Geometric And Functional Analysis 1016-443X/95/0x001-xx$1.50+0.20/0
Vol. 5, No. x (1995)

COMPLEX CONTACT STRUCTURES AND THE FIRST EIGENVALUE OF THE DIRAC OPERATOR ON KÄHLER MANIFOLDS

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Abstract

In this paper Kählerian Killing spinors on manifolds of complex dimension $m = 4l + 3$ are constructed. The construction is based on a theorem which states that a closed Kähler Einstein manifold of complex dimension $4l+3$ and positive scalar curvature admits a Kählerian Killing spinor if and only if there is a complex $(2l + 1)$-contact structure. In particular, any complex contact structure in the usual sense gives rise to such a generalized contact structure. Using this, new examples of Kählerian Killing spinors are obtained.

0. Introduction

It is known from [Ki1] that each eigenvalue λ of the Dirac operator D on a closed spin Kähler manifold M^{2m} of odd complex dimension m and positive scalar curvature R satisfies the inequality

$$\lambda^2 \geq \frac{m+1}{4m} R_0 \ , \tag{1}$$

where $R_0 > 0$ is the minimum of R on M. The limiting case of this inequality is characterized by the fact that M is Einstein and that all eigenspinors of D corresponding to the first eigenvalue $\lambda_1 = \sqrt{(m+1)R_0/4m}$ are Kählerian Killing spinors.

Conversely, if a closed spin Kähler manifold M admits a Kählerian Killing spinor, then M is an Einstein manifold of odd complex dimension m and $\sqrt{(m+1)R/4m}$ is the first eigenvalue of the Dirac operator D (see [Ki1, Th. 5]). Hence, for closed spin Kähler manifolds M with odd complex dimension m and positive scalar curvature, the classification problem corresponding to the limiting case of the inequality (1) is equivalent to the classification problem of Kähler manifolds admitting Kählerian Killing spinors.

Up to now this classification problem has been solved only for the complex dimensions $m = 1$ and $m = 3$. For $m = 1$, only $\mathbb{C}P^1 \simeq S^2$ is possible and, for $m = 3$, the complete list is given by the complex projective space $\mathbb{C}P^3$ and the flag manifold $F(1, 2)$ with respect to their natural Kähler structures (cf. [Ki2]). In higher dimensions the complex projective spaces $\mathbb{C}P^m$ with m odd were the only known examples admitting Kählerian Killing spinors (cf. [CFG], [Ki3], [SeSem]).

In this paper we describe a new general construction principle for Kählerian Killing spinors on manifolds of complex dimension $m = 4l + 3$.

This construction principle is based on Theorem 6, which states that a closed Kähler Einstein manifold M^{2m} with $m = 4l + 3$ and positive scalar curvature admits a Kählerian Killing spinor if and only if there is a complex r-contact structure on M, with $r = 2l+1$. The notion of a complex r-contact structure appearing here is a natural generalisation of the usual concept of a complex contact structure.

A complex r-contact structure is defined to be a family of local holomorphic forms of degree r which satisfies certain conditions. For $r = 1$ we regain the definition of a complex contact structure.

In particular, using results of S. Salamon ([S]) concerning the existence of complex contact structures on the twistor space of quaternionic Kähler manifolds, we obtain new examples of manifolds admitting Kählerian Killing spinors.

In some sense the result presented in this article is the Kählerian counterpart of the corresponding result in the Riemannian spin geometry, which states a relation between Killing spinors and Einstein-Sasaki structures for odd dimensions (cf. [FrK]).

A relation between Kählerian Killing spinors and complex contact structures was conjectured by Th. Friedrich some time ago. Our paper shows that, for complex dimension $m = 4l + 3$, a modified form of this conjecture is true. The construction was motivated by the results in dimension 3 (there are Killing spinors on $\mathbb{C}P^3$ and $F(1,2)$ defining complex contact structures, cf. [Ki3, Prop. 3]).
In this paper we use the notations introduced in [Ki3].

2. Kählerian Killing Spinors

In this section we want to associate Kählerian Killing spinors with a certain geometric structure, which will be closer investigated in the next section.
We consider a closed spin Kähler manifold (M, g, J) of complex dimension m. The spin structure corresponds to a fixed root $K^{1/2}$ of the canonical bundle $K = \Lambda^{m,0}$ of M. The spinor bundle S possesses a canonical decomposition $S = S_0 \oplus S_1 \oplus \cdots \oplus S_m$ into holomorphic subbundles S_r with $\mathrm{rank}_{\mathbb{C}} \mathrm{S}_\mathrm{r} = \binom{\mathrm{m}}{\mathrm{r}}$. We assume further the existence of a Kählerian Killing spinor $\Psi = \Psi_k + \Psi_{k+1} \in \Gamma(S_k \oplus S_{k+1})$ with Killing number μ, i.e. by definition, the spinors Ψ_k and Ψ_{k+1} satisfy the differential equations

$$\nabla_X \Psi_k = -\mu \bar{p}(X) \cdot \Psi_{k+1} \qquad \text{and} \qquad \nabla_X \Psi_{k+1} = -\mu p(X) \cdot \Psi_k \ , \tag{2}$$

where $p(X) = 1/2(X - iJX)$ denotes the projection of X onto the holomorphic tangent bundle $T^{1,0}M$. These equations imply that (M, g, J) is a Kähler-Einstein manifold of odd complex dimension $m = 2k + 1$.

Using the isomorphism $S_r \cong K^{1/2} \otimes \Lambda^{0,r}M$ we can associate with Ψ local complex forms which are defined as follows.
Let $\Psi_0 \in \Gamma(S_0|_U)$ be any local holomorphic section that is nowhere zero on $U \subset M$. Then there exist uniquely determined forms $\xi_k \in \Gamma(\Lambda^{k,0}|_U)$ and $\xi_{k+1} \in \Gamma(\Lambda^{k+1,0}|_U)$ with $\Psi_k = \bar{\xi}_k \cdot \Psi_0$ and $\Psi_{k+1} = \bar{\xi}_{k+1} \cdot \Psi_0$.
We will call the set $\{\Psi_0, \xi_k, \xi_{k+1}\}$ a local holomorphic representation of the spinor $\Psi = \Psi_k + \Psi_{k+1}$.
The next step is to reformulate the defining equations (2) of a Kählerian Killing spinor in terms of its local holomorphic representation.

Proposition 1. *The spinor $\Psi = \Psi_k + \Psi_{k+1}$ is a Kählerian Killing spinor iff each local holomorphic representation $(\Psi_0, \xi_k, \xi_{k+1})$ satisfies the differential equations*

(i) $\nabla_Z \xi_k = 2\mu Z \lrcorner\, \xi_{k+1}$,
(ii) $\nabla_{\bar{Z}} \xi_k = -\bar{\partial} \log |\Psi_0|^2(\bar{Z}) \cdot \xi_k$,
(iii) $\nabla_Z \xi_{k+1} = 0$,
(iv) $\nabla_{\bar{Z}} \xi_{k+1} = -\bar{\partial} \log |\Psi_0|^2(\bar{Z}) \cdot \xi_{k+1} - \mu g(\bar{Z}) \wedge \xi_k$,

where $Z \in \Gamma(T^{1,0}M|_U)$ is any section of the holomorphic tangent bundle and $g(\bar{Z})$ denotes the $(1,0)$-form defined by $g(\bar{Z})(X) = g(\bar{Z}, X)$.

Proof: Let $\Psi = \Psi_k + \Psi_{k+1}$ be a Kählerian Killing spinor with local holomorphic representation $(\Psi_0, \xi_k, \xi_{k+1})$. Since Ψ_0 is a holomorphic section in $S_0|_U$ and we have fixed the canonical connection in S_0, we have

$$\begin{aligned}\nabla_X \Psi_k &= \nabla_X(\bar{\xi}_k \cdot \Psi_0)\\ &= \nabla_X \bar{\xi}_k \cdot \Psi_0 + \bar{\xi}_k \cdot \partial \log |\Psi_0|^2(X)\Psi_0\\ &= \big(\nabla_X \bar{\xi}_k + \partial \log |\Psi_0|^2(X)\bar{\xi}_k\big) \cdot \Psi_0\ .\end{aligned}$$

The spinor $\Psi = \Psi_k + \Psi_{k+1}$ was assumed to be a Kählerian Killing spinor, hence

$$\begin{aligned}\nabla_X \Psi_k &= -\mu \bar{p}(X) \cdot \Psi_{k+1}\\ &= -\mu \bar{p}(X) \cdot \bar{\xi}_{k+1} \cdot \Psi_0\\ &= -2\mu\big(\bar{p}(X) \lrcorner\, \bar{\xi}_{k+1}\big) \cdot \Psi_0\ .\end{aligned}$$

Here we have used the formula

$$\bar{Z} \cdot \bar{\xi}_{k+1} = (-1)^{k+1} \bar{\xi}_{k+1} \cdot \bar{Z} - 2\bar{Z} \lrcorner\, \bar{\xi}_{k+1}$$

and the equation $\bar{Z} \cdot \Psi_0 = 0$, where $\bar{Z}$ is a section in the anti-holomorphic tangent bundle $T^{0,1}M$. From these two equations for $\nabla_X \Psi_k$ we obtain

$$\nabla_X \bar{\xi}_k = -\partial \log |\Psi_0|^2(X)\bar{\xi}_k - 2\mu \bar{p}(X) \lrcorner\, \bar{\xi}_{k+1}$$

and hence

$$\begin{aligned}\nabla_Z \bar\xi_k &= -\partial \log |\Psi_0|^2(Z)\bar\xi_k \\ \nabla_{\bar Z}\bar\xi_k &= -2\mu \bar Z \lrcorner \bar\xi_{k+1} \ .\end{aligned}$$

The equations (i) and (ii) follow after conjugating both sides of the latter two equations.

To prove (iii) and (iv) we compute analogously

$$\begin{aligned}\nabla_X \Psi_{k+1} &= \nabla_X(\bar\xi_{k+1}\cdot\Psi_0) \\ &= \nabla_X\bar\xi_{k+1}\cdot\Psi_0 + \bar\xi_{k+1}\cdot\partial\log|\Psi_0|^2(X)\Psi_0 \\ &= \big(\nabla_X\bar\xi_{k+1} + \partial\log|\Psi_0|^2(X)\bar\xi_{k+1}\big)\cdot\Psi_0 \ .\end{aligned}$$

and

$$\begin{aligned}\nabla_X\Psi_{k+1} &= -\mu p(X)\cdot\Psi_k \\ &= -\mu p(X)\cdot\bar\xi_k\cdot\Psi_0 \\ &= -\mu\big(g(p(X))\wedge\bar\xi_k\big)\cdot\Psi_0 \ .\end{aligned}$$

Here we have used the formulas

$$Z\cdot\bar\xi_k = g(Z)\wedge\bar\xi_k - Z\lrcorner\bar\xi_k$$

and $Z\lrcorner\bar\xi_k = 0$, where Z is a section in the holomorphic tangent bundle $T^{1,0}M$ and $g(Z)(X) = g(Z,X)$. ¿From these two equations for $\nabla_X\Psi_{k+1}$ we obtain

$$\nabla_X\bar\xi_{k+1} = -\partial\log|\Psi_0|^2(X)\bar\xi_{k+1} - \mu g\big(p(X)\big)\wedge\bar\xi_k$$

and also

$$\begin{aligned}\nabla_Z\bar\xi_{k+1} &= -\partial\log|\Psi_0|^2(Z)\bar\xi_{k+1} - \mu g(Z)\wedge\bar\xi_k \\ \nabla_{\bar Z}\bar\xi_k &= 0 \ .\end{aligned}$$

Conjugating these two equations we obtain (iii) and (iv). Conversely, it is easy to see that a spinor $\Psi\in\Gamma(S_k\oplus S_{k+1})$ with the local holomorphic representation (Ψ_0,ξ_k,ξ_{k+1}) satisfying the equations (i) – (iv) is a Kählerian Killing spinor. □

A simple calculation gives the following equivalent formulation of equations (ii) and (iv):

(ii)' $\qquad \nabla_{\bar Z}(|\Psi_0|^2\xi_k) = 0$

(iv)' $\qquad \nabla_{\bar Z}(|\Psi_0|^2\xi_{k+1}) = -\mu g(\bar Z)\wedge(|\Psi_0|^2\xi_k)$.

Since it is possible to express the differential $d = \partial + \bar\partial$ using the covariant derivative ∇, we immediately obtain

Corollary 1.1. *Let $\Psi = \Psi_k + \Psi_{k+1}$ be a Kählerian Killing spinor with Killing number μ. Then each local holomorphic representation (Ψ_0,ξ_k,ξ_{k+1}) of Ψ satisfies the differential equations*

$$\begin{aligned}\bar\partial\big(|\Psi_0|^2\xi_k\big) &= 0\ ,\\ \partial\xi_k &= \mu(m+1)\xi_{k+1}\ ,\\ \partial\xi_{k+1} &= 0\ ,\\ \bar\partial\big(|\Psi_0|^2\xi_{k+1}\big) &= i\mu\Omega\wedge\big(|\Psi_0|^2\xi_k\big)\ ,\end{aligned}$$

where Ω denotes the Kähler form of M.

Let (M,g,J) be a Kähler spin manifold of complex dimension $m=2k+1$ which admits a Kählerian Killing spinor $\Psi=\Psi_k+\Psi_{k+1}$. Now, we assume k to be an odd number. Under this assumption the canonical anti-unitary bundle map $j:S\to S$ (cf. [ABoSh], [Ki3]) is a real structure, i.e. $j^2=id$. Hence, we can consider the real part of Ψ or, in case it vanishes, i times the imaginary part of Ψ. This spinor is again a Kählerian Killing spinor and is fixed under the map j. Thus, without loss of generality, we can assume the existence of a real Kählerian Killing spinor, i.e. a Kdhlerian Killing spinor with the additional property $j\Psi=\Psi$. With such a spinor Ψ we associate a family $\mathcal{C}_\Psi=\{(U_i,\omega_i)\}$ of open sets U_i and differential forms $\omega_i\in\Gamma(\Lambda^{k,0}M|_{U_i})$.
To introduce this family $\mathcal{C}_\Psi$ we fix a holomorphic trivialisation of the line bundle S_0, i.e. an open covering $\{U_i\}$ of M and nonvanishing local holomorphic sections $\Psi_{0,i}\in\Gamma(S_0|_{U_i})$. We define $\omega_i:=|\Psi_{0,i}|^2\xi_k$, where ξ_k is the k-form on U_i with $\Psi|_{U_i}=\bar\xi_k\cdot\Psi_{0,i}$.

Proposition 2. *Let (M,g,J) be a Kähler spin manifold of complex dimension $m=2k+1$ (k odd) which admits a Kählerian Killing spinor. Then, the family $\mathcal{C}_\Psi$ associated with a real Kählerian Killing spinor Ψ satisfies the following conditions:*
(i) *$\{U_i\}$ is an open covering of M.*
(ii) *ω_i is a holomorphic k-form on U_i, i.e. $\bar\partial\omega_i=0$.*
(iii) *$\omega_i\wedge\partial\omega_i\in\Gamma(\Lambda^{m,0}M|_{U_i})$ is different from zero at each point of U_i.*
(iv) *$\omega_i=f_{ij}\omega_j$ in $U_i\cap U_j$, where f_{ij} is a holomorphic function on $U_i\cap U_j$.*

Proof: The definition of $\mathcal{C}_\Psi$ and Corollary 1.1 immediately imply (i) and (ii). To prove (iv) we consider two open sets U and U' of $\mathcal{C}_\Psi$ with $V:=U\cap U'\neq\emptyset$. The corresponding sections Ψ_0 and Ψ_0' are holomorphic sections of the line bundle S_0 restricted to the open set V. Hence, there exists a holomorphic function f on V with $\Psi_0'=\mathrm{f}\Psi_0$. Since $\Psi_k=\bar\xi_k'\cdot\Psi_0'=\bar\xi_k\cdot\Psi_0$, we have $\xi_k'=\bar{\mathrm{f}}^{-1}\xi_k$ and hence

$$\omega'=|\Psi_0'|^2\xi_k'=|\mathrm{f}|^2|\Psi_0|^2\bar{\mathrm{f}}^{-1}\,\xi_k=\mathrm{f}\big(|\Psi_0|^2\xi_k\big)=\mathrm{f}\omega\ .$$

Thus we have (iv). It remains to show that $\omega\wedge\partial\omega$ is nowhere zero on U. Using again Corollary 1.1 we obtain

$$\begin{aligned}\omega\wedge\partial\omega &= |\Psi_0|^2\xi_k\wedge\partial\big(|\Psi_0|^2\xi_k\big)\\ &= |\Psi_0|^2\xi_k\wedge\big(\partial|\Psi_0|^2\wedge\xi_k+|\Psi_0|^2\partial\xi_k\big)\\ &= |\Psi_0|^4\xi_k\wedge\partial\xi_k\\ &= |\Psi_0|^4\mu(m+1)\xi_k\wedge\xi_{k+1}\ .\end{aligned}$$

To obtain this expression for $\omega\wedge\partial\omega$ we need the assumption $k\equiv 1(2)$, since $\eta\wedge\eta=0$ if η is a form of odd degree.

Now it is sufficient to show that $\xi_k\wedge\xi_{k+1}$ is nowhere zero on U. To do this we introduce a local form $\xi_m\in\Gamma(\Lambda^{m,0}M|_U)$ defined by

$$j\Psi_0=\bar\xi_m\cdot\Psi_0\ .$$

This definition is correct since the isomorphism $S_r\cong K^{1/2}\otimes\Lambda^{0,r}M$ is given by Clifford multiplication and $j(S_r)=S_{m-r}$. The Kählerian Killing spinor Ψ was assumed to satisfy $j\Psi=\Psi$, i.e.

$$j\Psi_k=\Psi_{k+1}\quad\text{and}\quad j\Psi_{k+1}=\Psi_k\ . \tag{3}$$

In particular, Ψ_{k+1} and Ψ_k have the same constant length. Hence, we can assume that

$$|\Psi_k|^2=|\Psi_{k+1}|^2=1\ .$$

Next we see that ξ_m is nowhere zero on U since, by

$$|\Psi_0|^2=|j\Psi_0|^2=|\bar\xi_m\Psi_0|^2=2^m|\xi_m|^2|\Psi_0|^2\ ,$$

we find $|\xi_m|^2=2^{-m}$. Using (3) we obtain

$$\begin{aligned}\bar\xi_{k+1}\cdot\Psi_0=\Psi_{k+1}=j\Psi_k &= j(\bar\xi_k\cdot\Psi_0)=\xi_k\cdot j\Psi_0\\ &= \xi_k\cdot\bar\xi_m\cdot\Psi_0\\ &= (-1)^{k(k+1)/2}2^k(\xi_k\lrcorner\bar\xi_m\cdot\Psi_0\ .\end{aligned}$$

Thus we have the equation

$$\bar\xi_k\lrcorner\xi_m=2^{-k}(-1)^{k(k+1)/2}\xi_{k+1}\ .$$

Since $\xi_k\wedge\xi_{k+1}\in\Gamma(\Lambda^{m,0}M|_U)$ and $\xi_m\neq 0$, there exists a function λ on U with

$$\xi_k\wedge\xi_{k+1}=\lambda\xi_m\ .$$

Hence, to prove that $\xi_k\wedge\xi_{k+1}$ is nowhere zero on U it remains to show that $\lambda\neq 0$.

It holds that

$$\begin{aligned}\lambda|\xi_m|^2 &= \langle\xi_k\wedge\xi_{k+1},\xi_m\rangle\\ &= \big\langle\xi_{k+1},\bar\xi_k\lrcorner\xi_m\big\rangle\\ &= 2^{-k}(-1)^{k(k+1)/2}|\xi_{k+1}|^2\\ &= 2^{-m}(-1)^{k(k+1)/2}|\Psi_0|^{-2}\ .\end{aligned}$$

The latter equation is true because of $1 = |\Psi_{k+1}|^2 = |\bar{\xi}_{k+1} \cdot \Psi_0|^2 = 2^{k+1}|\xi_{k+1}|^2|\Psi_0|^2$. Since $\xi_m \neq 0$, we have $\lambda \neq 0$ and, hence, $\xi_k \wedge \xi_{k+1}$ is nowhere zero on U. □

2. Generalized Complex Contact Structures

Let M be a complex manifold of complex dimension $m = 2k+1$. For a fixed odd number r we define a complex r-contact structure on M to be a family $\mathcal{C} = \{(U_i, \omega_i)\}$ satisfying the following conditions:

(i) $\{U_i\}$ is an open covering of M.
(ii) ω_i is a holomorphic r-form on U_i.
(iii) $\omega_i \wedge (\partial\omega_i)^s \in \Gamma(\Lambda^{m,0})$ is different from zero at every point of U_i, where $s = \frac{m-r}{r+1}$.
(iv) $\omega_i = f_{ij}\omega_j$ in $U_i \cap U_j$, where f_{ij} is a holomorphic function on $U_i \cap U_j$.
(v) The family $\mathcal{C}$ is maximal with these properties.

For $r = 1$ the above definition is just the definition of a complex contact structure as given in [Ko].

Such a structure only exists if $s = (m-r)/(r+1)$ is an integer. In particular, it is necessary for M to have odd complex dimension.

Let $\mathcal{C} = \{(U_i, \omega_i)\}$ be a complex r-contact structure. Then there exists an associated holomorphic line subbundle $L_{\mathcal{C}} \subset \Lambda^{r,0}(M)$ with transition functions $\{f_{ij}^{-1}\}$.

Proposition 3. *The holomorphic line bundle $L_{\mathcal{C}}$ is an $(s+1)$st root of the canonical bundle $K = \Lambda^{m,0}M$ of M, i.e. we have*

$$L_{\mathcal{C}}^{s+1} \cong K \ ,$$

where $s = (m-r)/(r+1)$. In particular, M admits a canonical spin structure if s is an odd number or, equivalently, if $k+1$ is divisible by $r+1$.

Proof: To prove the proposition we construct a global section of the line bundle $K \otimes (L_{\mathcal{C}}^*)^{s+1}$ that is nowhere zero and hence defines a trivialisation. Let $(U, \omega) \in \mathcal{C}$, then $\omega \in \Gamma(L_{\mathcal{C}}|_U)$ is nowhere zero on U and we can define the dual section $\zeta \in \Gamma(L_{\mathcal{C}}^*|_U)$ by $\zeta(\omega) = 1$. We consider the section

$$\mathrm{s}_\omega := \big(\omega \wedge (\partial\omega)^s\big) \otimes \zeta^{\otimes(s+1)} \in \Gamma\big(K \otimes (L_{\mathcal{C}}^*)^{s+1}|_U\big) \ ,$$

which is again nowhere zero on U.

Next we show that this definition is independent of the choice of (U, ω). To show this let $(U', \omega') \in \mathcal{C}$ with corresponding dual section ζ'. Then there exists a holomorphic function f on $U \cap U'$ with

$$\omega' = \mathrm{f}\omega \quad \text{and} \quad \zeta' = \mathrm{f}^{-1}\zeta \ .$$

Using the fact that ω is a form of odd degree and hence $\omega \wedge \omega = 0$ we have

$$\begin{aligned}
\mathrm{s}_{\omega'} &= \big(\omega' \wedge (\partial\omega')^s\big) \otimes (\zeta')^{\otimes(s+1)} \\
&= \mathrm{f}\big(\omega \wedge (\partial \mathrm{f} \wedge \omega + \mathrm{f}\partial\omega)^s\big) \otimes \mathrm{f}^{-(s+1)}\zeta^{\otimes(s+1)} \\
&= \big(\omega \wedge (\partial\omega)^s\big) \otimes \zeta^{\otimes(s+1)} \\
&= \mathrm{s}_{\omega} \ .
\end{aligned}$$

Thus we see that s_ω is globally defined and yields the desired trivialisation. □

In the following we will assume s to be an odd number to assure the existence of a canonical spin structure on M. The next proposition allows us to concentrate on the case of a complex k-contact structure, where $m = 2k+1$ is the complex dimension of M.

PROPOSITION 4. *Let M be a complex manifold of complex dimension $m = 2k+1$. If M admits a complex r-contact structure such that $s = (m-r)/(r+1)$ is odd, then M also admits a complex k-contact structure.*

Proof: Starting from a complex r-contact structure $\{(U_i, \omega_i)\}$ on M we define a complex k-contact structure as $\{(U_i, \omega_i')\}$ by $\omega_i' := \omega_i \wedge (\partial\omega_i)^{\frac{s-1}{2}}$. Since $\frac{s-1}{2} = \frac{k-r}{r+1}$, the form ω_i' is a form of degree $r + (r+1)\frac{k-r}{r+1} = k$ and is, of course, holomorphic.

Condition (iii) is satisfied because

$$\omega_i' \wedge \partial\omega_i' = \big(\omega_i \wedge (\partial\omega_i)^{\frac{s-1}{2}}\big) \wedge (\partial\omega_i)^{\frac{s+1}{2}} = \omega_i \wedge (\partial\omega_i)^s \neq 0 \ .$$

To check property (iv) we remark that

$$\begin{aligned}
\omega_i' &= \omega_i \wedge (\partial\omega_i)^{\frac{s-1}{2}} \\
&= f_{ij}\omega_j \wedge \big(\partial(f_{ij}\omega_j)\big)^{\frac{s-1}{2}} \\
&= f_{ij}\omega_j \wedge (\partial f_{ij} \wedge \omega_j + f_{ij}\partial\omega_j)^{\frac{s-1}{2}} \\
&= f_{ij}^{\frac{s+1}{2}}\omega_j' \ .
\end{aligned}$$

Here we have used again the fact that ω_i is a form of odd degree ($\omega_i \wedge \omega_i = 0$) and that f_{ij} was assumed to be a holomorphic function.

The assumption $s \equiv 1(2)$ or, equivalently, $k+1 \equiv 0(r+1)$ implies that k is odd as it is necessary for defining a complex k-contact structure. □

Now, we consider a Kähler manifold M of complex dimension $m = 2k+1$ with k odd admitting a complex k-contact structure $\mathcal{C} = \{(U_i, \omega_i)\}$. Associated with $\mathcal{C}$ we have the holomorphic line bundle $L_\mathcal{C} \subset \Lambda^{k,0}(M)$. The bundle $L_\mathcal{C}$ defines a canonical spin structure

$$L_\mathcal{C} \cong K^{1/2} \cong S_0 \ . \tag{4}$$

Using the k-contact structure $\mathcal{C}$ we define a section $\Psi_\mathcal{C}$ of the spinor bundle. For doing so we fix $(U,\omega)\in\mathcal{C}$ and define $\Psi_{\mathcal{C}}$ over the open set U by

$$\Psi_{\mathcal{C}}|_U := |\Psi_\omega|^{-2}\bar{\omega}\cdot\Psi_\omega\ , \tag{5}$$

where $\Psi_\omega\in\Gamma(S_0|_U)$ is the local section in S_0 corresponding to $\omega\in\Gamma(L_{\mathcal{C}}|_U)$ under the identification (4).
Since the definition is independent of the choice of ω, we have in fact a globally defined section $\Psi_{\mathcal{C}}\in\Gamma(S_k)$.

Our next aim is to compute $D^2\Psi_{\mathcal{C}}$, where D is the Dirac operator of (M,g). The Dirac operator of a complex manifold decomposes as $D = D_+ + D_-$, with $D^2 = D_-D_+ + D_+D_-$. Locally $D_\pm$ can be expressed by

$$D_- = \sum_{i=1}^m \mathrm{f}^i\cdot\nabla_{\mathrm{f}_i} \quad\text{and}\quad D_+ = \sum_{i=1}^m \bar{\mathrm{f}}^i\cdot\nabla_{\bar{\mathrm{f}}_i}\ ,$$

where $\{\mathrm{f}_i\}_{i=1\ldots m}$ is a local frame in $T^{1,0}M$ with corresponding dual frame $\{\mathrm{f}^i\}_{i=1\ldots m}$ in $\Lambda^{1,0}M$ and conjugated frame $\{\bar{\mathrm{f}}_i\}_{i=1\ldots m}$ in $T^{0,1}M$. On Kähler manifolds the operators D_+ and D_- are related to $\bar\partial$ and $\bar\partial^*$ (cf. [Ki3, Lemma 1.1]). In particular, for any $\omega\in\Gamma(\Lambda^{0,r}|_U)$ and any holomorphic section $\Psi_0\in\Gamma(S_0|_U)$, we have the formula

$$D_+(\omega\cdot\psi_0) = (\bar\partial\omega)\cdot\psi_0\ . \tag{6}$$

Proposition 5. *The spinor $\Psi_{\mathcal{C}}$ has the following properties:*
(i) *$\Psi_{\mathcal{C}}$ is nowhere zero on M.*
(ii) *$\Psi_{\mathcal{C}}$ is an anti-holomorphic section, i.e. $\nabla_Z\Psi_{\mathcal{C}}=0$ for all $Z\in\Gamma(T^{1,0}M)$.*
(iii) *It holds that $D_-\Psi_{\mathcal{C}}=0$.*
(iv) *$\Psi_{\mathcal{C}}$ satisfies the equation*

$$D_-D_+\Psi_{\mathcal{C}} = \frac{R}{4}\Psi_{\mathcal{C}} - \frac{i}{2}\rho\cdot\Psi_{\mathcal{C}}\ ,$$

where R is the scalar curvature and ρ the Ricci form of M.

Proof: Let $(U,\omega)\in\mathcal{C}$, then ω is nowhere zero on U and from the definition (5) the same is true for $\Psi_{\mathcal{C}}$. Since the open sets of $\mathcal{C}$ cover M, we conclude (i). The k-form ω and its corresponding section $\Psi_\omega\in\Gamma(S_0|_U)$ are holomorphic. Hence, for all $Z\in\Gamma(T^{1,0}M)$, we have

$$\begin{aligned}\nabla_{\bar Z}\omega &= 0\ ,\\ \nabla_{\bar Z}\Psi_\omega &= 0\ ,\\ \nabla_Z\Psi_\omega &= \big(\partial\log|\Psi_\omega|^2\big)(Z)\Psi_\omega\ .\end{aligned}$$

Applying these formulas we compute

$$
\begin{aligned}
\nabla_Z\Psi_{\mathcal{C}} &= \nabla_Z\big(|\Psi_\omega|^{-2}\bar{\omega}\Psi_\omega\big)\\
&= Z\big(|\Psi_\omega|^{-2}\big)\bar{\omega}\Psi_\omega + |\Psi_\omega|^{-2}(\nabla_Z\bar{\omega})\Psi_\omega + |\Psi_\omega|^{-2}\bar{\omega}\nabla_Z\Psi_\omega\\
&= -|\Psi_\omega|^{-2}\big(\partial\log|\Psi_\omega|^2\big)(Z)\bar{\omega}\Psi_\omega + |\Psi_\omega|^{-2}\bar{\omega}\big(\partial\log|\Psi_\omega|^2\big)(Z)\Psi_\omega\\
&= 0\ .
\end{aligned}
$$

Thus, $\Psi_{\mathcal{C}}$ is antiholomorphic. The definition of D_- Thus, $\Psi_{\mathcal{C}}$ is antiholomorphic. The definition of D_- implies $D_-\Psi_{\mathcal{C}} = 0$ and hence we have (ii) and (iii). Using formula (6) we find

$$
\begin{aligned}
D_+\Psi_{\mathcal{C}} &= D_+\big(|\Psi_\omega|^{-2}\bar{\omega}\Psi_\omega\big)\\
&= \bar{\partial}\big(|\Psi_\omega|^{-2}\big)\bar{\omega}\Psi_\omega + |\Psi_\omega|^{-2}D_+(\bar{\omega}\Psi_\omega)\\
&= -\bar{\partial}\log|\Psi_\omega|^2\Psi_{\mathcal{C}} + |\Psi_\omega|^{-2}(\bar{\partial}\bar{\omega})\Psi_\omega\ ,
\end{aligned}
$$

and we obtain the equation

$$D_+\Psi_{\mathcal{C}} = -\bar{\partial}\big(\log|\Psi_\omega|^2\big)\Psi_{\mathcal{C}} + |\Psi_\omega|^{-2}(\bar{\partial}\bar{\omega})\Psi_\omega\ .$$

Since $\partial\omega$ is holomorphic, we see (as in the proof of (1)) that

$$\nabla_Z\big(|\Psi_\omega|^{-2}\bar{\partial}\bar{\omega}\cdot\Psi_\omega\big) = 0\ .$$

The latter two equations and (ii) imply

$$\nabla_Z(D_+\Psi_{\mathcal{C}}) = -\nabla_Z\big(\bar{\partial}\log|\Psi_\omega|^2\big)\Psi_{\mathcal{C}}\ .$$

Using the local frame $\{\mathrm{f}_i\}_{i=1\ldots m}$ we can compute $D_-D_+\Psi_{\mathcal{C}}$.

$$
\begin{aligned}
D_-D_+\Psi_{\mathcal{C}} &= \sum_{i=1}^m \mathrm{f}^i\cdot\nabla_{\mathrm{f}_i}(D_+\Psi_{\mathcal{C}})\\
&= -\sum_{i=1}^m \mathrm{f}^i\cdot\nabla_{\mathrm{f}_i}\big(\bar{\partial}\log|\Psi_\omega|^2\big)\cdot\Psi_{\mathcal{C}}\\
&= -\sum_{i=1}^m \big(\mathrm{f}^i\wedge\nabla_{\mathrm{f}_i}(\bar{\partial}\log|\Psi_\omega|^2) - \mathrm{f}^i\lrcorner\nabla_{\mathrm{f}_i}(\bar{\partial}\log|\Psi_\omega|^2)\big)\cdot\Psi_{\mathcal{C}}\\
&= -\big(\partial\bar{\partial}(\log|\Psi_\omega|^2) + \bar{\partial}^*\bar{\partial}(\log|\Psi_\omega|^2)\big)\Psi_{\mathcal{C}}\ .
\end{aligned}
\tag{7}
$$

To prove (iv) we remark that Ψ_ω is a nowhere vanishing local holomorphic section of the hermitian line bundle $L_{\mathcal{C}}$. On $L_{\mathcal{C}}$ we have fixed the canonical connection and, hence, $\bar{\partial}\partial(\log|\Psi_\omega|^2)$ is just the curvature form of this line bundle. Moreover, we have the relation

$$-\partial\bar{\partial}\big(\log|\Psi_\omega|^2\big) = \bar{\partial}\partial\big(\log|\Psi_\omega|^2\big) = -\frac{i}{2}\rho \tag{8}$$

since $L_{\mathcal{C}} \cong K^{1/2}$ and the curvature form of the canonical bundle K is $-i$ times the Ricci form ρ of M. Taking the trace and using the assumption that M is a Kähler manifold, we obtain

$$\Delta \log |\Psi_\omega|^2 = 2\bar{\partial}^* \bar{\partial}\big(\log |\Psi_\omega|^2\big) = -\frac{R}{2} \ . \tag{9}$$

Inserting (8) and (9) into the result of the computation (7) gives (iv). □

Corollary 5.1. *Let (M, g, J) be a Kähler-Einstein manifold of complex dimension $m = 2k+1$ with k odd, and let $\mathcal{C}$ be a complex k-contact structure on M. Then the spinor $\Psi_\mathcal{C}$ associated with $\mathcal{C}$ satisfies the equation*

$$D^2\Psi_\mathcal{C} = D_- D_+ \Psi_\mathcal{C} = \frac{m+1}{4m} R\Psi_\mathcal{C} \ ,$$

where R is the scalar curvature of (M, g).

Proof: ¿From Proposition 5 we have

$$\begin{aligned} D^2\Psi_\mathcal{C} &= D_- D_+ \Psi_\mathcal{C} \\ &= -\frac{i}{2}\rho \cdot \Psi_\mathcal{C} + \frac{R}{4}\Psi_\mathcal{C} \ . \end{aligned}$$

Since the manifold (M, g) is Einstein, we have the relation

$$\rho = \frac{R}{2m}\Omega \ ,$$

where ρ denotes the Ricci form and Ω the Kähler form. ¿From the definition of $\Psi_\mathcal{C}$ it is clear that $\Psi_\mathcal{C}$ is a section of S_k. The definition of S_k implies that

$$-i\Omega \cdot \Psi_\mathcal{C} = (m - 2k)\Psi_\mathcal{C} \ .$$

Using this equation we obtain

$$-\frac{i}{2}\rho \cdot \Psi_\mathcal{C} = \frac{1}{2}\frac{R}{2m}(m-2k)\Psi_\mathcal{C} = \frac{R}{4m}\Psi_\mathcal{C}$$

and hence

$$D_- D_+ \Psi_\mathcal{C} = \left(\frac{R}{4m} + \frac{R}{4}\right)\Psi_\mathcal{C} = \frac{m+1}{4m} R\Psi_\mathcal{C} \ .$$

Thus, $\Psi_\mathcal{C}$ is an eigenspinor of $D^2 = D_+D_- + D_-D_+$ since $D_-\Psi_\mathcal{C} = 0$. □

Corollary 5.2. *Under the same assumptions as in Corollary 5.1, let M be closed and $R > 0$. Then the spinors $\Psi_\mathcal{C}^\pm := \lambda_1\Psi_\mathcal{C} \pm D_+\Psi_\mathcal{C}$ are Killing spinors with Killing number $\mu_\pm = \pm\frac{\lambda_1}{m+1}$, where $\lambda_1 = \sqrt{\frac{m+1}{4m}R}$.*

Proof: The spinors $\Psi_\mathcal{C}^\pm = \lambda_1\Psi_\mathcal{C} \pm D_+\Psi_\mathcal{C} = \lambda_1\Psi_\mathcal{C} \pm D\Psi_\mathcal{C}$ are eigenspinors of D for the eigenvalues $\pm\lambda_1$ because of

$$\begin{aligned} D\Psi_\mathcal{C}^\pm &= \lambda_1 D\Psi_\mathcal{C} \pm D^2\Psi_\mathcal{C} \\ &= \lambda_1 D\Psi_\mathcal{C} \pm (\lambda_1)^2\Psi_\mathcal{C} \\ &= \pm\lambda_1(\lambda_1\Psi_\mathcal{C} \pm D_+\Psi_\mathcal{C}) \\ &= \pm\lambda_1\Psi_\mathcal{C}^\pm \ . \end{aligned}$$

Now, the assertion of the corollary follows from Theorem 5 of [Ki3]. □

Combining the results of the sections 1 and 2 we have proved

THEOREM 6. *Let (M, g, J) be a closed Kähler manifold of complex dimension $m = 2k + 1$ (k odd) with scalar curvature $R \not\equiv 0$. Then the following two statements are equivalent:*

(1) *M is a spin manifold and there exists a Kählerian Killing spinor on M ,*

(2) *M is an Einstein manifold and M admits a complex k-contact structure.*

The next corollary is an immediate consequence of Proposition 4.

Corollary 6.1. *Let (M^{2m}, g, J) be a closed Kähler-Einstein manifold with positive scalar curvature admitting a complex r-contact structure such that $s = \frac{m-r}{r+1}$ is an odd number. Then M is a spin manifold and there exists a Kählerian Killing spinor on M.*

In the special case, of $r = 1$, i.e. for complex contact manifolds in the usual sense, we formulate the above corollary once again.

Corollary 6.2. *Let (M^{2m}, g, J) be a closed Kähler-Einstein manifold of complex dimension $m = 2k + 1$ (k odd) with positive scalar curvature admitting a complex contact structure. Then M is a spin manifold and there exists a Kählerian Killing spinor on M.*

Applying Theorem 6 in dimension $m = 3$ together with the classification result of K.-D. Kirchberg (cf. [Ki2]) we obtain:

Corollary 6.3. *Let (M^6, g, J) be a closed Kähler-Einstein manifold of positive scalar curvature admitting a complex contact structure. Then M^6 is analytically isometric either to the complex projective space $\mathbb{C}P^3$ or to the flag manifold $F(1, 2)$ with respect to their natural Kähler structures.*

Until now it was necessary to make the assumption $k \equiv 1(2)$, where $m = 2k + 1$ was the complex dimension of the considered manifold. One result for manifolds with $k \equiv 0(2)$ is the following proposition.

Proposition 7. *Let (M^{2m}, g, J) be a closed Kähler-Einstein manifold of complex dimension $m = 2k+1$ (k even) with positive scalar curvature. If M admits a complex contact structure and is, moreover, a spin manifold, then M is biholomorphically equivalent to the complex projective space $\mathbb{C}P^m$.*

Proof: Since M has a complex contact structure we have

$$c_1(M) = (k + 1)\alpha \ ,$$

where α is the first Chern class of the contact line bundle and, hence, is a cohomology class in $H^{1,1}(M, \mathbb{Z})$. Moreover, α is a positive class since $c_1(M)$ is positive. The manifold M is assumed to be a spin manifold and hence $c_1(M) \equiv 0 \bmod 2$. Since $(k + 1)$ is odd, we obtain

$$c_1(M) = 2(k+1)\alpha' = (m+1)\alpha' \ ,$$

where α' is again a positive class in $H^{1,1}(M, \mathbb{Z})$. Now, the assertion follows from a theorem of S. Kobayashi and T. Ochiai (cf. [KoO]). □

3. New Examples of Kählerian Killing Spinors

In this section we give examples of closed Kähler-Einstein manifolds of positive scalar curvature admitting a complex contact structure. In particular, these manifolds will supply new examples of manifolds with Kählerian Killing spinors.
The following proposition is a combination of results obtained by S. Salamon (cf. [S]).

PROPOSITION 8. *Let M be a quaternionic Kähler manifold with positive scalar curvature. Then its twistor space Z admits a Kähler Einstein metric of positive scalar curvature. Moreover, Z admits a complex contact structure.*

Explicit examples for this theorem we obtain by considering compact simply connected quaternionic Kähler manifolds that are Riemannian symmetric spaces. These spaces were classified by J.A. Wolf (cf. [W], [B]) and there is a 1-1 correspondence with compact simply connected homogeneous complex contact manifolds via the twistor fibration. Besides the exceptional cases there are the following three families:

- $\mathbb{H}P^n = Sp(n+1)/Sp(n) \times Sp(1)$
- $X^n = SU(n+2)/S(U(n) \times U(2))$
- $Y^n = SO(n+4)/S(O(n) \times O(4))$.

The corresponding twistor spaces are (cf. [W], [S])

- $Sp(n+1)/Sp(n) \times U(1) \cong \mathbb{C}P^{2n+1}$,
- $SU(n+2)/S(U(n) \times U(1) \times U(1))$,
- $SO(n+4)/S(O(n) \times O(3) \times O(2))$.

The complex dimension of these Kähler-Einstein manifolds is equal to $2n+1$. By Proposition 8, for odd n, all these manifolds satisfy the assumptions of Corollary 6.2. The only exceptional manifold satisfying these assumptions is

$$F_4/Sp(3)Sp(1) \quad \text{with twistor space} \quad F_4/Sp(3)U(1) \ .$$

Thus, in addition to $\mathbb{C}P^{2n+1}$, we have two new families of closed Kähler manifolds admitting Kählerian Killing spinors and one 15-dimensional manifold with this property.

4. Concluding Remarks

One problem is to find manifolds admitting complex r-contact structures which do not come from complex contact structures ($r = 1$). We conjecture that such manifolds can again be constructed as flag manifolds. Moreover, it would be interesting to have examples of non-homogeneous Kähler Einstein manifolds with complex contact structures.

Another problem is to investigate the case of complex dimension $m = 4l + 1$. The only result in this direction is Proposition 7. The only known examples of manifolds admitting Kählerian Killing spinors are the complex projective spaces. In this dimension it is possible to construct an eigenfunction of the Laplace operator from a Kählerian Killing spinor $\Psi = \Psi_k + \Psi_{k+1}$ (cf. [Ki3]). In fact, we have

$$\Delta \langle \Psi_{k+1}, j\Psi_k \rangle = \frac{R}{m} \langle \Psi_{k+1}, j\Psi_k \rangle \ .$$

The eigenvalue $\frac{R}{m}$ is the lowest possible, on a Kähler manifold of complex dimension m. There is an analogue of the Obata theorem, which is due to S. Kobayashi. By this theorem, a Kähler manifold is biholomorphically equivalent to the complex projective space if $\frac{R}{m}$ is an eigenvalue of the Laplace operator of multiplicity $m(m+2)$.

Motivated by these remarks we conjecture that in dimension $m = 4l+1$ the complex projective spaces are the only manifolds admitting Kählerian Killing spinors.

In this paper we have mainly considered compact manifolds. On non-compact Kähler-Einstein manifolds a complex r-contact structure only yields eigenspinors of D. These are Kählerian Killing spinors iff the length function of the spinor $\Psi_{\mathcal{C}}$ satisfies the following differential equation:

$$\Delta |\Psi_{\mathcal{C}}|^2 = \frac{R}{2m}|\Psi_{\mathcal{C}}|^2 - \frac{2}{m+1}|D_+\Psi_{\mathcal{C}}|^2$$

(cf. [Ki4]).

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Submitted: May 1994