SPINORS, SELF-DUALITY, AND IP ALGEBRAIC CURVATURE TENSORS

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ABSTRACT. We give a construction of an IP algebraic curvature tensor of rank 4 in dimension $m = 4$ using spinors and the decomposition of the 2 forms into self-dual and anti-self dual forms. In dimension 7, we can construct anti-symmetric IP tensors of rank 4 and 6 and show that they are not algebraic curvature tensors.

§1 INTRODUCTION

The Riemann curvature tensor R of a Riemannian metric on a manifold of dimension m satisfies the following identities

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z). \tag{1}$$

$$R(X,Y,Z,W) = R(Z,W,X,Y). \tag{2}$$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0. \tag{3}$$

We consider these symmetries on a purely algebraic level. Let $\mathbb{R}^m$ be Euclidean space with the usual inner product $(\cdot,\cdot)$. We say that a 4 tensor $R \in \otimes^4\mathbb{R}^m$ is *anti-symmetric if R satisfies (1).* Such a tensor R defines a linear map $\Phi = \Phi_R$ from $\Lambda^2(\mathbb{R}^m)$ to $\Lambda^2(\mathbb{R}^m)$ which is characterized by the identity:

$$(\Phi(X \wedge Y), Z \wedge W) = R(X,Y,Z,W).$$

Conversely, any such linear map Φ determines an anti-symmetric 4 tensor $R = R_\Phi$ which satisfies equation (2) if and only if Φ is self-adjoint. Identity (3) is the Bianchi identity. We say that R is an *algebraic curvature tensor* if all three identities hold.

As noted above, the curvature tensor of a Riemannian metric defines an algebraic curvature tensor at each point of the manifold. Conversely, fix a metric g_0 at a point P of M and suppose given an algebraic curvature tensor $R_0 \in \otimes^4 T_P M$. Then there exists a Riemannian metric g on M so that $g(P) = g_0$ and so that the Riemann curvature tensor of R at P is given by R_0. Thus the algebraic curvature tensors are important in Riemannian geometry.

Let $Gr_2^+(m)$ be the space of oriented 2 planes in $\mathbb{R}^m$ and let $\mathfrak{so}(m)$ be the Lie algebra of the special orthogonal group $SO(m)$; $\mathfrak{so}(m)$ is the vector space of skew-symmetric $m \times m$ real matrices. If R is an anti-symmetric 4 tensor, the

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skew-symmetric curvature operator S_R is a map from $Gr_2^+(m)$ to $\mathfrak{so}(m)$ which is characterized by the identity

$$(S_R(\pi)Z, W) = R(X, Y, Z, W)$$

where $\{X, Y\}$ is any oriented orthonormal basis for $\pi \in Gr_2^+(m)$. We say that an anti-symmetric 4 tensor R is IP if the eigenvalues of $S_R(\pi)$ are independent of the 2 plane π. Thus, in particular, the rank of $S_R(\pi)$ is independent of π; this number is called the rank of R. We say that a Riemannian metric is IP if the associated algebraic curvature tensor is IP at each point of M; the eigenvalues in question are permitted to vary with the point of M.

It is natural to ask what geometric properties arise from algebraic properties of the curvature tensor. Ivanov and Petrova [7] classified both the IP algebraic curvature tensors and the IP Riemannian metrics in dimension $m = 4$. This classification was extended by Gilkey, Leahy, and Sadofsky [5] to the dimensions $m = 5$, $m = 6$, and $m \geq 9$; subsequently the case $m = 8$ was dealt with in [6]. We summarize those results as follows:

Theorem A. *Let (M, g) be a Riemannian manifold of dimension $m \geq 4$ and $m \neq 7$ such that (M, g) is IP. Then either (M, g) has constant sectional curvature or (M, g) is locally isometric to a warped product of the form $ds^2 = dt^2 + f(t)ds_K^2$ on $(t_0, t_1) \times N$ where $f(t) := (\mathcal{K}t^2 + At + B)/2 > 0$ and where N has constant sectional curvature $\mathcal{K}$.*

Gilkey, Leahy, and Sadofsky [5] proved Theorem A for $m = 5$, $m = 6$, and $m = 8$ by establishing the following two results:

Theorem B. *Suppose that (M, g) is an IP Riemannian metric for $m \geq 5$ of rank at most 2. Then either (M, g) has constant sectional curvature or (M, g) is locally isometric to a warped product of the form $ds^2 = dt^2 + f(t)ds_K^2$ on $(t_0, t_1) \times N$ where $f(t) := (\mathcal{K}t^2 + At + B)/2 > 0$ and where N has constant sectional curvature $\mathcal{K}$.*

Theorem C. *Suppose that R is an anti-symmetric 4 tensor which is IP, and suppose that $m = 5$, $m = 6$, or that $m \geq 9$. Then R has rank at most 2.*

The first author [6] showed that there are no algebraic curvature tensors which are IP of rank greater than 2 if $m = 8$; Theorem A then followed from Theorem B in dimension 8. The case $m = 7$ is still open; it is not known if there exist algebraic curvature tensors of rank greater than 2 in dimension $m = 7$.

Ivanov and Petrova [7] defined the following 4 tensor in dimension $m = 4$:

$$\begin{aligned} &R_{1212} = a_2,\ R_{1313} = a_2,\ R_{2424} = a_2, R_{1414} = a_1,\ R_{2323} = a_1, \\ &R_{3434} = a_2,\ R_{1234} = a_1,\ R_{1324} = -a_1,\ R_{1423} = a_2,\ a_2 + 2a_1 = 0. \end{aligned} \tag{4}$$

Since this tensor satisfies the curvature symmetries (1)-(3), it is an algebraic curvature tensor. It is a bit more difficult to verify that this tensor is IP of rank 4. In §2, we will use spinors to construct this tensor, see Theorem 1 for details. the advantage of this construction is that it is quite conceptual in nature and that it is easy to verify the tensor is IP. Ivanov and Petrova use the second Bianchi identities to show that this tensor can not be extended to a neighborhood as an IP algebraic curvature tensor so this exceptional algebraic structure does not arise from a Riemannian metric.

In §3, we study seven dimensional geometry. Gilkey, Leahy, and Sadofsky proved Theorem C using methods of algebraic topology. They showed that it is sharp by exhibiting anti-symmetric tensors R which are IP of rank greater than 2 if $m = 7$ or if $m = 8$. The first author showed there are no IP algebraic curvature tensors of rank greater than 2 in dimension $m = 8$. In this note, we will show the construction of [5] does **not yield** algebraic curvature tensors; this was left open in that paper. The exceptional Lie group G_2 is at the heart of our analysis. The G_2 geometry involved makes the computations fairly easy and conceptually straightforward. We use at several points in the argument the result of Friedrich, Kath, Moroianu, & Semmelmann [4] that G_2 acts transitively on $Gr_2^+(7)$.

§2 Four Dimensional Geometry

We refer to Atiyah, Bott, and Shapiro [1] for details concerning spinors. The Clifford algebra $\mathrm{Clif}(m)$ is the universal unital algebra generated by $\mathbb{R}^m$ subject to the relations $x * y + y * x = -2(x, y)$. Let $\mathbb{H}$ be the quaternions. We have $\mathrm{Clif}(4) = \mathbb{H} \oplus \mathbb{H}$. The natural inclusion of $\mathbb{H} \oplus \mathbb{H} \subset M_8(\mathbb{R})$ defines the spin representation c of $\mathrm{Clif}(4)$ on $\mathbb{R}^8$;

$$c(x)(u, v) = (xv, -\bar{x}u) \text{ for } (u, v) \in \mathbb{R}^8 = \mathbb{H}_+ \oplus \mathbb{H}_- \text{ and } x \in \mathbb{R}^4 = \mathbb{H}.$$

We may decompose $c = c_+ \oplus c_-$ where $c_\pm(x) : \mathbb{H}_\pm \to \mathbb{H}_\mp$ are the half spin representations given by:

$$c_-(x)(0, v) = (xv, 0) \text{ and } c_-(x)(u, 0) = (0, -\bar{x}u).$$

If $\{\xi_1, \xi_2\}$ is an oriented orthonormal basis for a real 2 plane π in $\mathbb{H}$, define the map $c(\pi) = c(\xi_1)c(\xi_2)$ which preserves the chiral decomposition; the restriction of $c(\pi)$ to $\mathbb{H}_+$ defines a map

$$c(\xi_1, \xi_2) := -\xi_1\bar{\xi}_2 \text{ from } Gr_2^+(\mathbb{H}) \text{ to } \mathfrak{so}(\mathbb{H}) \text{ with } c(\pi)^2 = -1.$$

Let $x_- = (0, i) \in \mathbb{H}_-$. We define an orthogonal decomposition of $\mathbb{H}_+ = E_1 \oplus E_2$ which decomposes $c(\pi) = S^1 \oplus S^2$:

$$\begin{aligned} &E_1(\pi) := \mathrm{span}\{c(\xi_1)x_-, c(x_2)x_-\},\ S^1(\pi) := c(\xi_1)c(\xi_2)|_{E_1},\\ &E_2(\pi) := E_1(\pi)^\perp,\ S^2(\pi) := c(\xi_1)c(\xi_2)|_{E_2}. \end{aligned}$$

Theorem 1. *Let $S_R(\pi) := a_1S^1 + a_2S^2$ define an anti-symmetric 4 tensor R. If $a_2 + 2a_1 = 0$ and if $a_1 \neq 0$, then R is an algebraic IP curvature tensor of rank 4 equivalent to the one defined in equation (4).*

Proof. Let $x_1 = 1$, $x_2 = i$, $x_3 = j$, and $x_4 = k$ be the canonical orthonormal basis for $\mathbb{H}$. Let $\epsilon_{\xi_1\xi_2\xi_3\xi_4} = (\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4, \nu)$ where ν is the standard orientation of

$\mathbb{R}^4$. Let $S^\mu_{ij} = S^\mu(x_i, x_j)$. We have $c(1)x_- = i$, $c(i)x_- = -1$, $c(j)x_- = -k$, and $c(k)x_- = j$. We compute:

$$
(5)\qquad
\begin{array}{lllll}
\epsilon_{1ijk} = 1, & S^1_{12}i = -1, & S^2_{12}j = k, & R_{1212} = a_1, & R_{1234} = a_2,\\
\epsilon_{i,-k,1,j} = 1, & S^1_{13}i = -k, & S^2_{13}1 = j, & R_{1313} = a_2, & R_{1324} = -a_1,\\
\epsilon_{i,j,1,k} = 1, & S^1_{14}i = j, & S^2_{14}1 = k, & R_{1414} = a_2, & R_{1423} = a_1,\\
\epsilon_{1,k,i,j} = 1, & S^1_{23}1 = k, & S^2_{23}i = j, & R_{2323} = a_2, & R_{2314} = a_1,\\
\epsilon_{1,-j,i,k} = 1, & S^1_{24}1 = -j, & S^2_{24}i = k, & R_{2424} = a_2, & R_{2413} = -a_1,\\
\epsilon_{k,-j,1,i} = 1, & S^1_{34}k = -j, & S^2_{34}1 = i, & R_{3434} = a_1, & R_{3412} = a_2.
\end{array}
$$

The curvature $R(\cdot,\cdot,\cdot,\cdot)$ vanishes if there are exactly three distinct entries present. We check (2) holds

$$R_{1234} = a_2 = R_{3412}, R_{1324} = -a_1 = R_{2413}, R_{1423} = a_1 = R_{2314}.$$

Finally we check that the Bianchi identity holds if $2a_1 + a_2 = 0$.

$$
\begin{aligned}
&R_{1234} + R_{3124} + R_{2314} = a_2 + a_1 + a_1,\ R_{1243} + R_{4123} + R_{2413} = -a_2 - a_1 - a_1\\
&R_{1342} + R_{3412} + R_{4132} = a_1 + a_2 + a_1,\ R_{2341} + R_{3421} + R_{4231} = -a_1 - a_2 - a_1.
\end{aligned}
$$

It is immediate from the definition that $S_R(\pi)$ has rank 4 if $a_1a_2 \neq 0$. We can convert the curvature tensor defined in (5) to the one defined in (4) by making a change of notation to change the indices: $1 \to 3$, $2 \to 1$, $3 \to 2$, and $4 \to 4$. □

We have used spinors to construct this curvature tensor. There is an alternate viewpoint which is group theoretic in nature. We decompose the vector space Λ^2 of 2 forms on $\mathbb{R}^4$ as a direct sum $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ where $\Lambda^2_\pm$ are the self dual and the anti-self dual 2 planes. Let $\Pi_\pm$ be the associated orthogonal projections. We define the endomorphism $\Phi = \Phi^{a_+,a_-} := a_+\Pi_+ + a_-\Pi_-$ of $\Lambda^2(\mathbb{R}^4)$. The action of the special orthogonal group on Λ^2 preserves this decomposition and consequently the associated 4 tensor R_Φ is equivariant with respect to the action of $SO(4)$. Conversely, any anti-symmetric 4 tensor which is equivariant with respect to the action of $SO(4)$ must have this form. Since $SO(4)$ acts transitively on $Gr_2^+(4)$, $S_R(\pi)$ has constant rank so R is IP. If $g \in SO(4)$, let $\Lambda^2 g$ be the extension to $\Lambda^2(\mathcal{R}^4)$ and let $\mathcal{R} = \mathcal{R}_{a_+,a_-,g}$ be defined by the endomorphism $\Phi^{a_+,a_-} \circ \Lambda^2(g)$ of $\Lambda^2(\mathbb{R}^4)$. It is then immediate that $\mathcal{R}$ satisfies the curvature symmetry (1) and has constant rank. Furthermore, $\mathcal{R}$ satisfies the curvature symmetry (2) if and only if $\Phi \circ \Lambda^2(g)$ is self-adjoint. Since Φ commutes with $\Lambda^2(g)$ and since Φ is self-adjoint, $\mathcal{R}$ satisfies (2) if and only if $\Lambda^2(g)$ is self-adjoint or equivalently if and only if $g^2 = \pm 1$. Let $g \in SO(4)$ send $x \to x \cdot i$; note that $g^2 = -1$. It is immediate from this discussion that $\mathcal{R} = \mathcal{R}_{a_1+a_2,a_1-a_2,g}$ gives the tensor in Theorem 1; it is an IP algebraic curvature of rank 4 if $a_1 \neq 0$ and if $2a_1 + a_2 = 0$.

There is another construction of this curvature tensor which is useful. We define

$$
\begin{aligned}
&R_0(X,Y)Z := g(Y,Z)X - g(X,Z)Y, \text{ and}\\
&R_i(X,Y)Z := g(Y,iZ)iX - g(X,iZ)iY - 2g(X,iY)iZ.
\end{aligned}
$$

The tensor R_0 corresponds to constant sectional curvature and is an algebraic curvature tensor. We show these are algebraic curvature tensors by computing:

$$\begin{array}{llll}
R_0(1,i)1=-i, & R_0(1,i)i=1, & R_0(1,i)j=0, & R_0(1,i)k=0,\\
R_0(1,j)1=-j, & R_0(1,j)i=0, & R_0(1,j)j=1, & R_0(1,j)k=0,\\
R_0(1,k)1=-k, & R_0(1,k)i=0, & R_0(1,k)j=0, & R_0(1,k)k=1,\\
R_0(i,j)1=0, & R_0(i,j)i=-j, & R_0(i,j)j=i, & R_0(i,j)k=0,\\
R_0(i,k)1=0, & R_0(i,k)i=-k, & R_0(i,k)j=0, & R_0(i,k)k=i,\\
R_0(j,k)1=0, & R_0(j,k)i=0, & R_0(j,k)j=-k, & R_0(j,k)k=j
\end{array}$$

and

$$\begin{array}{llll}
R_i(1,i)1=3i, & R_i(1,i)i=-3, & R_i(1,i)j=2k, & R_i(1,i)k=-2j,\\
R_i(1,j)1=0, & R_i(1,j)i=k, & R_i(1,j)j=0, & R_i(1,j)k=-i,\\
R_i(1,k)1=0, & R_i(1,k)i=-j, & R_i(1,k)j=i, & R_i(1,k)k=0,\\
R_i(i,j)1=-k, & R_i(i,j)i=0, & R_i(i,j)j=0, & R_i(i,j)k=1,\\
R_i(i,k)1=j, & R_i(i,k)i=0, & R_i(i,k)j=-1, & R_i(i,k)k=0,\\
R_i(j,k)1=2i, & R_i(j,k)i=-2, & R_i(j,k)j=3k, & R_i(j,k)k=-3j.
\end{array}$$

The curvature tensor $R_0 - R_i$ is complex linear and is the curvature tensor of a metric of constant holomorphic sectional curvature $+4$; modulo a suitable normalizing constant, it is the curvature tensor of the Fubini-Study metric on $\mathbb{CP}^2$. Let $\check{R} := R_i + 2R_0$. Then

$$\begin{array}{llll}
\check{R}(1,i)1=i, & \check{R}(1,i)i=-1, & \check{R}(1,i)j=2k, & \check{R}(1,i)k=-2j,\\
\check{R}(1,j)1=-2j, & \check{R}(1,j)i=k, & \check{R}(1,j)j=2, & \check{R}(1,j)k=-i,\\
\check{R}(1,k)1=-2k, & \check{R}(1,k)i=-j, & \check{R}(1,k)j=i, & \check{R}(1,k)k=2,\\
\check{R}(i,j)1=-k, & \check{R}(i,j)i=-2j, & \check{R}(i,j)j=2i, & \check{R}(i,j)k=1,\\
\check{R}(i,k)1=j, & \check{R}(i,k)i=-2k, & \check{R}(i,k)j=-1, & \check{R}(i,k)k=2i,\\
\check{R}(j,k)1=2i, & \check{R}(j,k)i=-2, & \check{R}(j,k)j=k, & \check{R}(j,k)k=-j.
\end{array}$$

If we change the index '4' to the index '-4', this is the curvature tensor given in equation (5) and hence is an IP curvature tensor. This shows that the IP curvature tensor is a non-trivial linear combination of the constant sectional curvature tensor and the constant holomorphic sectional curvature tensor.

§3 Seven Dimensional Geometry

If $m = 7$, then one can construct anti-symmetric IP tensors of higher rank using the exceptional Lie group G_2. We will show, however, that this construction does not lead to a tensor satisfying identity (3); one does not construct an algebraic curvature tensor which is IP of higher rank in this fashion. The following construction is due to Bryant and Salamon [2]. Identify $\mathbb{R}^7 = \operatorname{Imag}\mathbb{H} \oplus \mathbb{H}$ to introduce coordinates $(y_1, y_2, y_3; y_4, y_5, y_6, y_7)$ on $\mathbb{R}^7$. Define $e_i := dy_i = dy^i$ where we identify vectors and covectors using the metric on $\mathbb{R}^7$. Let $e_{ij} := e_i \wedge e_j$ and $e_{ijk} := e_i \wedge e_j \wedge e_k$. Let

$$\begin{aligned}
&\omega := e_{123} + e_1 \wedge \omega_1 + e_2 \wedge \omega_2 + e_3 \wedge \omega_3 \text{ where}\\
&\omega_1 := e_{45} - e_{67},\ \omega_2 := e_{46} + e_{57}, \text{ and } \omega_3 := e_{47} - e_{56}
\end{aligned} \tag{6}$$

are a basis for the anti-self dual 2 forms on $\mathbb{R}^4$. The exceptional Lie group G_2 is the subgroup of $SO(7)$ which preserves the 3 form ω. Let $Spin(4) := S^3 \times S^3$. We embed a copy of $SO(4)$ in G_2 by the action $(p,q)\cdot(u;v) = (pu\bar{p}; qv\bar{p})$. Let int denote interior multiplication. We define $\Lambda_7^+ := \operatorname{int}(\mathbb{R}^7)\omega = \operatorname{span}\{u_i\}$ where

$$\begin{aligned}
&u_1 := e_{23} + e_{45} - e_{67}, \quad u_2 := -e_{13} + e_{46} + e_{57}, \; u_3 := e_{12} + e_{47} - e_{56},\\
&u_4 := -e_{15} - e_{26} - e_{37}, \; u_5 := e_{14} - e_{27} + e_{36}, \quad u_6 := e_{17} + e_{24} - e_{35},\\
&u_7 := -e_{16} + e_{25} + e_{34}.
\end{aligned}$$

Theorem 2. *Let $\Phi := 3a\Pi_7 + bI$ for $(a,b) \neq (0,0)$ where Π_7 is orthogonal projection on Λ_7^+. Let $g \in SO(7)$. Define $\mathcal{R} := R_{\Phi\circ\Lambda^2 g}$.*

1. *The tensor $\mathcal{R}$ is an anti-symmetric IP tensor of rank 6 for $a+b \neq 0$ and $a \neq 0$, of rank 4 for $a+b=0$ and $a\neq 0$, and of rank 2 for $b \neq 0$ and $a = 0$.*
2. *If $\operatorname{Rank}\mathcal{R} > 2$, then $\mathcal{R}$ does not satisfy equation (3).*
3. *The anti-symmetric IP tensors of rank greater than 2 defined in [5] are equivalent to the tensors $\mathcal{R}$ defined here.*

Proof. We use [4] to see that G_2 acts transitively on $Gr_2^+(\mathbb{R}^7)$. Let $R = R_\Phi$. Since G_2 preserves ω, G_2 commutes with Φ so the rank of R_Φ is constant; consequently $\mathcal{R}$ is IP. We compute the rank as follows. Let $\bar{S} := S_R(e_1,e_2)$. Then

$$\begin{aligned}
&\bar{S}e_1 = (a+b)e_2, \; \bar{S}e_2 = -(a+b)e_1, \; \bar{S}e_3 = 0,\\
&\bar{S}e_4 = ae_7, \; \bar{S}e_7 = -ae_4, \; \bar{S}e_5 = -ae_6, \; \bar{S}e_6 = ae_5.
\end{aligned}$$

We assume $(a,b) \neq (0,0)$. This shows that the rank is 2 if $a = 0$, that the rank is 4 if $a+b=0$, and that the rank is 6 otherwise. The first assertion now follows.

If $\operatorname{Rank}(\mathcal{R}) > 2$, then $a \neq 0$; we suppose henceforth $a = 1$. Suppose $\mathcal{R}$ satisfies equation (3). Since $\mathcal{R}(\eta_1,\eta_2,\eta_3,\eta_4) = R(g\eta_1, g\eta_2, \eta_3, \eta_4)$, we set $\eta_i := g^{-1}\xi_i$ to see that $\mathcal{R}$ satisfies (3) if and only if we have the identity:

$$0 = S_R(\xi_1,\xi_2)g^{-1}\xi_3 + S_R(\xi_2,\xi_3)g^{-1}\xi_1 + S_R(\xi_3,\xi_1)g^{-1}\xi_2$$

for all orthonormal sets $\{\xi_1,\xi_2,\xi_3\}$. Suppose first that $+1$ is a double eigenvalue of g so that we can find an orthonormal set $\{\xi_1,\xi_2\}$ so that $g\xi_i = \xi_i$. Since G_2 acts transitively on $Gr_2^+(7)$, by changing the basis for $\mathbb{R}^7$ but perserving the form of ω as given above, we may assume that $\xi_1 = e_1$ and $\xi_2 = e_2$. We use the Bianchi identities to see

$$\begin{aligned}
0 &= S_R(e_1,e_2)g^{-1}e_4 + S_R(e_2,e_4)e_1 + S_R(e_4,e_1)e_2\\
&= \operatorname{int}(g^{-1}e_4)\{be_{12} + e_{47} - e_{56}\} + 2e_7.
\end{aligned}$$

This implies that $(g^{-1}e_4, e_4) = -2$ which is impossible as g is an isometry. A similar argument shows that -1 can not be a double eigenvalue of g. Consequently, the isometry g must have at least one complex eigenvalue which is not real.

Let π be a 2 plane left invariant by g corresponding to a complex eigenvalue $\lambda = c + s\sqrt{-1}$ where $s \neq 0$. Again, we may change coordinates appropriately to assume that π is spanned by e_1 and e_2 and that

$$\begin{aligned} ge_1 &= ce_1 + se_2,\ ge_2 = ce_2 - se_1 \\ g^{-1}e_1 &= ce_1 - se_2,\ g^{-1}e_2 = ce_2 + se_1. \end{aligned}$$

Let $P_3 := \operatorname{span}\{e_1, e_2, e_3\}$. This 3 plane is preserved by $S_R(e_1, e_2)$, $S_R(e_2, e_3)$, and $S_R(e_3, e_1)$. Thus the identity

$$0 = S_R(e_1, e_2)g^{-1}e_3 + S_R(e_2, e_3)(ce_1 - se_2) + S_R(e_3, e_1)(ce_2 + se_3)$$

implies $S_R(e_1, e_2)g^{-1}e_3 \in P_3$ and consequently $g^{-1}e_3 = \pm e_3$. Furthermore, since $S_R(e_1, e_2)e_3 = 0$, we have

$$0 = S_R(e_2, e_3)(ce_1 - se_2) + S_R(e_3, e_1)(ce_2 + se_1) = -2s(a + b)e_3.$$

Since $s \neq 0$, this implies $a + b = 0$. Thus $S_R(\pi)$ is zero on π and R has rank 4. Next we compute

$$\begin{aligned} 0 &= S_R(e_1, e_2)g^{-1}e_4 + S_R(e_2, e_4)(ce_1 - se_2) + S_R(e_4, e_1)(ce_2 + se_1) \\ &= S_R(e_1, e_2)g^{-1}e_4 + ce_7 + ce_7, \text{ and} \\ 0 &= S_R(e_1, e_2)g^{-1}e_7 + S_R(e_2, e_7)(ce_1 - se_2) + S_R(e_7, e_1)(ce_2 + se_1) \\ &= S_R(e_1, e_2)g^{-1}e_7 - ce_4 - ce_4. \end{aligned}$$

Consequently $S_R(e_1, e_2)g^{-1}e_4 = 2ce_7$ and $S_R(e_1, e_2)g^{-1}e_7 = -2ce_4$. This shows that $c = \pm\frac{1}{2}$, that $g^{-1}e_4 = \pm e_7$, and that $g^{-1}e_7 = - \pm e_4$. Since λ was arbitrary, we conclude $\operatorname{Real}(\lambda) = \pm\frac{1}{2}$ for any complex eigenvalue of g which is not real. On the other hand, the eigenvalues of g on the 2 plane spanned by e_4 and e_7 are $\pm\sqrt{-1}$. This contradiction establishes the second assertion.

To prove the final assertion, we must review the construction given in [5] in dimension 7. The isomorphism $\operatorname{Clif}(\mathbb{R}^8) = M_{16}(\mathbb{R})$ defines the spin representation. As in the case $m = 4$, we may decompose $\mathbb{R}^{16} = \mathbb{R}^8_+ \oplus \mathbb{R}^8_+$ into the half spin representations and decompose $c = c_+ \oplus c_-$. If $\{\xi_1, \xi_2\}$ is an oriented orthonormal basis for a 2 plane π, then $c(\pi)^2 = -1$ where $c(\pi) := c(\xi_1)c(\xi_2)$. Choose a unit vector $x_- \in \mathbb{R}^8_-$ and a unit vector $e_8 \in \mathbb{R}^8$. Let $\mathbb{R}^7 := e_8^\perp$ and $\mathbb{R}^7_+ = (c(e_8)x_-)^\perp$. Let $\pi \subset \mathbb{R}^7$. We define the following orthogonal decomposition $\mathbb{R}^7_+ = E_1 \oplus E_2 \oplus E_3$ and a corresponding decomposition $c(\pi) = c(\xi_1)c(\xi_2) = S^1 \oplus S^2 \oplus S^3$:

$$\begin{aligned} &E_1(\pi) := \operatorname{span}\{c(\xi_1)x_-, c(\xi_2)x_-\},\ S^1(\pi) := c(\xi_1)c(\xi_2)|_{E_1}, \\ &E_2(\pi) := \operatorname{span}\{c(e_8)x_-, c(x_8)c(\xi_1)c(\xi_2)x_-\},\ S^2(\pi) := c(\xi_1)c(\xi_2)|_{E_2}, \\ &E_3(\pi) := (E_1 \oplus E_2)^\perp,\ S^3(\pi) := c(\xi_1)c(\xi_2)|_{E_3}. \end{aligned}$$

Let $S^{\alpha,\beta} := \alpha S^1 + \beta S^3$. Let $\mathbb{R}^7_+ := (c(e_8)x_-)^\perp$. Since $c(e_8)x_- \subset E_2$, we have

$$S^{\alpha,\beta} : Gr_2^+(\mathbb{R}^7) \to \mathfrak{so}(\mathbb{R}^7_+).$$

It is immediate that $\mathrm{Rank}(S^{\alpha,\beta}(\pi))$ is constant. This rank is 6 if $(\alpha,\beta)\neq(0,0)$, this rank is 4 if $\alpha=0$ and $\beta\neq 0$, and this rank is 2 if $\alpha\neq 0$ and $\beta=0$. Let h be any isometry between $\mathbb{R}^7$ and $\mathbb{R}^7_+$. We use h to identify $\mathfrak{so}(\mathbb{R}^7)$ and $\mathfrak{so}(\mathbb{R}^7_+)$ and to regard $S: Gr_2^+(7)\rightarrow\mathfrak{so}(7)$. Let R_S be the associated anti-symmetric 4 tensor.

Let Θ be the 3 co-tensor defined by:

$$\Theta(\xi_1,\xi_2,\xi_3):=(c(\xi_1)c(\xi_2)c(\xi_3)x_-,c(e_8)x_-).$$

If $\{\xi_i\}$ is an orthonormal set, the Clifford commutation relations show

$$\Theta(\xi_{\epsilon(1)},\xi_{\epsilon(2)},\xi_{\epsilon(3)})=\mathrm{sign}(\epsilon)\Theta(\xi_1,\xi_2,\xi_3)$$

for any permutation ϵ. Let ω be the associated 3 form defined by antisymmetrizing Θ. Then there exists a suitably chosen basis e_i for $\mathbb{R}^7$ so that ω has the form given above in equation (6). Let $h_0(\xi):=c(\xi)x_-$ define the canonical isometry between $\mathbb{R}^7$ and $\mathbb{R}^7_+$. Then $R_S=R_{a\Pi_7+bI}$ for suitably chosen $a=a(\alpha,\beta)$ and $b=b(\alpha,\beta)$. Furthermore, replacing h_0 by an arbitrary isomorphism just twists R by an element g. This shows the construction using spinors given in [5] is equivalent to the construction given above using G_2. $\square$

Remark. It is tempting to try to construct an IP manifold of higher rank in dimension 7 using a homogeneous G_2 manifold. Such a manifold would necessarily be Osserman since G_2 acts transitively on the 1 planes. Hence the manifold would have constant sectional curvature (see Chi [3]) and would not be IP of rank greater than 2.

Conclusion. We have shown that that it is not possible to use spinors to construct and IP algebraic curvature tensor of higher rank in dimension 7. There are no known other ways to construct anti-symmetric IP tensors of higher rank but it has not been shown they do not exist.

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