SYMMETRIES OF CONTACT METRIC MANIFOLDS

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Abstract. We study the Lie algebra of infinitesimal isometries on compact Sasakian and K–contact manifolds. On a Sasakian manifold which is not a space form or 3–Sasakian, every Killing vector field is an infinitesimal automorphism of the Sasakian structure. For a manifold with K–contact structure, we prove that there exists a Killing vector field of constant length which is not an infinitesimal automorphism of the structure if and only if the manifold is obtained from the Konishi bundle of a compact pseudo–Riemannian quaternion–Kähler manifold after changing the sign of the metric on a maximal negative distribution. We also prove that non–regular Sasakian manifolds are not homogeneous and construct examples with cohomogeneity one. Using these results we obtain in the last section the classification of all homogeneous Sasakian manifolds.

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1. Introduction

There are many interesting geometric situations, which are characterized by the existence of a Killing vector field with special properties. Examples are Sasakian manifolds, K-contact manifolds or total spaces of S^1-bundles.

In this article we will study the isometry group of a compact Riemannian manifold admitting a special unit Killing vector field.

The starting point is the simple observation that on a compact Riemannian manifold (M, g) a Killing vector field ξ induces a decomposition of the Lie algebra $\mathfrak{g} := Lie(Iso(M))$ of the isometry group into eigenspaces of the Lie derivative L_ξ. The zero eigenspace $\mathfrak{g}_0$ can be identified with the space of Killing vector fields commuting with ξ.

In the regular case, i.e. where all orbits of ξ are closed and have the same length, it is easy to show (under some technical restrictions) that $\mathfrak{g}_0$ is spanned by ξ and the Killing vector fields of the quotient space M/S^1, cf. Lemma 2.3.

The space $\mathfrak{g}_0$ is thus well understood, at least in the regular case. We then concentrate on the complement of $\mathfrak{g}_0$ in $\mathfrak{g}$ with respect to the decomposition mentioned above. Two

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interesting remarks can be made (cf. Lemma 2.4 below): on the one hand, each Killing vector field from this complement is orthogonal to ξ at every point of M, and on the other hand, the non–zero eigenvalues for the action of $\mathcal{L}_\xi$ on $\mathfrak{g}$ are determined by those of the field of endomorphisms φ corresponding to $d\xi$ on M. In particular, if φ has no constant non–zero eigenvalues on M, then $\mathfrak{g} = \mathfrak{g}_0$.

In order to study the complement of $\mathfrak{g}_0$ in $\mathfrak{g}$ it is thus useful to make further assumptions on φ. In the remaining part of the paper we consider the case where φ defines at every point of M a complex structure on the orthogonal complement of ξ. The manifolds admitting such a structure are called *K–contact*, and among them, those satisfying a further integrability condition are called *Sasakian*.

For these manifolds the Lie algebra $\mathfrak{g}$ splits (as a vector space) into only two subspaces: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$. We provide an easy proof for the fact that a compact Sasakian manifold with non-trivial $\mathfrak{g}_2$ has to be a space of constant curvature or a 3-Sasakian manifold. Moreover we describe the decomposition of $\mathfrak{g}$ on 3-Sasakian manifolds.

In section 4 we characterize K-contact structures carrying Killing vector fields of constant length in $\mathfrak{g}_2$. It turns out that these manifolds are obtained from $SO(3)$-bundles over pseudo-Riemannian quaternion Kähler manifolds by changing the sign of the metric. In addition we give an example of K-contact manifold having Killing vector fields of *non-constant length* in $\mathfrak{g}_2$.

In section 5 we study the isometry group on irregular Sasakian manifolds and use this to obtain in the last section the classification of simply connected compact homogeneous Sasakian manifolds. Note that this classification was already obtained in [7] under the slightly stronger assumption that the automorphism group of the Sasakian structure (rather than the isometry group of the manifold) acts transitively.

2. Manifolds with unit Killing vector fields

Throughout this article we identify vectors and 1–forms on Riemannian manifolds using the metric.

Let (M^n, g) be a compact Riemannian manifold carrying a Killing vector field ξ. We denote by $G = Iso(M)$ the isometry group of M and by $\mathfrak{g} = \mathfrak{iso}(M)$ the Lie algebra of G, identified with the Lie algebra of Killing vector fields on M. Since G is compact, the adjoint representation of G on $\mathfrak{g}$ is orthogonal with respect to some scalar product on $\mathfrak{g}$. In particular, the endomorphism $\mathcal{L}_\xi$ of $\mathfrak{g}$ is antisymmetric, so $i\mathcal{L}_\xi$ is a Hermitian endomorphism of $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_{\lambda_1}^{\mathbb{C}} \oplus \ldots \oplus \mathfrak{g}_{\lambda_s}^{\mathbb{C}}$ be the decomposition of $\mathfrak{g}^{\mathbb{C}}$ in eigenspaces for $i\mathcal{L}_\xi$, where $\mathfrak{g}_{\lambda_k}^{\mathbb{C}}$ corresponds to the eigenvalue λ_k and $\mathfrak{g}_0^{\mathbb{C}}$ to the eigenvalue 0. This induces a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\lambda_1} \oplus \ldots \oplus \mathfrak{g}_{\lambda_s}, \tag{1}$$

where $\mathcal{L}_\xi|\mathfrak{g}_0 = 0$ and $\mathcal{L}_\xi \circ \mathcal{L}_\xi|\mathfrak{g}_{\lambda_k} = -\lambda_k^2$. Note that all λ_k's can be chosen to be strictly positive real numbers. For such a choice, (1) will be called the *standard decomposition* of $\mathfrak{g}$.

From now on we will always assume that ξ has constant length 1.

Definition 2.1. *The vector field ξ is called* regular *if its flow has closed orbits of constant length. It is called* quasi–regular *if the flow has closed orbits whose lengths have jumps. Finally, ξ is called* irregular *if its flow has a non–closed orbit.*

It is easy to see that ξ is regular or quasi–regular if and only if there exists some $p > 0$ such that $\exp_e(p\xi) = e$, where e is the unit of G.

We now study in more detail the case where ξ is regular. The flow of ξ defines an isometric circle action on M and the quotient $N := M/S^1$ of M by this action carries a Riemannian metric such that the projection $M \to N$ is a Riemannian submersion with minimal fibers. Conversely, every connection θ on a principal S^1–bundle M over a Riemannian manifold N induces a 1–parameter family of metrics g^t on M turning the bundle projection into a Riemannian submersion with minimal fibers via the following formula:

$$g^t = g_N + t^2\theta \otimes \theta. \tag{2}$$

The metric g^t carries a unit Killing vector field whose orbits are closed and have constant length $2\pi t$. The following "folkloric" lemma relates the exterior derivative $d\xi$ and the curvature of the connection of the S^1–bundle $M \to N$.

Lemma 2.2. *Let (M^n, g, ξ) be as above and denote by $2\pi\ell$ the length of the orbits of ξ. On the S^1–principal fibration $M \to N := M/S^1$ we define a connection whose horizontal distribution is $\xi^\perp$. Then the curvature form of this connection is equal to $id\xi/\ell$.*

Conversely, if F is a 2–form in $H^2(N, \mathbb{Z})$, let M be the S^1–bundle over N with Chern class $[F]$ and let ξ denote the vector field dual to the connection form of a connection on M whose curvature form is iF. Then for each $t > 0$, $\xi_t := \xi/t$ is a unit Killing vector field for the metric g^t defined by (2), its orbits have length $2\pi t$ and $d\xi_t = d(t\theta) = tF$.

By rescaling the metric on M, we may assume that the orbits of the Killing vector field ξ have constant length 2π (this amounts to take $t = 1$ in the construction above). The infinitesimal isometries of M which preserve ξ (i.e. the space $\mathfrak{g}_0$) can be easily described in the following way:

Lemma 2.3. (cf. [10]) *With the notations from Lemma 2.2, let A be a Killing vector field on M commuting with ξ. Then there exists a unique Killing vector field X on N and a function f on N (unique up to a constant) such that*

$$df = -X \lrcorner F \tag{3}$$

and $A = X^* + f\xi$. Conversely, if F is harmonic and N is simply connected, then for every Killing vector field X on N the equation (3) has a solution f unique up to a constant and $X^* + f\xi$ is a Killing vector field on M commuting with ξ.

Note: For a vector field X on N, X^* denotes the horizontal lift of X to M, i.e. the unique vector field on M orthogonal to ξ which projects onto X.

Proof. Write $A = f\xi + B$ with $\langle B, \xi\rangle = 0$. Since $\mathcal{L}_\xi$ preserves the decomposition $TM = \xi \oplus \xi^\perp$, f is actually a function on N and B is projectable on some $X \in \Gamma(TN)$. The condition $\nabla A(\xi,\xi) = 0$ is automatically satisfied, $\nabla A(Y^*, Y^*) = 0$ for every $Y \in TN$ is equivalent to X being Killing on N and finally $\nabla A(Y^*,\xi) + \nabla A(\xi, Y^*) = 0$ for every $Y \in TN$ is equivalent to (3).

To prove the converse we only have to check that $d(X \lrcorner F) = 0$ for every Killing vector field X on N and harmonic 2–form F. This follows directly from the Hodge decomposition: the harmonic form $\mathcal{L}_X F$ ($\mathcal{L}_X$ commutes with Δ, because X is Killing) is equal to $d(X \lrcorner F) + X \lrcorner dF = d(X \lrcorner F)$, which is exact, so they both vanish. ◻

The previous Lemma actually shows that there is a canonical exact sequence

$$0 \to \mathbb{R}\xi \to \mathfrak{g}_0 \to \mathfrak{iso}(N) \to 0.$$

We now return to the general setting and prove the following simple but very useful lemma concerning the complement of $\mathfrak{g}_0$ in $\mathfrak{g}$ with respect to the standard decomposition described above.

Lemma 2.4. *Let A be a Killing vector field in $\mathfrak{g}_{\lambda_1} \oplus \ldots \oplus \mathfrak{g}_{\lambda_s}$. Then*

1. *A is orthogonal to ξ at every point of M.*
2. *$\mathcal{L}_\xi A = -A \lrcorner d\xi$.*

Proof. 1. It is enough to prove this for A in some fixed $\mathfrak{g}_{\lambda_k}$. As before we can write $A = f\xi + B$ with $\langle B, \xi\rangle = 0$, and from the invariance w.r.t. $\mathcal{L}_\xi$ of the decomposition $TM = \xi \oplus \xi^\perp$ we deduce that $\xi(\xi(f)) = -\lambda_k^2 f$. On the other hand $0 = \langle \nabla_\xi A, \xi\rangle = \xi(f)$, so $f = 0$ (as $\lambda_k \neq 0$).

2. Using the first part of the lemma we get

$$\mathcal{L}_\xi A = -\mathcal{L}_A \xi = -d(A \lrcorner \xi) - A \lrcorner d\xi = -A \lrcorner d\xi. \tag{4}$$

◻

If we identify $d\xi$ with an endomorphism of TM, we see that A has to be an eigenvector of the symmetric endomorphism $d\xi \circ d\xi$ for the eigenvalue $-\lambda_k^2$. This shows that the non-zero coefficients of X in the above "Fourier–type" decomposition are completely determined by the algebraic behavior of $d\xi$.

Example. If $d\xi \circ d\xi$ has no constant negative eigenvalue on M, then every Killing vector field on M is an automorphism of the structure (M, ξ) (i.e. $\mathfrak{g} = \mathfrak{g}_0$).

Example. For the Hopf fibration $(S^{2n+1}, \xi) \to \mathbb{C}P^n$, $d\xi$ is twice the pull–back of the Kähler form of the base, thus showing that $\mathfrak{g}$ is actually reduced to $\mathfrak{g}_0 \oplus \mathfrak{g}_2$.

Example. Let N be equal to the product of s copies of $\mathbb{C}P^1$ with the Fubini–Study metric (i.e. round spheres of radius 1/2). Take $\lambda_1, \dots, \lambda_s$ to be distinct positive integers and let M be the Riemannian manifold induced (via the procedure described above) by the S^1 bundle over N with curvature $i(\lambda_1\omega_1 + \ldots + \lambda_s\omega_s)$, where ω_i is the Kähler form of the i-th factor. Then the Lie algebra of infinitesimal isometries of M is equal (as vector space) to $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\lambda_1} \oplus \ldots \oplus \mathfrak{g}_{\lambda_n}$, where $\mathfrak{g}_0$ is isomorphic to $\mathbb{R}\xi$ plus the direct sum of s copies of $\mathfrak{su}_2$ and each $\mathfrak{g}_{\lambda_i}$ is 2–dimensional.

So far we obtained a description of $\mathfrak{g}_0$ in the regular case and made some general remarks on the Killing vector fields in the orthogonal complement of $\mathfrak{g}_0$. In order to get further results on this complement it is necessary to impose additional assumptions on ξ, more precisely on the algebraic behavior of $d\xi$. In the remaining sections we will consider the case where the skew–symmetric endomorphism corresponding to $d\xi$ defines a complex structure on the orthogonal complement $\xi^\perp$. This leads to K-contact structures or, with a further integrability condition to Sasakian structures.

3. Sasakian Manifolds

A Sasakian structure is a special contact structure on a Riemannian manifold. These structures were studied in the seventies by the Japanese school and, in the last decade, after the work of Bär [2] and Friedrich et al. [3] on manifolds with Killing spinors and that of Boyer et al. on 3–Sasakian manifolds [6], they turned out to constitute one of the most important special geometries, being the odd–dimensional analogues of Kähler manifolds.

Definition 3.1. *A vector field ξ on a Riemannian manifold (M, g) is called a Sasakian structure if ξ is a Killing vector field of unit length and*

$$\nabla_{\cdot}\nabla\xi = \xi \wedge \cdot. \tag{5}$$

In particular, if we apply (5) to two arbitrary vectors and then take the scalar product with ξ we find that the tensors $\varphi := \nabla\xi$ and $\eta := g(\xi, .)$ are related by

$$\varphi^2 = -Id + \eta \otimes \xi.$$

It is an easy exercise (see e.g. [2]) to check that (M, g) is Sasakian if and only if the metric cone $(\bar{M}, \bar{g})$ defined by $\bar{M} = M \times \mathbb{R}^+$ and $\bar{g} = dr^2 + r^2 g$ is Kähler. Most experts today actually prefer to take this last statement as the definition of Sasakian structures, because it is more geometrical and intuitive.

Definition 3.2. *A triple $\{\xi_1, \xi_2, \xi_3\}$ of Sasakian structures is called a 3–Sasakian structure on M if the following conditions are satisfied:*

1. The frame (ξ_1, ξ_2, ξ_3) is orthonormal;

2. For each permutation (i, j, k) of signature δ, the tensors $\varphi_i := \nabla\xi_i$ and $\eta_i := g(\xi_i, .)$ are related by $\varphi_j\,\varphi_i = (-1)^\delta\,\varphi_k - \eta_j \otimes \xi_i$.

Equivalently, M is 3–Sasakian if and only if the cone $\bar{M}$ is hyperkähler, and this can be taken to be the definition as before.

The next lemma gives a sufficient condition for a manifold to be 3–Sasakian. It was originally proved in [3].

Lemma 3.3. *If ξ_1 and ξ_2 are two orthogonal vector fields defining Sasakian structures, then the triple $\{\xi_1, \xi_2, \xi_3 := \nabla_{\xi_1}\xi_2\}$ is a 3–Sasakian structure.*

It turns out that on compact manifolds, the norm of a Killing vector field satisfying (5) is a "characteristic function of the sphere". In other words, we have:

Lemma 3.4. (see [14]) *Let ξ be a Killing vector field on a compact manifold M satisfying (5). If M has non–constant sectional curvature, then ξ has constant length (so it defines a Sasakian structure on M after a homothetic change of metric).*

The following result is a synthesis of several papers studying the isometries of Sasakian manifolds [14], [13]. We will include here a short proof using the considerations in Section 2.

Theorem 3.5. *Let M be a Sasakian manifold.*

- *If M is neither 3–Sasakian nor a space form, then every infinitesimal isometry of M is an infinitesimal automorphism of the Sasakian structure (i.e. $\mathfrak{g} = \mathfrak{g}_0$).*
- *If M is 3–Sasakian but has non–constant sectional curvature, then there is a Lie algebra decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{su}_2$, where $\mathfrak{su}_2$ is generated by the three Sasakian vector fields and $\mathfrak{g}'$ consists of the automorphisms of the 3–Sasakian structure (i.e. Killing vector fields commuting with the Sasakian vector fields).*

Proof. 1. From Section 2 it follows that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$ (since $d\xi \circ d\xi = -4$). Let X be a non-zero element of $\mathfrak{g}_2$ and $Y := \frac{1}{2}\mathcal{L}_\xi X$. Then $\mathcal{L}_Y \xi = -\mathcal{L}_\xi Y = -2\mathcal{L}_\xi^2 X = 2X$, so taking the Lie derivative with respect to Y in (5) yields that X satisfies (5), too. From Lemma 3.4 we deduce that either M is a space form, or X has constant length (and consequently defines a Sasakian structure). By Lemma 3.3, this implies that M is 3–Sasakian, a contradiction. So $\mathfrak{g}_2 = \emptyset$.

2. Let now M be 3–Sasakian. From the first part of the above proof, we deduce that for each $i = 1, 2, 3$, $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{g}_0^i \oplus \mathfrak{g}_2^i$. Now suppose that there is some vector field in say $\mathfrak{g}_2^1$ which is not a linear combination of ξ_2 and ξ_3. The arguments above show that X defines a Sasakian structure, so the cone $\bar{M}$ admits both a hyperkähler structure and another Kähler structure. On the other hand, the cone is either irreducible or flat (see [8]). In the last case, M has to be a space form. Otherwise, it cannot be locally symmetric (being Ricci–flat), so the existence of another Kähler structure on a hyperkähler manifold yields a further holonomy reduction, which is impossible from the Berger Holonomy Theorem. This proves that for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, $\mathfrak{g}_2^i = \langle \xi_j, \xi_k \rangle$. Denote by $\mathfrak{g}' := \cap \mathfrak{g}_0^i$. It is clear that each ξ_i preserves $\mathfrak{g}'$ and that $\mathfrak{g} = \mathfrak{g}' \oplus \langle \xi_1, \xi_2, \xi_3 \rangle$. □

4. K–contact manifolds

Definition 4.1. *1. A* contact metric structure *on a Riemannian manifold* (M, g) *is a unit length vector field* ξ *such that the endomorphism* φ *defined by* $g(\varphi\cdot, \cdot) := \frac{1}{2}d\xi(\cdot, \cdot)$ *and the 1–form* $\eta := \langle \xi, \cdot \rangle$ *are related by*

$$\varphi^2 = -1 + \eta \otimes \xi. \tag{6}$$

In other words, φ *defines a complex structure on the distribution orthogonal to* ξ*.*

2. A contact metric structure (M, g, ξ, φ) *is called* K–contact *if* ξ *is Killing.*

3. A contact metric 3–structure is an orthonormal frame of contact metric structures (ξ_1, ξ_2, ξ_3) *such that for each permutation* (i, j, k) *of signature* δ*, the tensors* $\varphi_i := -\nabla \xi_i$ *and* $\eta_i := g(\xi_i, .)$ *are related by* $\varphi_i\,\varphi_j = (-1)^\delta\,\varphi_k + \eta_j \otimes \xi_i$*. Equivalently, the endomorphisms* φ_i *satisfy the quaternionic relations on the distribution orthogonal to* $\{\xi_1, \xi_2, \xi_3\}$*.*

It is well–known (and straightforward to prove) that M has a contact metric structure if and only if the cone $\bar{M}$ is almost Kähler. Using this simple observation we can easily retrieve (with a shorter proof) the following result of Kashiwada [11]: a contact metric 3–structure is necessarily 3–Sasakian! Indeed, the cone $\bar{M}$ of a contact metric 3–structure M is almost hyperkähler and a lemma by Hitchin [9] (see below) states that every almost hyperkähler manifold is hyperkähler. So $\bar{M}$ is hyperkähler, i.e. M is 3–Sasakian.

We now construct a family of examples of Riemannian manifolds admitting 3 orthogonal (non–integrable) K–contact metric structures which will play an important role in this section. This construction makes essential use of pseudo–Riemannian geometry.

Fundamental example. Let (Q, h) be a pseudo–Riemannian manifold with holonomy $Sp_{p,q} \cdot Sp_1 \subset SO_{4p,4q}$. The simplest examples of such manifolds are, as in the Riemannian case, discrete quotients of the pseudo–Riemannian quaternionic hyperbolic spaces $\mathbb{H}H^{p,q} := Sp_{p,q+1}/Sp_{p-1,q+1} \cdot Sp_1$ (the case $p = 0$, which is excluded in these examples, can be thought of as being the quaternionic projective space $(\mathbb{H}P^q, -can)$, with the negative metric). Other explicit homogeneous examples were constructed by Alekseevski and Cortes (see [1]).

The holonomy condition is equivalent to the existence of a parallel sub–bundle E of $End(Q)$ locally spanned by three endomorphisms satisfying the quaternionic relations. The Konishi construction (see [12]) carries over verbatim to the pseudo–Riemannian situation. The SO_3–principal bundle S associated to E admits a *pseudo 3–Sasakian* metric of signature $(4p, 4q + 3)$. (The definition of a pseudo 3–Sasakian structure is the same as that of a 3–Sasakian structure, except that the metric is pseudo–Riemannian and the 3 Killing vector fields are time–like).

Let us now choose an orthogonal decomposition of the tangent space of Q, $TQ = D_+ \oplus D_-$, such that h is positive on D_+ and negative on D_-. On S there is an induced orthogonal decomposition $TS = D_+^* \oplus D_-^* \oplus V$, where V denotes the vertical distribution. Obviously, this decomposition is invariant under each of the Sasakian Killing vector fields. We define a Riemannian metric g by changing the sign of h on

$D_-^* \oplus V$. The Sasakian vector fields, say ξ_i, are still Killing vector fields for g, and the endomorphisms associated to $\frac{1}{2}d\xi_i$ via the new metric are still complex structures on $\xi_i^\perp$.

Nevertheless, this change of metric has definitely altered the integrability of the contact structures $(\xi_i, \frac{1}{2}d\xi_i)$, and moreover, the three K–contact structures ξ_i on (Q, g) do not define a contact metric 3–structure, since the corresponding endomorphisms φ_i do not satisfy the quaternionic relations on the horizontal distribution (they satisfy the quaternionic relations on D_- and the anti–quaternionic relations on D_+). The Riemannian structures obtained in this way are called *weakly K–contact 3–structures.*

The following result was proved by Hitchin in the Riemannian case [9]. The proof given in [9] works without changes in the pseudo–Riemannian setting.

Lemma 4.2. (pseudo–Riemannian Hitchin Lemma) *A pseudo–Riemannian almost hyperkähler manifold is hyperkähler.*

(Three almost complex structures J_i on a pseudo–Riemannian manifold (M, h) define an almost hyperkähler structure if the J_i satisfy the quaternionic relations and the associated Kähler forms $h(J_i\cdot, \cdot)$ are closed.)

Consider now a K–contact structure, denoted ξ_1 for later convenience, on a compact manifold M. The results in Section 2 show that one can decompose the Lie algebra $\mathfrak{iso}(M)$ as $\mathfrak{iso}(M) = \mathfrak{g}_0 \oplus \mathfrak{g}_2$, where $\mathcal{L}_{\xi_1}|_{\mathfrak{g}_0} = 0$ and $(\mathcal{L}_{\xi_1} \circ \mathcal{L}_{\xi_1})|_{\mathfrak{g}_2} = -4$.

Theorem 4.3. *Let (M, g, ξ_1, φ_1) be a K–contact manifold and let $\mathfrak{iso}(M) = \mathfrak{g}_0 \oplus \mathfrak{g}_2$ be the above decomposition of the Lie algebra of infinitesimal isometries of M. Then $\mathfrak{g}_2$ contains a Killing vector field of constant length if and only if M admits a weakly K–contact 3–structure.*

Proof. The reverse implication is clear from the previous example. Conversely, let $\xi_2 \in \mathfrak{g}_2$ be a Killing vector field of unit length, and denote by $\xi_3 := \frac{1}{2}\mathcal{L}_{\xi_1}\xi_2$, which obviously has unit length, too. By the definition of $\mathfrak{g}_2$ we have $\mathcal{L}_{\xi_1}\xi_3 = -2\xi_2$. The Killing vector field $\zeta := [\xi_2, \xi_3]$ can be computed as follows. On one hand, by the Jacobi identity we have $\mathcal{L}_{\xi_1}\zeta = 0$, and

$$-\langle \zeta, \xi_1 \rangle = d\xi_1(\xi_2, \xi_3) = 2\langle \nabla_{\xi_2}\xi_1, \xi_3 \rangle = -2.$$

Finally, for every $Y \perp \xi_1$, we have

$$\begin{aligned} \langle \zeta, \nabla_Y \xi_1 \rangle &= -\langle \nabla_Y \zeta, \xi_1 \rangle = \langle \nabla_{\xi_1}\zeta, Y \rangle = \langle \nabla_\zeta \xi_1, Y \rangle \\ &= \langle -\varphi_1(\zeta), Y \rangle = \langle \zeta, \varphi_1(Y) \rangle = -\langle \zeta, \nabla_Y \xi_1 \rangle, \end{aligned}$$

thus showing that $\zeta = 2\xi_1$. Consequently, for each permutation (i, j, k) of $(1, 2, 3)$ of signature δ we have $[\xi_i, \xi_j] = 2\delta\xi_k$. Now, if we denote by $\varphi_i := -\nabla\xi_i = \frac{1}{2}d\xi_i$, $\eta_i := \langle \xi_i, \cdot \rangle$ and take the Lie derivative with respect to ξ_2 and ξ_3 in (6) we get

$$\varphi_1\varphi_3 + \varphi_3\varphi_1 = \eta_1 \otimes \xi_3 + \eta_3 \otimes \xi_1, \tag{7}$$

and

$$\varphi_1\varphi_2 + \varphi_2\varphi_1 = \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1. \tag{8}$$

Taking $\mathcal{L}_{\xi_2}$ in (7) and using (6) yields $\varphi_2^2 = -1 + \eta_2 \otimes \xi_2$ and similarly $\varphi_3^2 = -1 + \eta_3 \otimes \xi_3$. Finally, taking $\mathcal{L}_{\xi_3}$ in (7) gives

$$\varphi_2\varphi_3 + \varphi_3\varphi_2 = \eta_2 \otimes \xi_3 + \eta_3 \otimes \xi_2, \tag{9}$$

which shows that the φ_i's are three anti–commuting complex structures on the distribution D orthogonal to ξ_1, ξ_2, ξ_3. From now on we restrict our attention to the distribution D. The endomorphism $\varphi_1\varphi_2\varphi_3$ has square 1, so we can decompose D as $D = D_+ \oplus D_-$, where $D_\pm$ is the eigenspace of $\varphi_1\varphi_2\varphi_3$ corresponding to the eigenvalue ± 1. That is, the restrictions of φ_i to D_+ (resp. D_-) satisfy the anti–quaternionic (resp. quaternionic) relations. If D_+ is empty, we are done, since in that case (ξ_1, ξ_2, ξ_3) define a contact metric 3–structure, so by the theorem of Kashiwada, M has to be 3–Sasakian.

The interesting case is when D_+ is not empty. We then define a pseudo–Riemannian metric h on M by changing the sign of g on D_+. The three vector fields ξ_i are still Killing vector fields for h and the endomorphisms associated to $\frac{1}{2}d\xi_i$ via h satisfy the quaternionic relations on D. The cone $(\bar{M}, \bar{h})$ is then a pseudo–Riemannian almost hyperkähler manifold. From Lemma 4.2 we deduce that $(\bar{M}, \bar{h})$ is hyperkähler, so (M, h) is pseudo 3–Sasakian. From the construction of h it follows immediately that (M, g) has a weakly K–contact 3–structure. ☐

In the Sasakian case we have seen that the component $\mathfrak{g}_2$ of the Lie algebra of Killing fields is either 0– or 2–dimensional, or the manifold has constant sectional curvature. Moreover, in the first case, any Killing vector field in $\mathfrak{g}_2$ has constant length. The K-contact structures do not have, in general, such a rigidity property:

Proposition 4.4. *There exist deformations of the round metric on S^{2n+1}, $n \geq 3$, which are non Sasakian K-contact structures for which the component $\mathfrak{g}_2$ of the Lie algebra of the isometry group contains Killing fields of non-constant length.*

Proof. Let $V = V_1 \oplus V_2 \simeq \mathbb{R}^{2n+2}$, be decomposed in the $(2n-2)$-dimensional subspace generated by the first coordinates, and the 4-dimensional subspace generated by the last ones, and let S^{2n+1} be the standard sphere in V.

Let $\mathfrak{g} \cong \mathfrak{so}(V_1) \oplus \mathbb{R}\xi$ be the Lie subalgebra of $\mathfrak{so}(V)$ consisting of the Killing vector fields of S^{2n+1} depending only on the first $2n-2$ coordinates, plus the Killing vector field ξ, defined by the standard Sasakian structure of the sphere (and corresponding to the standard complex structure J on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$). We will deform the metric on S^{2n+1}, keeping it G-invariant (where G is the associated Lie subgroup to $\mathfrak{g}$). Because $2n-2 \geq 4$, there are elements in $\mathfrak{g}$ which do not commute with ξ. On the other hand, these Killing vector fields do not have constant length, because they all vanish on some totally geodesic 3-sphere.

We proceed as follows: Let X_0, J_0X_0 be non-trivial vector fields on S^3, seen as the standard sphere in $V_2 \cong \mathbb{R}^4$, commuting with and orthogonal to ξ_0, the Killing field defined by the standard Sasakian structure on S^3 (and corresponding to the matrix J_0, the standard complex structure on $\mathbb{R}^4$). Such vectors exist, and are the lifts of some vector fields on S^2, the orbit space of ξ_0 on S^3. We consider now the vector fields X, JX on S^{2n+1}, depending only on the last 4 coordinates in $\mathbb{R}^{2n+2}$, and which project onto X_0, J_0X_0 via the projection $V \to V_2$. The vectors X, JX are then G-invariant. Let U be the open set of S^{2n+1} where X and JX do not vanish, and let $F : S^{2n+1} \to \mathbb{R}$ be a non-trivial smooth G-invariant function whose support lies in U.

We construct a new metric, g_F, on S^{2n+1} defined by the field A of symmetric endomorphisms of TS^{2n+1} relating it to the standard metric. On U, A has eigenvalues e^F, e^{-F} and 1, corresponding respectively to the eigenvectors X, JX and the vectors orthogonal to them, and on $S^{2m+1} \setminus U$, A is the identity. Since A is clearly G-invariant, g_F is a G-invariant metric. The pair (g_F, ξ) is a K-contact structure because ξ (viewed both as a vector and as a 1-form) and $d\xi$ (viewed as a 2-form) remain the same after the above deformation, and the endomorphism ϕ_F of $\xi^\perp$, identified with the form $d\xi$, changes on X and on JX, but remains a complex structure on $\xi^\perp$ (namely, $\phi_F(X) = e^{-2F}JX$ and $\phi_F(JX) = -e^{2F}X$).

Of course, this induces an almost complex structure on $\mathbb{CP}^n$, the space of orbits of ξ, which is never integrable, because it is not analytic (it coincides with the standard complex structure on the non-empty interior of the set where F vanishes, and is different on the open set where $F \neq 0$). So the constructed K-structure is not Sasakian (this is also a consequence of Theorem 3.5). □

5. Irregular Sasakian manifolds

In this section (M, g, ξ) denotes an irregular Sasakian manifold. Consider the exponential orbit O of ξ in $G := Iso(M)$ and denote by T the closure of O in G. Since G is compact, T is a compact subgroup of G. As the closure of an Abelian subgroup, T is itself Abelian, thus isomorphic to a quotient of a torus $\tilde{T}$ by a finite subgroup $\Gamma \subset \tilde{T}$. Let $\Lambda \subset \mathfrak{g}$ be the Lie algebra of T and $\tilde{T}$.

Lemma 5.1. *The subalgebra Λ is central in $\mathfrak{g}$.*

Proof. As every 3–Sasakian manifold (or Sasakian space form) is at least quasi–regular, Theorem 3.5 shows that $\mathfrak{g} = \mathfrak{g}_0$. Thus, for every $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, $Ad_{\exp(t\xi)}X = 0$. By continuity we obtain $Ad_gX = 0$ for every $g \in T$, so $ad_YX = 0$ for every $Y \in \Lambda$. □

Lemma 5.2. *The closure of the generic orbit on M of the Sasakian vector field has dimension larger than 1.*

Proof. This amounts to say that the generic orbit of the compact Lie group T defined above cannot be a circle. Let $\zeta \in \Lambda$ be not collinear to ξ. Suppose that the T–orbit of each point in an open set U of M is a circle. Then ζ is a multiple of ξ on U, say $\zeta = f\xi$.

As ζ is Killing as well, we deduce that the symmetric part of $df \otimes \xi$ vanishes on U, so ζ is a constant multiple of ξ on U, and hence on M, a contradiction. □

A classical and non–trivial result by Boothby and Wang [5] states that any contact structure whose automorphism group acts transitively has to be regular. We give here a simple proof of a slightly stronger version of this result for the case of Sasakian structures.

Proposition 5.3. *An irregular Sasakian manifold is not homogeneous as Riemannian manifold.*

Proof. Let $\zeta \in \Lambda$ be as before and consider a point $x \in M$ where ζ_x is not collinear to ξ_x (the existence of x follows from the previous lemma). Consider an arbitrary Killing vector field X. By Lemma 5.1 X, ξ and ζ are commuting Killing vector fields, hence $\langle \nabla_\zeta \xi, X\rangle = 0$ (trivial application of the Koszul formula). In other words, *every* Killing vector field is orthogonal at x to the vector $\phi(\zeta)_x = -(\nabla_\zeta \xi)_x$, which is non-zero by the assumption on ζ. Thus M cannot be homogeneous. □

In the remaining part of this section we construct an example of irregular Sasakian manifold of cohomogeneity one, showing that the theorem above is optimal.

Examples of cohomogeneity one irregular Sasakian manifolds. We use the following equivalent definition for a Sasakian structure: *it is a K-contact structure for which the underlying CR structure is integrable.* Recall that a CR structure is — in the most general setting — a field of complex structures on a field of hyperplanes, and $J := \nabla\xi$ on $Q := \xi^\perp$ gives us such a structure on a K-contact manifold (M, g, ξ). Moreover, the integrability condition for the CR structure (M, Q, J) is given by

$$N(X, Y) = 0,\ \forall X, Y \in Q,\ \text{where } N : \Lambda^2 Q \to Q,$$

$$4N(X, Y) := [JX, JY] - J[JX, Y]^Q - J[X, JY]^Q - [X, Y],$$

where X, Y are extended to vector fields contained in Q, and A^Q denotes the component in Q of the vector field A. It is then clear that $N \equiv 0$ is equivalent to the integrability of the space of local orbits of ξ, which is almost Kähler for a K-contact structure, and Kähler for a Sasakian one.

Let $\mathbb{C}^{n+1} \cong V = V_1 \oplus V_2$ be a splitting in two complex vector spaces of dimensions 1, resp. n, and let J_0 be the element of $\mathfrak{u}(n+1) \subset \mathfrak{so}(2n+2)$ corresponding to the multiplication by i on V, and let J_1 be the element of $\mathfrak{u}(n+1)$ corresponding to the multiplication by i on V_1, and acting trivially on V_2. They correspond to Killing vector fields on the round sphere $(S^{2n+1}, g_0) \subset \mathbb{C}^{n+1}$ denoted by T_0, resp. T_1.

The first one is the Killing vector field associated to the standard Sasakian structure on S^{2n+1} and it is easy to see that $T := T_0 + aT_1$, where $a \in \mathbb{R} \setminus \mathbb{Q} \cap (0, 1)$ is a nowhere vanishing Killing vector field, whose generic orbits are dense in a torus generated by the action of T_0 and T_1. Moreover, T is obviously transverse to $Q := T_0^\perp$ and commutes with T_0. We define a new metric g on the sphere such that T has unit length, is orthogonal to Q, and $g|_Q := \alpha g_0|_Q$, where $\alpha := g_0(T, T_0)^{-1}$ and g_0 is the standard metric on S^{2n+1}. This new metric is T_0- and T-invariant, so ∇T is a skew-symmetric endomorphism of Q.

If we extend two arbitrary orthogonal vectors X, Y in Q by local vector fields contained in Q, orthogonal to each other and commuting with T_0, we get by Koszul's formula

$$2g(\nabla_X T, Y) = g([X, T], Y) - g([T, Y], X) - g(T, [X, Y]) = -g(T, [X, Y]).$$

Here the other terms vanish as T is Killing and X is orthogonal to Y. Similarly we get $2g_0(\nabla^0_X T_0, Y) = -g_0(T_0, [X, Y])$, and we can decompose $[X, Y] = g_0(T_0, [X, Y])T_0 + W$, where $W \in Q$. Then $g(T, [X, Y]) = g_0(T_0, [X, Y])g(T, T_0)$. It follows then that $\nabla_X T = \nabla^0_X T_0 = J_0(X)$, i.e. it is the standard CR structure on the sphere, which is integrable. Hence g is Sasakian.

On the other hand, the metric g is invariant under the action of $U(1) \times U(n)$, where the two factors act on V_1, resp. on V_2, and the generic orbits of this group on S^{2n+1} have codimension 1, thus g has cohomogeneity one, as claimed.

6. Homogeneous Sasakian Manifolds

The aim of this section is to prove the following classification result.

Theorem 6.1. *A simply connected compact Sasakian manifold is homogeneous (as Riemannian manifold) if and only if it is the canonical S^1–bundle of a homogeneous simply connected compact Hodge manifold with the metric described in Lemma 2.2.*

Proof. If N is a homogeneous simply connected compact Hodge manifold, we can assume (after possibly rescaling the metric by an integer constant) that the cohomology class $[F]$ of the Kähler form is not an integer multiple of any class in $H^2(N, \mathbb{Z})$. The Gysin sequence then shows that the canonical S^1–bundle M of N is simply connected. It is well–known that M has a Sasakian structure, and Lemma 2.3 shows that every Killing vector field X on N induces a Killing vector field X^* on M, projectable onto X. As the vertical vector field is itself a (non–vanishing) Killing vector field, it follows that for every point $x \in M$ and for every vector $A \in T_xM$ there exists a Killing vector field ζ on M such that $\zeta_x = A$. Thus the orbit through every point of the (compact) isometry group of M is open, hence M is homogeneous.

Conversely, let (M, g, ξ) be a homogeneous simply connected compact Sasakian manifold. We can assume that M is not 3–Sasakian, since in that case M has to be the canonical S^1–bundle of a generalized flag manifold (see [6]). From Proposition 5.3, M is either regular or quasi–regular (i.e. the exponential orbit of ξ in the isometry group is a circle). Denote by φ_t the isometry induced by $\exp(t\xi)$ on M and let p be the least positive number such that $\varphi_p = Id_M$. If M is quasi–regular, there exists a positive number $q < p$ and a point $x \in M$ such that $\varphi_q(x) = x$. For every vector $A \in T_xM$ there exists a Killing vector field ζ on M such that $\zeta_x = A$. By our assumption, every Killing vector field on M commutes with ξ, so $(\varphi_q)_*\zeta = \zeta$. This shows that φ_q has to be the identity on M, a contradiction.

We have thus proved that the Sasakian structure is regular. The quotient N of M by the Sasakian flow is Kähler (see [3]) and simply connected (by the exact homotopy sequence). From Lemma 2.3 and Theorem 3.5 we see that every Killing vector field on M is of the form $X^* + f\xi$, where X is a Killing vector field on N. For every point $x \in N$ and vector $Y \in T_xN$, take some y in the fiber over x; as M is homogeneous, there exists a Killing vector field ζ on M such that $\zeta_y = Y^*$. If we write $\zeta = X^* + f\xi$ with X Killing vector field on N, this last equation shows that $X_x = Y$. So N is homogeneous. $\square$

Remark. As Charles Boyer pointed out to us, the above result was also obtained in [7] under the slightly stronger assumption that the automorphism group of the Sasakian structure acts transitively.

Remark. The classification of simply connected compact homogeneous Kähler manifolds can be found in [4]. Every such manifold has to be an orbit of the co–adjoint representation of a compact connected Lie group, endowed with its canonical complex structure. Note that every such co–adjoint orbit carries several homogeneous Kähler metrics, but there is always a canonically defined invariant Kähler–Einstein metric, thus at least one Hodge structure.

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