphere (S^6, can) can be stated as follows:

Theorem 5.1. *Let $G := (g, J, \omega, \psi^+, \psi^-)$ be a Gray structure on S^6 such that g is the round metric. Then the space of infinitesimal deformations of G is isomorphic to the eigenspace for the eigenvalue 6 of the Laplace operator Δ on functions, and is, in particular, 7-dimensional.*

Proof. Since there are no Einstein deformations on the sphere, we may assume $\dot{g} = 0$ from the beginning. From (21) we get $\varphi = 0, h = 0, S = 0$ and in particular $\lambda = 0$ too. Then equation (46) gives $\xi = Jd\mu$ and (47) shows that μ is an eigenfunction of Δ on (S^6, can) corresponding to its first non-zero eigenvalue, 6. These eigenfunctions (also called *first spherical harmonics*) satisfy $\nabla_X d\mu = -\mu X$ for all tangent vectors X. We define the infinitesimal SU_3 deformation $\dot{\omega} := \xi \lrcorner \psi^+$, $\dot{\psi}^+ := -\xi \wedge \omega + \mu\psi^-$ and $\dot{\psi}^- := -J\xi \wedge \omega - \mu\psi^+$. A short calculation easily shows that this indeed is a solution of the linearized system (31). $\square$

References

[1] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, *Twistor and Killing Spinors on Riemannian Manifolds,* Teubner-Verlag, Stuttgart-Leipzig, (1991).

[2] M. Berger, D. Ebin *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Differential Geom. **3**, 379–392 (1969).

[3] A. Besse *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 10. Springer-Verlag, Berlin, (1987).
[4] J.-B. Butruille, *Classification des variétés approximativement kähleriennes homogènes*, Ann. Global Anal. Geom. **27**, 201–225 (2005).
[5] Th. Friedrich, *Nearly Kähler and nearly parallel G_2-structures on spheres*, math.DG/0509146.
[6] A. Gray, *The Structure of Nearly Kähler manifolds,* Math. Ann. **223**, 233–248 (1976).
[7] P.-A. Nagy, *Nearly Kähler geometry and Riemannian foliations*, Asian J. Math. **3**, 481–504 (2002).
[8] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere,* J. Math. Soc. Japan, **14**, 333–339 (1962).
[9] R. Reyes-Carrión, *Some special geometries defined by Lie groups,* PhD Thesis, Oxford, (1993).
[10] M. Verbitsky *Hodge theory on nearly Kähler manifolds* math. DG/0510618.
[11] M. Wang *Preserving parallel spinors under metric deformations*, Indiana Univ. Math. J. **40**, 815–844 (1991).
[12] J. Wolf, A. Gray, *Homogeneous spaces defined by Lie group automorphisms I, II,* J. Differential Geom. **2**, 77–114, 115–159 (1968).

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