Conformal Killing forms on Riemannian manifolds

Uwe Semmelmann

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Abstract

Conformal Killing forms are a natural generalization of conformal vector fields on Riemannian manifolds. They are defined as sections in the kernel of a conformally invariant first order differential operator. We show the existence of conformal Killing forms on nearly Kähler and weak G_2-manifolds. Moreover, we give a complete description of special conformal Killing forms. A further result is a sharp upper bound on the dimension of the space of conformal Killing forms.

1 Introduction

A classical object of differential geometry are *Killing vector fields*. These are by definition infinitesimal isometries, i.e. the flow of such a vector field preserves a given metric. The space of all Killing vector fields forms the Lie algebra of the isometry group of a Riemannian manifold and the number of linearly independent Killing vector fields measures the degree of symmetry of the manifold. It is known that this number is bounded from above by the dimension of the isometry group of the standard sphere and, on compact manifolds, equality is attained if and only if the manifold is isometric to the standard sphere or the real projective space. Slightly more generally one can consider *conformal vector fields*, i.e. vector fields with a flow preserving a given conformal class of metrics. There are several geometric conditions which force a conformal vector field to be Killing.

Much less is known about a rather natural generalization of conformal vector fields, the so-called *conformal Killing forms*. These are p-forms ψ satisfying for any vector field X the differential equation

$$\nabla_X \psi \;-\; \tfrac{1}{p+1} X \lrcorner\, d\psi \;+\; \tfrac{1}{n-p+1} X^* \wedge d^*\psi \;=\; 0\;, \tag{1.1}$$

where n is the dimension of the manifold, ∇ denotes the covariant derivative of the Levi-Civita connection, X^* is 1-form dual to X and $\lrcorner$ is the operation dual to the wedge product. It is easy to see that a conformal Killing 1-form is dual to a conformal vector field. Coclosed conformal Killing p-forms are called *Killing forms*. For $p = 1$ they are dual to Killing vector fields.

The left hand side of equation (1.1) defines a first order elliptic differential operator T, which was already studied in the context of Stein-Weiss operators (c.f. [6]). Equivalently one can describe a conformal Killing form as a form in the kernel of T. From this point of view conformal Killing forms are similar to twistor spinors in spin geometry. One shared property is the conformal invariance of the defining equation. In particular, any form which is parallel for some metric g, and thus a Killing form for trivial reasons, induces non-parallel conformal Killing forms for metrics conformally equivalent to g (by a non-trivial change of the metric).

Killing forms, as a generalization of the Killing vector fields, were introduced by K. Yano in [30]. Later S. Tachibana (c.f. [25]), for the case of 2–forms, and more generally T. Kashiwada (c.f. [17], [18]) introduced conformal Killing forms generalizing conformal vector fields.

Already K. Yano noted that a p–form ψ is a Killing form if and only if for any geodesic γ the $(p-1)$–form $\dot{\gamma} \lrcorner \psi$ is parallel along γ. In particular, Killing forms give rise to quadratic first integrals of the geodesic equation, i.e. functions which are constant along geodesics. Hence, they can be used to integrate the equation of motion. This was first done in the article [22] of R.Penrose and M. Walker, which initiated an intense study of Killing forms in the physics literature. In particular, there is a local classification of Lorentz manifolds with Killing 2-forms. More recently Killing forms and conformal Killing forms have been successfully applied to define symmetries of field equations (c.f. [3], [4]).

Despite this longstanding interest in (conformal) Killing forms there are only very few global results on Riemannian manifolds. Moreover the number of the known non-trivial examples on compact manifolds is surprisingly small. The aim of this article is to fill this gap and to start a study of global properties of conformal Killing forms.

As a first contribution we will show that there are several classes of Riemannian manifolds admitting Killing forms, which so far did not appear in the literature. In particular, we will show that there are Killing forms on nearly Kähler manifolds and on manifolds with a weak G_2–structure. All these examples are related to Killing spinors and nearly parallel vector cross products. Moreover, they are all so-called *special Killing forms*. The restriction from Killing forms to special Killing forms is analogous to the definition of a Sasakian structure as a unit length Killing vector field satisfying an additional equation. One of our main results in this paper is the complete description of manifolds admitting special Killing forms.

Since conformal Killing forms are sections in the kernel of an elliptic operator it is clear that they span a finite dimensional space in the case of compact manifolds. Our second main result is an explicit upper bound for the dimension of the space of conformal Killing forms on arbitrary connected Riemannian manifolds. The upper bound is provided by the dimension of the corresponding space on the standard sphere. It is also shown that if the upper bound is attained the manifold has to be conformally flat.

In our paper we tried to collect all that is presently known for conformal Killing forms on Riemannian manifolds. This includes some new proofs and new versions of known results.

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2 The definition of conformal Killing forms

In this section, we will introduce conformal Killing forms, give integrability conditions and state several well-known elementary properties, including equivalent characterizations.

Let $(V, \langle\cdot,\cdot\rangle)$ be an n–dimensional Euclidean vector space. Then the $O(n)$–representation $V^* \otimes \Lambda^p V^*$ has the following decomposition:

$$V^* \otimes \Lambda^p V^* \;\cong\; \Lambda^{p-1} V^* \oplus \Lambda^{p+1} V^* \oplus \Lambda^{p,1} V^* \;, \tag{2.2}$$

where $\Lambda^{p,1}V^*$ is the intersection of the kernels of wedge product and contraction map. The highest weight of the representation $\Lambda^{p,1}V^*$ is the sum of the highest weights of V^* and of $\Lambda^p V^*$. Elements of $\Lambda^{p,1}V^* \subset V^* \otimes \Lambda^p V^*$ can be considered as 1-forms on V with values in $\Lambda^p V^*$. For any $v \in V$, $\alpha \in V^*$ and $\psi \in \Lambda^p V^*$, the projection $\mathrm{pr}_{\Lambda^{p,1}} : V^* \otimes \Lambda^p V^* \to \Lambda^{p,1} V^*$ is then explicitly given by

$$[\mathrm{pr}_{\Lambda^{p,1}}(\alpha \otimes \psi)]\, v \;:=\; \alpha(v)\,\psi \;-\; \tfrac{1}{p+1}\, v \lrcorner (\alpha \wedge \psi) \;-\; \tfrac{1}{n-p+1}\, v^* \wedge (\alpha^\sharp \lrcorner \psi) \;, \tag{2.3}$$

where v^* denotes the 1-form dual to v, i.e. $v^*(w) = \langle v, w\rangle$, $\alpha^\sharp$ is the vector defined by $\alpha(v) = \langle \alpha^\sharp, v\rangle$ and $v \lrcorner$ denotes the interior multiplication which is dual to the wedge product $v \wedge$.

This decomposition immediately translates to Riemannian manifolds (M^n, g), where we have the decomposition

$$T^*M \otimes \Lambda^p T^*M \;\cong\; \Lambda^{p-1} T^*M \oplus \Lambda^{p+1} T^*M \oplus \Lambda^{p,1} T^*M \tag{2.4}$$

with $\Lambda^{p,1}T^*M$ denoting the vector bundle corresponding to the representation $\Lambda^{p,1}$. The covariant derivative $\nabla\psi$ of a p–form ψ is a section of $T^*M \otimes \Lambda^p T^*M$, projecting it onto the summands $\Lambda^{p+1}T^*M$ resp. $\Lambda^{p-1}T^*M$ yields $d\psi$ resp. $d^*\psi$. The projection onto the third summand $\Lambda^{p,1}T^*M$ defines a natural first order differential operator T, which we will call the *twistor operator*. The twistor operator $T : \Gamma(\Lambda^p T^*M) \to \Gamma(\Lambda^{p,1} T^*M) \subset \Gamma(T^*M \otimes \Lambda^p T^*M)$ is given for any vector field X by the following formula

$$[\,T\psi\,]\,(X) \;:=\; [\mathrm{pr}_{\Lambda^{p,1}}(\nabla\psi)]\,(X) \;=\; \nabla_X\,\psi \;-\; \tfrac{1}{p+1}\, X \lrcorner\, d\psi \;+\; \tfrac{1}{n-p+1}\, X^* \wedge d^*\psi \;.$$

This definition is similar to the definition of the twistor operator in spin geometry, where one has the decomposition of the tensor product of spinor bundle and cotangent bundle into the sum of spinor bundle and kernel of the Clifford multiplication. The twistor operator is defined as the projection of the covariant derivative of a spinor onto the kernel of the Clifford multiplication, which, as a vector bundle, is associated to the representation given by the sum of highest weights of spin and standard representation.

Definition 2.1 *A p-form ψ is called a conformal Killing p-form if and only if ψ is in the kernel of T, i.e. if and only if ψ satisfies for all vector fields X the equation*

$$\nabla_X\,\psi \;=\; \tfrac{1}{p+1}\, X \lrcorner\, d\psi \;-\; \tfrac{1}{n-p+1}\, X^* \wedge d^*\psi \;. \tag{2.5}$$

If the p-form ψ is in addition coclosed it is called a Killing p-form. This is equivalent to $\nabla\psi \in \Gamma(\Lambda^{p+1}T^*M)$ or to $X \lrcorner \nabla_X\psi = 0$ for any vector field X.

Closed conformal Killing forms will be called $*$–*Killing forms*. In the physics literature, equation (2.5) defining a conformal Killing form is often called the *Killing–Yano equation*. A further natural notation is *twistor forms*, which is also motivated by the following observation. Let (M, g) be a spin manifold and let ψ be a *twistor spinor*, i.e. a section of the spinor bundle lying in the kernel of the spinorial twistor operator or, equivalently, a spinor satisfying for all vector fields X the equation $\nabla_X \psi \;=\; -\frac{1}{n} X \cdot D\psi$, where D is the Dirac operator and $\cdot$ denotes the Clifford multiplication. Given two such twistor spinors, ψ_1 and ψ_2, we can introduce k-forms ω_k, which are on any tangent vectors $X_1, \ldots, X_k$ defined by

$$\omega_k(X_1, \ldots, X_k) := \langle (X_1^* \wedge \ldots \wedge X_k^*) \cdot \psi_1, \psi_2 \rangle \ .$$

It is well-known that for $k = 1$ the form ω_1 is dual to a conformal vector field. Moreover, if ψ_1 and ψ_2 are Killing spinors the form ω_1 is dual to a Killing vector field. Recall that a Killing spinor is a section ψ of the spinor bundle satisfying for all vector fields X and some constant c the equation $\nabla_X \psi = cX \cdot \psi$, i.e. Killing spinors are special solutions of the twistor equation. More generally we have

Proposition 2.2 *Let (M^n, g) be a Riemannian spin manifold with twistor spinors ψ_1 and ψ_2. Then for any k the associated k-form ω_k is a conformal Killing form.*

The proof, which follows from a simple local calculation, is given in the appendix.

Decomposition (2.4) implies that the covariant derivative $\nabla\psi$ splits into three components. Using the twistor operator T we can write the covariant derivative of a p-form ψ as

$$\nabla_X \psi \;=\; \tfrac{1}{p+1}\, X \lrcorner\, d\psi \;-\; \tfrac{1}{n-p+1}\, X^* \,\wedge\, d^*\psi \;+\; [\,T\psi\,]\,(X) \ . \tag{2.6}$$

This formula leads to the following pointwise norm estimate together with a further characterization of conformal Killing forms (c.f. [13]).

Lemma 2.3 *Let $(M^n,\, g)$ be a Riemannian manifold and let ψ be any p–form. Then*

$$|\,\nabla\psi\,|^2 \;\geq\; \tfrac{1}{p+1}\,|\,d\psi\,|^2 \;+\; \tfrac{1}{n-p+1}\,|\,d^*\psi\,|^2 \ , \tag{2.7}$$

with equality if and only if ψ is a conformal Killing p–form.

As an application of Lemma 2.3 one can prove that the Hodge star-operator $*$ maps conformal Killing p–forms into conformal Killing $(n-p)$–forms. In particular, $*$ interchanges closed and coclosed conformal Killing form.

Differentiating equation (2.6) we obtain two Weitzenböck formulas, which play an important role in the proof of many global results. Similar characterizations were obtained in [17]. For any p-form ψ we have the equations

$$\nabla^*\nabla\psi \;\;=\;\; \tfrac{1}{p+1}\, d^*d\,\psi \;\;+\;\; \tfrac{1}{n-p+1}\, dd^*\,\psi \;\;+\;\; T^*T\,\psi \ , \tag{2.8}$$

$$q(R)\,\psi \;\;=\;\; \tfrac{p}{p+1}\, d^*d\,\psi \;\;+\;\; \tfrac{n-p}{n-p+1}\, dd^*\,\psi \;\;-\;\; T^*T\,\psi \ , \tag{2.9}$$

where $q(R)$ is the curvature term appearing in the classical Weitzenböck formula for the Laplacian on p-forms: $\Delta = d^*d + dd^* = \nabla^*\nabla + q(R)$. It is the symmetric endomorphism of the bundle of differential forms defined by

$$q(R) \;=\; \sum\, e_j^* \,\wedge\, e_i \lrcorner\, R_{e_i, e_j},$$

where $\{e_i\}$ is any local ortho-normal frame and R_{e_i,e_j} denotes the curvature of the form bundle. On forms of degree one and two one has an explicit expression for the action of $q(R)$. Indeed, if ξ is any 1–form, then $q(R)\,\xi = \mathrm{Ric}\,(\xi)$ and if ω is any 2–form then

$$q(R)\,\omega \;=\; \mathrm{Ric}\,(\omega) - 2\mathcal{R}\,(\omega), \tag{2.10}$$

where Ric denotes the symmetric endomorphism of the form bundle obtained by extending the Ricci curvature as derivation. Moreover, $\mathcal{R}$ denotes the Riemannian curvature operator defined on vector fields X, Y, Z, U by $g(\mathcal{R}(X\wedge Y),\, Z\wedge U) = -g(R(X,\,Y)\,Z,\,U)$.

Integrating the second Weitzenböck formula (2.9) gives rise to an important integrability condition and a characterization of conformal Killing forms on compact manifolds. Indeed we have

Proposition 2.4 *Let $(M^n,\,g)$ a compact Riemannian manifold. Then a p–form is a conformal Killing p–form, if and only if*

$$q(R)\,\psi \;=\; \tfrac{p}{p+1}\,d^*d\,\psi \;+\; \tfrac{n-p}{n-p+1}\,dd^*\,\psi\;. \tag{2.11}$$

This proposition implies that there are no conformal Killing forms on compact manifolds, where $q(R)$ has only negative eigenvalues. This is the case on manifolds with constant negative sectional curvature or on conformally flat manifolds with negative-definite Ricci tensor. For coclosed forms, Proposition 2.4 is a generalization of the well-known characterization of Killing vector fields on compact manifolds, as divergence free vector fields in the kernel of $\Delta - 2\,\mathrm{Ric}$. In the general case, it can be reformulated as

Corollary 2.5 *Let $(M^n,\,g)$ a compact Riemannian manifold with a coclosed p–form ψ. Then ψ is a Killing form if and only if*

$$\Delta\,\psi \;=\; \frac{p+1}{p}\,q(R)\,\psi\;.$$

We note that there are similar results for $*$–Killing forms and for conformal Killing m-forms on 2m-dimensional manifolds.

A further interesting property of the equation defining conformal Killing forms is its conformal invariance (c.f. [4]). The precise formulation is

Proposition 2.6 *Let $(M^n,\,g)$ be a Riemannian manifold with a conformal Killing p–form ψ. Then $\widehat{\psi} := e^{(p+1)\lambda}\psi$ is a conformal Killing p–form with respect to the conformally equivalent metric $\hat{g} := e^{2\lambda}g$.*

In particular, it follows from this proposition that the Lie derivative with respect to conformal vector fields preserves the space of conformal Killing forms.

There is still another characterization of conformal Killing forms which is often given as the definition.

Proposition 2.7 *Let $(M^n,\,g)$ be a Riemannian manifold. A p–form ψ is a conformal Killing form if and only if there exists a $(p-1)$–form θ such that*

$$\begin{aligned}
&(\nabla_Y\,\psi)(X, X_2, \ldots, X_p) \;+\; (\nabla_X\,\psi)(Y, X_2, \ldots, X_p)\\
&\qquad = 2g(X,\,Y)\,\theta(X_2,\ldots,X_p) \;-\; \sum_{a=2}^{p}(-1)^a\,\Big(g(Y,\,X_a)\,\theta(X,\,X_2,\ldots,\hat{X}_a,\ldots,\,X_p)\\
&\qquad\qquad +\; g(X,\,X_a)\,\theta(Y,\,X_2,\ldots,\hat{X}_a,\ldots,\,X_p)\Big)
\end{aligned}$$

for any vector fields $Y, X, X_1, \dots X_p$, where $\hat{X}_a$ means that X_a is omitted.

It was already mentioned in the introduction that the interest in Killing forms in relativity theory stems from the fact that they define first integrals of the geodesic equation. At the end of this chapter, we will now describe this construction in more detail. Let ψ be a Killing p-form and let γ be a geodesic, i.e. $\nabla_{\dot\gamma}\dot\gamma = 0$. Then

$$\nabla_{\dot\gamma}(\dot\gamma \lrcorner \psi) \;=\; (\nabla_{\dot\gamma}\dot\gamma) \lrcorner \psi \;+\; \dot\gamma \lrcorner \nabla_{\dot\gamma}\psi \;=\; 0 \;,$$

i.e. $\dot\gamma \lrcorner \psi$ is a $(p-1)$–form parallel along the geodesic γ and in particular its length is constant along γ. The definition of this constant can be given in a more general context. Indeed for any p-form ψ we can consider a symmetric bilinear form K_ψ defined for any vector fields X, Y as

$$K_\psi(X,\,Y) \;:=\; g(X \lrcorner \psi,\, Y \lrcorner \psi\,) \;.$$

For Killing forms the associated bilinear form has a very nice property.

Lemma 2.8 *If ψ is a Killing form, then the associated symmetric bilinear form K_ψ is a Killing tensor, i.e. for any vector fields X, Y, Z it satisfies the equation*

$$(\nabla_X K_\psi)(Y,\,Z) \;+\; (\nabla_Y K_\psi)(Z,\,X) \;+\; (\nabla_Z K_\psi)(X,\,Y) \;=\; 0 \;. \tag{2.12}$$

In particular, $K_\psi(\dot\gamma,\,\dot\gamma)$ is constant along any geodesic γ.

In general, a $(0, k)$–tensor $\mathcal{T}$ is called *Killing tensor* if the complete symmetrization of $\nabla\mathcal{T}$ vanishes. This is equivalent to $(\nabla_X \mathcal{T})(X, \dots, X) = 0$. It follows again that for such a Killing tensor, the expression $\mathcal{T}(\dot\gamma, \dots, \dot\gamma)$ is constant along any geodesic γ and hence defines a k-th order first integral of the geodesic equation. Note that the length of the $(p-1)$–form $X \lrcorner \psi$ is $K_\psi(X, X)$ and that $\mathrm{tr}(K_\psi) = p\,|\psi|^2$.

3 Examples of conformal Killing forms

We start with parallel forms which are obviously in the kernel of the twistor operator and thus are conformal Killing forms. Using Proposition 2.6, we see that with any parallel p-form ψ, the form $\widehat{\psi} := e^{(p+1)\lambda}\,\psi$ is a conformal Killing p–form with respect to the conformally equivalent metric $\widehat{g} := e^{2\lambda}\,g$. This new form $\widehat{\psi}$ is in general no longer parallel.

Conformal Killing forms were introduced as a generalization of conformal vector fields, i.e. we have the following well-known result.

Proposition 3.1 *Let $(M,\,g)$ be a Riemannian manifold. Then a vector field ξ is dual to a conformal Killing 1-form if and only if it is a conformal vector field, i.e. if there exists a function f such that $\mathcal{L}_\xi\, g = f\,g$. Moreover, ξ is dual to a Killing 1-form if and only if it is a Killing vector field, i.e. if $\mathcal{L}_\xi\, g = 0$.*

The simplest examples of manifolds with conformal Killing forms are the spaces of constant curvature. We will recall the result for the standard sphere $(S^n,\,g)$ with scalar curvature $s = n(n-1)$. The spectrum of the Laplace operator on p-forms consists of two series:

$$\lambda'_k \;=\; (p+k)(n-p+k+1) \qquad\qquad \text{and} \qquad\qquad \lambda''_k \;=\; (p+k+1)(n-p+k) \;,$$

where $k = 0, 1, 2, \ldots$. The eigenvalues λ'_k correspond to closed eigenforms, whereas the eigenvalues λ''_k correspond to coclosed eigenforms. The multiplicities of the eigenvalues are well–known. In particular, we find for the minimal eigenvalues λ'_0 and λ''_0 that

$$\lambda'_0 \quad \text{has multiplicity} \quad \binom{n+1}{p} \qquad \text{and} \qquad \lambda''_0 \quad \text{has multiplicity} \quad \binom{n+1}{p+1} .$$

The conformal Killing forms turn out to be sums of eigenforms of the Laplacian corresponding to the minimal eigenvalues on $\ker(d)$ resp. $\ker(d^*)$.

Proposition 3.2 *A p–form ω on the standard sphere (S^n, g) is a conformal Killing form, if and only if it is a sum of eigenforms for the eigenvalue λ'_0 resp. of eigenforms for the eigenvalue λ''_0.*

The first interesting class of manifolds admitting conformal Killing forms are Sasakian manifolds. These are contact manifolds satisfying a normality (or integrability) condition. In the context of conformal Killing forms, it is convenient to use the following

Definition 3.3 *A Riemannian manifold (M, g) is called a Sasakian manifold, if there exists a unit length Killing vector field ξ satisfying for any vector field X the equation*

$$\nabla_X \, (d\,\xi^*) \;=\; -\,2\,X^* \wedge \,\xi^* \,. \tag{3.1}$$

Note that in the usual definition of a Sasakian structure, as a special contact structure one has the additional condition $\phi^2 \,=\, -\,\mathrm{id} \,+\, \eta \otimes \xi$ for the associated endomorphism $\phi = -\nabla\xi$ and the 1-form $\eta := \xi^*$. But this equation is implied by (3.1), if we write (3.1) first as

$$(\nabla_X \phi)(Y) \;=\; g(X,\,Y)\,\xi \;-\; \eta(Y)\,X \;, \tag{3.2}$$

and take then the scalar product with ξ. It follows that the dimension of a Sasakian manifolds has to be odd and if $\dim(M) = 2n+1$, then $\xi^* \wedge \,(d\,\xi^*)^n$ is the Riemannian volume form on M.

There are many examples of Sasakian manifolds, e.g. given as S^1–bundles over Kähler manifolds. Even in the special case of *3-Sasakian* manifolds, where one has three unit length Killing vector fields, defining Sasakian structures satisfying the $SO(3)$-commutator relations, one knows that there are infinitely many diffeomorphism types (c.f. [8]).

On a manifold with a Killing vector field ξ we have the Killing 1-form ξ^*. It is then natural to ask whether $d\,\xi^*$ is also a conformal Killing form. The next proposition shows that for Einstein manifolds this is the case, if and only if ξ defines a Sasakian structure. Slightly more general, we have

Proposition 3.4 *Let (M, g) be a Riemannian manifold with a Sasakian structure defined by a unit length vector field ξ. Then the 2–form $d\,\xi^*$ is a conformal Killing form. Moreover, if (M^n, g) is an Einstein manifold with scalar curvature s normalized to $s = n(n-1)$ and if ξ is a unit length Killing vector field such that $d\,\xi^*$ is a conformal Killing form, then ξ defines a Sasakian structure.*

Proof. We first prove that for a Killing vector field ξ defining a Sasakian structure, the 2-form $d\xi^*$ is a conformal Killing form. From the definition (3.1) of the Sasakian structure we obtain: $d^*d\xi^* = 2(n-1)\,\xi^*$. Substituting ξ^* in (3.1) using this formula yields:

$$\nabla_X\,(d\xi^*) \;=\; -\,2X^* \wedge \tfrac{1}{2(n-1)}\,d^*d\xi^* \;=\; -\tfrac{1}{n-1}\,X^* \wedge d^*d\xi^*\;.$$

But since $d\xi^*$, is closed this equation implies that $d\xi^*$ is indeed a conformal Killing form. To prove the second statement, we first note that $d^*d\xi^* = \Delta\xi^* = 2\mathrm{Ric}\,(\xi^*) = 2(n-1)\xi^*$ because of equation (2.11) for Killing 1-forms and the assumption that $(M,\,g)$ is an Einstein manifold with normalized scalar curvature. Then we can reformulate the condition that $d\xi^*$ is a closed conformal Killing form to obtain

$$\nabla_X\,(d\xi^*) \;=\; -\tfrac{1}{n-1}\,X^* \wedge d^*d\xi^* \;=\; -\tfrac{1}{n-1}\,X^* \wedge 2(n-1)\xi^* \;=\; -2\,X^* \wedge \xi^*\;,$$

i.e. the unit length Killing vector field ξ also satisfies the equation (3.1) and thus defines a Sasakian structure. □

We know already that on a Sasakian manifold defined by a Killing vector field ξ, the dual 1-form ξ^* and the 2-form $d\xi^*$ are both conformal Killing forms. In fact, the same is true for all possible wedge products of ξ^* and $d\xi^*$. We have

Proposition 3.5 *Let $(M^{2n+1},\,g,\,\xi)$ be a Sasakian manifold with Killing vector field ξ. Then*

$$\omega_k \;:=\; \xi^* \wedge (d\xi^*)^k$$

is a Killing $(2k+1)$–form for $k=0,\ldots n$. Moreover, ω_k satisfies for any vector field X and any k the additional equation

$$\nabla_X(d\omega_k) \;=\; -\,2\,(k+1)\,X^* \wedge \omega_k.$$

In particular, ω_k is an eigenform of the Laplace operator corresponding to the eigenvalue $4(k+1)(n-k)$.

This can be proved by a simple local calculation. However, it is also part of a more general property which we will further discuss in Section 4.

Recall that a form ψ on a Sasakian manifold is called *horizontal* if $\xi \lrcorner \psi = 0$, where ξ is the vector field defining the Sasakian structure. In [28] resp. [29] S. Yamaguchi proved the following

Theorem 3.6 *Let $(M,\,g)$ be a compact Sasakian manifold, then*

1. *any horizontal conformal Killing form of odd degree is Killing, and*
2. *any conformal Killing form of even degree has a unique decomposition into the sum of a Killing form and a $*$–Killing form.*

We will now describe a general construction which provides new examples of Killing forms in degrees 2 and 3. For this aim we have to recall the notion of a vector cross product (c.f. [14]). Let V be a finite dimensional real vector space and let $\langle\cdot,\cdot\rangle$ be a non-degenerate bilinear form on V. Then a *vector cross product* on V is defined as a linear map $P : V^{\otimes r} \to V$ satisfying the axioms

$$\begin{array}{llcll} (i) & \langle P(v_1, \ldots, v_r), v_i\rangle & = & 0 & (1 \le i \le r), \\ (ii) & |P(v_1, \ldots, v_r)|^2 & = & \det(\langle v_i, v_j\rangle) \ . & \end{array}$$

Vector cross products are completely classified. There are only four possible types: 1-fold and (n-1)-fold vector cross products on n-dimensional vector spaces, 2-fold vector cross products on 7-dimensional vector spaces and 3-fold vector cross products on 8-dimensional vector spaces. We will consider r-fold vector cross products on Riemannian manifolds (M, g). These are tensor fields of type $(r, 1)$ which are fibrewise r-fold vector cross products. As a special class, one has the so-called *nearly parallel* vector cross products. By definition they satisfy the differential equation

$$(\nabla_{X_1} P)(X_1, \ldots, X_r) \;=\; 0$$

for any vector fields $X_1, \ldots, X_r$. Together with an r-fold vector cross product P, one has an associated $(r+1)$-form ω defined as

$$\omega(X_1, \ldots, X_{r+1}) \;=\; g(P(X_1, \ldots, X_r), X_{r+1}) \ .$$

The definition of a nearly parallel vector cross product is obviously equivalent to the condition $X \lrcorner \nabla_X \omega = 0$ for the associated form. Hence, we obtain

Lemma 3.7 *Let P be a nearly parallel r-fold vector cross product with associated form ω. Then ω is a Killing $(r+1)$-form.*

We will examine the four possible types of vector cross products to see which examples of manifolds with Killing forms one can obtain. We start with 1-fold vector cross products, which are equivalent to almost complex structures compatible with the metric. Hence, a Riemannian manifold (M, g) with a nearly parallel 1-fold vector cross product J is the same as an almost Hermitian manifold, where the almost complex structure J satisfies $(\nabla_X J) X = 0$ for all vector fields X. Such manifolds are called *nearly Kähler*. It follows from Lemma 3.7 that the associated 2-form ω defined by $\omega(X, Y) = g(JX, Y)$ is a Killing 2-form. On a Kähler manifold, ω is the Kähler form and thus parallel by definition. But there are also many non-Kähler, nearly Kähler manifolds, e.g. the 3-symmetric spaces which were classified by A. Gray and J. Wolf (c.f. [15]). Due to a result of S. Salamon (c.f. [7]) nearly Kähler, non-Kähler manifolds are never Riemannian symmetric spaces.

Next, we consider 2-fold vector cross products. They are defined on 7 dimensional Riemannian manifolds and exist, if and only if the structure group of the underlying manifold M can be reduced to the group $G_2 \subset O(7)$, i.e. if M admits a topological G_2-structure. Riemannian manifolds with a nearly parallel 2-fold vector cross product are called *weak G_2–manifolds.* There are many examples of homogeneous and non-homogeneous G_2–manifolds, e.g. on any 7-dimensional 3-Sasakian manifold. Here exists a canonically defined (additional) Einstein metric which is weak-G_2 (c.f. [11]).

Finally, we have to consider the $(n-1)$-fold and 3-fold vector cross products. But in these cases, results of A. Gray show that the associated forms have to be parallel (c.f. [14]). Hence, they yield only trivial examples of conformal Killing forms.

We have seen that nearly Kähler manifolds are special almost Hermitian manifolds where the Kähler form ω, defined by $\omega(X,Y) = g(JX,Y)$, is a Killing 2-form. This leads to the natural question whether there are other almost Hermitian manifolds, where the Kähler form is a conformal Killing form. The following proposition gives an answer to this question.

Proposition 3.8 *Let (M^{2n}, g, J) be an almost Hermitian manifold. Then the Kähler form ω is a conformal Killing 2-form if and only if the manifold is nearly Kähler or Kähler.*

Proof. Let Λ denote the contraction with the 2-form ω, i.e. $\Lambda = \frac{1}{2}\sum Je_i \lrcorner\, e_i \lrcorner$. On an almost Hermitian manifold (with Kähler form ω), one has the following well known formulas:

$$\Lambda(d\omega) \;=\; J(d^*\omega) \qquad \text{and} \qquad d\omega \;=\; (d\omega)_0 \;+\; \tfrac{1}{n-1}\,(Jd^*\omega)\,\wedge\,\omega\;,$$

where $(d\omega)_0$ denotes the effective or primitive part, i.e. the part of $d\omega$ in the kernel of Λ. We will show that if ω is a conformal Killing 2-form, then it has to be coclosed. The defining equation of a Killing 2-form reads

$$(\nabla_X\,\omega)(A,\,B) \;=\; \tfrac{1}{3}\,d\omega(X,\,A,\,B) \;-\; \tfrac{1}{2n-1}\;(g(X,\,A)\,d^*\omega(B) \;-\; g(X,\,B)\,d^*\omega(A))\;.$$

Because $\nabla_X J\circ J + J\circ\nabla_X J = 0$ we see that $\nabla_X\omega$ is an anti-invariant 2-form. Setting $X = e_i$ and $A = Je_i$ and summing over an ortho-normal basis $\{e_i\}$ we obtain

$$\begin{aligned} -\,d^*\omega(JB) \;&=\; \tfrac{1}{3}\sum d\omega(e_i,\,Je_i,\,B) \;+\; \tfrac{1}{2n-1}\sum g(e_i,\,B)\,d^*\omega(Je_i)\\ &=\; \tfrac{2}{3}\,\Lambda(d\omega) \;+\; \tfrac{1}{2n-1}\,d^*\omega(JB)\\ &=\; (\tfrac{1}{2n-1} \;-\; \tfrac{2}{3})\,d^*\omega(JB)\;. \end{aligned}$$

From this equation follows immediately $d^*\omega = 0$, i.e. ω is already a Killing 2-form. But this is equivalent for (M,g,J) to be nearly Kähler, where we consider Kähler manifolds as a special case of nearly Kähler manifolds. $\Box$

4 Special Killing forms

In [24] S. Tachibana and W. Yu introduced the notion of special Killing forms. This definition seemed to be rather restrictive and indeed the only discussed examples were spaces of constant curvature. Nevertheless, it turns out that almost all examples of Killing forms described in the preceding section are special and we will now show that there are only a few further examples.

From the following definition it becomes clear that the restriction from Killing forms to special Killing forms is analogous to the restriction from Killing vector fields to Sasakian structures.

Definition 4.1 *A special Killing form is a Killing form ψ which for some constant c and any vector field X satisfies the additional equation*

$$\nabla_X\,(d\psi) \;=\; c\,X^*\,\wedge\,\psi\;. \qquad (4.1)$$

There is an equivalent version of equation (4.1), which gives a definition closer to the original one. Indeed, a special Killing form can be defined equivalently as a Killing form satisfying for some (new) constant c and for any vector fields X, Y the equation

$$\nabla^2_{X,Y}\psi \;=\; c\,(g(X,Y)\,\psi \;-\; X \wedge Y \lrcorner \psi)\;. \tag{4.2}$$

From equation (4.1) it follows immediately that special Killing p–forms are eigenforms of the Laplacian corresponding to the eigenvalue $-c(n-p)$. Hence, on compact manifolds the constant c has to be negative.

Our first examples of special Killing forms came from Sasakian manifolds. Here the defining equation (3.1) coincides with equation (4.1) for the constant $c = -2$, i.e. a Killing vector field ξ defining a Sasakian structure is dual to a special Killing 1-form with constant $c = -2$. Moreover, we have seen in Proposition 3.5 that on a Sasakian manifold also the forms $\omega_k := \xi^* \wedge (d\xi^*)^k$ are special Killing forms. All other known examples are given in

Proposition 4.2 *The following manifolds admit special Killing forms:*

1. *Sasakian manifolds with defining Killing vector field* ξ*. Here all the Killing forms* $\omega_k = \xi^* \wedge (d\xi^*)^k$ *are special with constant* $c = -\,2(k+1)$.
2. *Nearly Kähler non-Kähler manifolds in dimension 6. Here the associated 2–form* ω *is special with constant* $c = -\,\frac{s}{10}$ *and the 3-form* $*\,d\,\omega$ *is special with constant* $c = -\,\frac{2s}{15}$*, where* s *denotes the scalar curvature.*
3. *Weak* G_2*–manifolds of scalar curvature* s*. Here the associated 3–form is a special Killing form of constant* $c = -\,\frac{2\,s}{21}$.
4. *The standard sphere* S^n *of scalar curvature* $s = n(n-1)$*. Here all Killing* p*–forms, i.e. all coclosed minimal eigenforms of the Laplacian are special with constant* $c = -\,(p+1)$.

Note that weak G_2–manifolds and the nearly Kähler non-Kähler manifolds in dimension 6 are Einstein manifolds, hence they have constant scalar curvature. One can easily see that the associated 2–form on a nearly Kähler manifold of dimension different from 6 is never special.

We will now give a complete description of compact Riemannian manifolds admitting special Killing forms. It turns out that a p-form on M is a special Killing form if and only if it induces a $(p+1)$–form on the metric cone $\widehat{M}$ which is parallel. Since the metric cone is either flat or irreducible, the description of special Killing forms is reduced to a holonomy problem, i.e. to the question which holonomies admit parallel forms. This question can be completely answered and the existence of parallel forms on the cone can be retranslated into the existence of special geometric structures on the base manifold. The result will be that special Killing forms can exist only on Sasakian manifolds, nearly Kähler manifolds or weak G_2–manifolds. Our approach here is similar to the one of Ch. Bär in [2] which lead to the classification of Killing spinors.

The metric cone $\widehat{M}$ over a Riemannian manifold (M, g) is defined as a warped product, i.e. $\widehat{M} = M \times \mathbb{R}_+$ with metric $\widehat{g} := r^2 g + dr^2$. An easy calculation shows that the

Levi-Civita connection on 1–forms is given by

$$\widehat{\nabla}_X Y^* \;=\; \nabla_X Y^* \;-\; \tfrac{1}{r}\, g(X,\,Y)\,dr, \qquad \widehat{\nabla}_X\,dr \;=\; r\,X^*\;,$$
$$\widehat{\nabla}_{\partial_r} X^* \;=\; -\tfrac{1}{r}\,X^*, \qquad \widehat{\nabla}_{\partial_r}\,dr \;=\; 0\;,$$

where X, Y are vector fields tangent to M with g–dual 1–forms X^*, Y^*, and where ∂_r is the radial vector field on $\widehat{M}$ with $dr(\partial_r) = 1$. From this we immediately obtain the following useful formulas

$$\widehat{\nabla}_X\,\psi \;=\; \nabla_X\,\psi \;-\; \tfrac{1}{r}\,dr \,\wedge\, (X \lrcorner\, \psi), \qquad \widehat{\nabla}_{\partial_r}\,\psi \;=\; -\tfrac{p}{r}\,\psi\;,$$

where ψ is a p-form on M considered as p–form on $\widehat{M}$. For any p–form ψ on M, we define an associated $(p+1)$–form $\widehat{\psi}$ on $\widehat{M}$ by

$$\widehat{\psi} \;:=\; r^p\,dr \,\wedge\, \psi \;+\; \tfrac{r^{p+1}}{p+1}\,d\psi\;. \tag{4.3}$$

The next lemma is our main technical tool for the classification of special Killing forms. It states that special Killing forms are exactly those forms which translate into parallel forms on the metric cone.

Lemma 4.3 *Let $(M,\,g)$ be a Riemannian manifold and let ψ be a p–form on M. Then the associated $(p+1)$–form $\widehat{\psi}$ on the metric cone $\widehat{M}$ is parallel with respect to $\widehat{\nabla}$ if and only if*

$$\nabla_X\,\psi \;=\; \tfrac{1}{p+1}\,X \lrcorner\, d\psi \qquad \textit{and} \qquad \nabla_X\,(d\psi) \;=\; -\,(p+1)\,X^* \,\wedge\, \psi$$

i.e. $\widehat{\psi}$ is parallel if and only if ψ is a special Killing form with constant $c = -(p+1)$.

Proof. We will first show that a $(p+1)$–form $\hat{\psi}$ defined on the metric cone as in (4.3) is always parallel in radial direction. Indeed we have

$$\begin{aligned}
\widehat{\nabla}_{\partial_r}\,\psi \;&=\; p\,r^{p-1}\,dr \wedge \psi \;+\; r^p\,dr \wedge \widehat{\nabla}_{\partial_r}\,\psi \;+\; r^p\,d\omega \;+\; \tfrac{r^{p+1}}{p+1}\,\widehat{\nabla}_{\partial_r}\,(d\psi)\\
&=\; (p\,r^{p-1} - r^p\,\tfrac{p}{r})\,dr \,\wedge\, \psi \;+\; (r^p - \tfrac{r^{p+1}}{p+1}\,\tfrac{1}{r}\,(p+1))\,d\psi\\
&=\; 0\;.
\end{aligned}$$

Next, we compute the covariant derivative of $\widehat{\psi}$ in direction of a horizontal vector field X. This yields

$$\begin{aligned}
\widehat{\nabla}_X\,\psi \;&=\; r^p\,\widehat{\nabla}_X\,(dr) \wedge \psi \;+\; r^p\,dr \wedge \widehat{\nabla}_X\,\psi \;+\; \tfrac{r^{p+1}}{p+1}\,\widehat{\nabla}_X\,(d\psi)\\
&=\; r^{p+1}\,X^* \wedge \psi \;+\; r^p\,dr \wedge \nabla_X\,\psi \;+\; \tfrac{r^{p+1}}{p+1}\,\nabla_X\,(d\psi) \;-\; \tfrac{r^p}{p+1}\,dr \wedge (X \lrcorner\, d\psi)\\
&=\; r^{p+1}\,\Big(X^* \wedge \psi \;+\; \tfrac{1}{p+1}\,\nabla_X\,(d\psi)\Big) \;+\; r^p\,dr \wedge \Big(\nabla_X\,\psi \;-\; \tfrac{1}{p+1}\,X \lrcorner\, d\psi\Big)\;.
\end{aligned}$$

From this equation it becomes clear that $\widehat{\psi}$ is parallel, if and only if the two brackets vanish, i.e. if and only if the form ψ on M is a special Killing form. $\Box$

We already know that on Sasakian manifolds, the Killing 1-form ξ^* together with all forms $\xi^* \wedge (d\xi^*)^k$ are special Killing forms. As an immediate corollary of Lemma 4.3 we see that a similar statement is true for all manifolds admitting special Killing forms of odd degree. Note that we have to assume the Killing form ψ to be of odd degree, since otherwise $d\psi \wedge d\psi = 0$ and we could not obtain a new Killing form.

Lemma 4.4 *Let ψ be a special Killing form of odd degree p, then all the forms*

$$\psi_k \;:=\; \psi \wedge (d\psi)^k \qquad\qquad k = 0, \dots$$

are special Killing forms of degree $p + k(p+1)$.

Proof. Let $\widehat{\psi}$ be the parallel form associated with the special Killing form ψ. Then the form $\widehat{\psi}_k$ associated to ψ_k turns out to be $\frac{(p+1)^k}{k+1}\,\widehat{\psi}^{\,k+1}$, which is again parallel. Hence, ψ_k is a special Killing form. $\square$

In the proof of the lemma we have used that the power of the associated form $\widehat{\psi}$ is again parallel and can be written as associated form for some other special Killing form. The following lemma will show that this is a general fact, i.e. we have an simple characterization of all parallel forms on the metric cone. It turns out that there are no other parallel forms on the cone as the ones corresponding to special Killing forms on the base manifold.

Lemma 4.5 *Let ω be a form on the metric cone $\widehat{M}$. Then ω is parallel with respect to $\widehat{\nabla}$ if and only if there exists a special Killing form ψ on M such that $\omega = \hat{\psi}$.*

Proof. We know already that $\widehat{\psi}$ is parallel on the metric cone, provided that ψ is a special Killing form on M. It remains to verify the opposite direction. Assuming ω to be a parallel form on the cone we write it as

$$\omega \;=\; \omega_0 \;+\; dr \wedge \omega_1 \;,$$

where we consider ω_0 and ω_1 as a r-dependent family of forms on M. It is clear that ω is parallel in the radial direction ∂_r if and only if the same is true for the two forms ω_0 and ω_1. Let $\eta = \eta(r)$ be any horizontal p-form on $\widehat{M}$ considered as family of forms on M. Locally we can write $\eta = \sum r^p f_I(r,x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, with multi index $I = (i_1, \ldots, i_p)$. Then η is parallel in radial direction if and only if

$$\begin{aligned}
0 \;&=\; \partial_r(r^p f_I(r,x)) \;+\; r^p f_I(r,x)\,(-\tfrac{p}{r}\,) \\
&=\; p\, r^{p-1} f_I(r,x) \;+\; r^p\, \partial_r(f_I(r,x)) \;-\; r^{p-1} p\, f_I(r,x) \\
&=\; r^p\, \partial_r(f_I(r,x)) \;.
\end{aligned}$$

It follows that $f_I(r,x)$ does not depend on r. Hence, we can write $\eta = r^p\,\eta_0$, where η_0 is a p-form on M. In particular, we have $\omega_0 = r^{p+1}\omega_0^M$ and $\omega_1 = r^p\,\omega_1^M$, where ω_0^M and ω_1^M are forms on M. Next, we consider the covariant derivative of the parallel form ω in direction of a horizontal vector field X. Here we obtain

$$\begin{aligned}
\widehat{\nabla}_X\,\omega \;&=\; r^{p+1}\widehat{\nabla}_X\,\omega_0^M \;+\; r^{p+1}\,X^* \wedge \omega_1^M \;+\; r^p dr \wedge \widehat{\nabla}_X\,\omega_1^M \\
&=\; r^{p+1}\left(\nabla_X\,\omega_0^M \;-\; \tfrac{1}{r}\,dr \wedge (X \lrcorner\; \omega_0^M)\right) \\
&\qquad\qquad + r^{p+1} X^* \wedge \omega_1^M \;+\; r^p dr \wedge \nabla_X\,\omega_1^M \;.
\end{aligned}$$

From this we conclude that the form $\omega = r^p\, dr \wedge \omega_1^M + r^{p+1}\omega_0^M$ is parallel if and only if the following two equations are satisfied for all vector fields X on M

$$\nabla_X\, \omega_1^M \;=\; X \lrcorner\, \omega_0^M \qquad \text{and} \qquad \nabla_X\, \omega_0^M \;=\; -\,X^* \,\wedge\, \omega_1^M\ . \tag{4.4}$$

Using these equations we immediately find:

$$d\,\omega_0^M \;=\; 0 \;=\; d^*\omega_1^M,\quad d\,\omega_1^M = (p+1)\,\omega_0^M,\quad d^*\omega_0^M = (n-p)\omega_0^M\ .$$

In particular, we have $\Delta\,\omega_1^M \;=\; (p+1)(n-p)\,\omega_1^M$ and it is clear that $\omega = \widehat{\psi}$ for the special Killing p-form $\psi = \omega_1^M$. $\square$

We have seen that the map $\psi \mapsto \widehat{\psi}$ defines a 1-1-correspondence between special Killing p-forms on M and parallel $(p+1)$-forms on the metric cone $\widehat{M}$. We will use this fact to describe manifolds admitting special Killing forms. Let M be a compact oriented simply connected manifold, then the metric cone $\widehat{M}$ is either flat, and the manifold M has to be isometric to the standard sphere, or the cone is irreducible (c.f. [2] or [9]). In the latter case we know from the holonomy theorem of M. Berger that $\widehat{M}$ is either symmetric or its holonomy is one of the the following groups: $\mathrm{SO}(m)$, $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$, $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 or Spin_7. An irreducible symmetric space as well as a manifold with holonomy $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ is automatically Einstein (c.f. [5]). But it follows from the O'Neill formulas applied to the cone, that $\widehat{\mathrm{Ric}}\,(\partial_r,\, \partial_r) = 0$, i.e. the metric cone can only be Einstein if it is Ricci-flat. In this case the symmetric space has to be flat and the holonomy $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ restricts further to $\mathrm{Sp}(m)$ (this again can be found in c.f. [5]).

Let $(M,\, g)$ be a compact oriented simply connected manifold not isometric to the sphere. If ψ is a special Killing form on M then the metric cone $\widehat{M}$ is an irreducible manifold with a parallel form $\widehat{\psi}$. Since any parallel form induces a holonomy reduction, we see that the above list of possible holonomies is further reduced to $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2, or Spin_7. We will now go through this list and determine what are the possible parallel forms and how they translate into special Killing forms on M. The description of possible parallel forms can be found in [5], with the only exception of holonomy $\mathrm{Sp}(m)$. Nevertheless, in this case the parallel forms can be described using the realization of $\mathrm{Sp}(m)$–representation due to H. Weyl (the result is also contained in [12]). Concerning the translation from special holonomy on $\widehat{M}$ to special geometric structures on M we refer to [2], where the explicit constructions are described.

The first case, i.e. holonomy $\mathrm{U}(m)$, is equivalent to $\widehat{M}$ being a Kähler manifold. In this case all parallel forms are linear combinations of powers of the Kähler form. On the other hand, it is well-known that $\widehat{M}$ is Kähler, if and only if M is a Sasakian manifold. If the Killing vector field ξ defines the Sasakian structure on M, then $\widehat{\xi} = rdr \wedge \xi^* + \frac{r^2}{2}d\xi^*$ defines the Kähler form on $\widehat{M}$. Hence, all special Killing forms on a Sasakian manifold are spanned by the forms ω_k given in Proposition 3.5, and they all correspond to the powers of the Kähler form on $\widehat{M}$.

In the next case, $\widehat{M}$ has holonomy $\mathrm{SU}(m)$ and equivalently is Ricci-flat and Kähler. In this situation, there are two additional parallel forms given by the complex volume form and its conjugate. As real forms we obtain the real part resp. the imaginary part of the complex volume form. Because of the O'Neill formulas, the cone is Ricci-flat, if and only if the base manifold is Einstein, i.e. in this case our manifold is Einstein-Sasakian. As special Killing forms we have the forms ω_k and two additional forms of degree m, which can also be described using the Killing spinors of an Einstein-Sasakian manifold.

In the third case, $\widehat{M}$ has holonomy $\mathrm{Sp}(m)$ and is by definition a hyper-Kähler manifold, i.e. there are three Kähler forms compatible with the metric and such that the corresponding complex structures satisfy the quaternionic relations. Here, all parallel forms are linear combinations of wedge products of powers of the three Kähler forms (c.f. [12]). The metric cone is hyper-Kähler if and only if the base manifold has a 3-Sasakian structure and the possible special Killing forms are described by

Proposition 4.6 *Let (M, g) be a manifold with a 3-Sasakian structure defined by the Killing 1–forms η_1, η_2 and η_3. Then all special Killing forms on M are linear combinations of the forms $\psi_{a,b,c}$ defined for any integers (a,b,c) by*

$$\begin{aligned}\psi_{a,b,c} := \tfrac{a}{a+b+c}\,[\eta_1 \wedge (d\eta_1)^{a-1}] \wedge (d\eta_2)^b \wedge (d\eta_3)^c \\ + \tfrac{b}{a+b+c}\,(d\eta_1)^a \wedge [\eta_2 \wedge (d\eta_2)^{b-1}] \wedge (d\eta_3)^c \\ + \tfrac{c}{a+b+c}\,(d\eta_1)^a \wedge (d\eta_2)^b \wedge [\eta_3 \wedge (d\eta_3)^{c-1}]\ .\end{aligned}$$

Proof. Let ϕ_i be the parallel 2-form associated with the Sasakian structure η_i, for $i = 1, 2, 3$, i.e.

$$\phi_i \;=\; r\,dr \wedge \eta_i \;+\; \tfrac{r^2}{2}\,d\eta_i\ .$$

Then it follows from a simple computation that $\phi_1^a \wedge \phi_2^b \wedge \phi_3^c$ is a parallel form which is, up to a factor, associated to the form $\psi_{a,b,c}$ defined above. $\Box$

Next, we have to consider the two exceptional holonomies G_2 resp. Spin_7. These holonomies are defined by the existence of a parallel 3– resp. 4–form ψ and the only non-trivial parallel forms on such a manifold are the linear combinations of ψ and $*\psi$. The metric cone has holonomy G_2 if and only if the base manifold is a 6-dimensional nearly Kähler manifold. Here, the parallel 3-form ψ translates into the Kähler form ω and the parallel 4-form $*\psi$ translates, up to a constant, into the 3-form $*d\omega$. To make this more precise, we note the following simple fact

Lemma 4.7 *Let ω be a p-form on M considered as p-form on the metric cone $\widehat{M}$. Then the Hodge star operators of M and $\widehat{M}$ are related by*

$$*_{\widehat{M}}\,\omega \;=\; r^{n-2p}(*_M\omega) \wedge dr\ .$$

Now, back to the nearly Kähler case, let $\psi = r^2 dr \wedge \omega + \frac{r^3}{3}d\omega$ be the parallel 3-form associated with the Kähler form ω. As in the proof of Lemma 4.5 we conclude $\Delta\omega = 12\omega$. Hence, the scalar curvature s_M of the 6-dimensional nearly Kähler manifold is normalized to $s_M = 30$. Applying the lemma above yields

$$\begin{aligned}*_{\widehat{M}}\,\psi \;&=\; r^2 *_{\widehat{M}}(dr \wedge \omega) \;+\; \tfrac{r^3}{3} *_{\widehat{M}}(d\omega) \\ &=\; r^2\,\partial_r \lrcorner\,(*_{\widehat{M}}\,\omega) \;+\; \tfrac{r^3}{3} *_{\widehat{M}}(d\omega) \;=\; r^4 *_M \omega \;+\; \tfrac{r^3}{3}\,(*_M\, d\omega) \wedge dr\ .\end{aligned}$$

Since $\Delta\,\omega = 12\,\omega$ and $d^*\omega = 0$ it follows $d^*d\omega = - *_M d *_M d\omega = 12\omega$ and we obtain $d(*_M d\omega) = -12 *_M \omega$. Substituting this into the equation for $*_{\widehat{M}}\,\psi$, we find

$$*_{\widehat{M}}\,\psi \;=\; -\,\tfrac{r^4}{12}\,d(*_M d\omega) \;-\; \tfrac{r^3}{3}\,dr \wedge (*_M d\omega)\ .$$

From where we conclude that $*_M d\omega$ is the special Killing form on the nearly Kähler manifold M corresponding to the parallel 4-form $-3 *_{\widehat{M}} \psi$ on $\widehat{M}$.

Finally we have to consider the case of holonomy Spin_7. The metric cone has holonomy Spin_7 if and only if M is a 7-dimensional manifold with a weak G_2-structure. Here the parallel 4-form ψ on the cone is self-dual, i.e. $*\psi = \psi$, and the corresponding special Killing form is just the 3-form defining the weak G_2-structure.

Summarizing our description of compact manifolds with special Killing forms we have the following

Theorem 4.8 *Let (M^n, g) be a compact, simply connected manifold admitting a special Killing form. Then M is either isometric to S^n or M is a Sasakian, 3-Sasakian, nearly Kähler or weak G_2–manifold. Moreover, on these manifolds any special Killing form is a linear combination of the Killing forms described above.*

5 The dimension bound

It is well-known and easy to check that for twistor operator T the operator T^*T is elliptic. Hence, the space of conformal Killing forms is finite dimensional on compact manifolds. However, in this section we will prove that the space of conformal Killing forms is finite dimensional on any connected manifold. More precisely, we have

Theorem 5.1 *Let (M, g) be an n-dimensional connected Riemannian manifold and denote with $\mathcal{C}K^p(M)$ the space of conformal Killing p-forms, then*

$$\dim \mathcal{C}K^p(M) \quad \leq \quad \binom{n+2}{p+1}$$

with equality attained on the standard sphere. Moreover, if a manifold admits the maximal possible number of linear independent conformal Killing p-forms, with $1 < p < n-1$, then it is conformally flat.

The idea of the proof is to construct a vector bundle together with a connection, called *Killing connection*, such that conformal Killing forms are in a 1-1-correspondence to parallel sections for this connection. It then follows immediately that the dimension of the space of conformal Killing forms is bounded by the rank of the constructed vector bundle.

By definition, the covariant derivative of a conformal Killing p–form ψ involves $d\psi$ and $d^*\psi$. Computing the covariant derivatives of $d\psi$ and $d^*\psi$ we obtain an expression involving only ψ and $dd^*\psi$. Finally we have to compute the covariant derivative of $dd^*\psi$ which leads to an expression involving only $\psi, d\psi$ and $d^*\psi$. Collecting the covariant derivatives we can formulate the result of the computations as follows. Let $\hat{\psi} := (\psi,\, d\psi,\, d^*\psi,\, dd^*\psi)$, then $\hat{\psi}$ is a section of $\mathcal{E}^p(M) := \Lambda^p T^*M \oplus \Lambda^{p+1}T^*M \oplus \Lambda^{p-1}T^*M \oplus \Lambda^p T^*M$ and we have $\nabla_X \hat{\psi} = A(X)\,\hat{\psi}$, where $A(X)$ is a certain 4×4-matrix with coefficients which are endomorphisms of the form bundle depending on the vector field X. Here the components of $\nabla_X \hat{\psi}$ are the covariant derivatives of the components of $\hat{\psi}$. The Killing connection $\widetilde{\nabla}$ is then a connection on $\mathcal{E}^p(M)$ defined as $\widetilde{\nabla}_X := \nabla_X - A(X)$ and the conformal Killing forms are by definition the first components of parallel sections of $\mathcal{E}^p(M)$. Hence, the rank of the bundle $\mathcal{E}^p(M)$ is an upper bound on the dimension of the space of conformal Killing forms, i.e.

$$\dim \mathcal{C}K^p(M) \;\leq\; 2\binom{n}{p} \;+\; \binom{n}{p-1} \;+\; \binom{n}{p+1} \;=\; \binom{n+1}{p} \;+\; \binom{n+1}{p+1} \;=\; \binom{n+2}{p+1} .$$

It follows from Proposition 3.2 that this upper bound is attained on the standard sphere. Moreover, if on M exists the maximal possible number of linearly independent conformal Killing forms then the map $\mathcal{E}^p(M) \to \Lambda^p(T^*_xM)$, defined as projection onto the first component and evaluation in the point x, is obviously surjective, i.e. any p-form in $\Lambda^p(T^*_xM)$ can be extended to a conformal Killing form. In this situation, and with $1 < p < n-1$, a curvature calculation shows that the manifold has to be conformally flat (c.f. [17]).

In the remaining part of this section we will show the existence of the Killing connection, which then concludes the proof the Theorem 5.1. The covariant derivatives of $d\psi$ and $d^*\psi$ for a conformal Killing form ψ can be obtained by a direct calculation starting from the definition. In order to give the explicit formulas we introduce the notation $R^+(X)\psi := \sum e_j \wedge R_{X,e_j}\psi$ and $R^-(X)\psi := \sum e_j \lrcorner R_{X,e_j}\psi$, where $\{e_i\}$ is a local ortho-normal basis and ψ is any differential form.

Proposition 5.2 *Let ψ be a conformal Killing p-form, then for all vector fields X*

$$\begin{aligned}\nabla_X(d\psi) &= \tfrac{p+1}{p} R^+(X)\,\psi \;+\; \tfrac{p+1}{p(n-p+1)} X \wedge d\,d^*\psi \\ \nabla_X(d^*\psi) &= -\tfrac{n-p+1}{n-p} R^-(X)\,\psi \;+\; \tfrac{1}{p} X \lrcorner\, d\,d^*\psi \;-\; \tfrac{n-p+1}{p(n-p)} X \lrcorner\, q(R)\,\psi \;.\end{aligned}$$

It remains to show that the covariant derivative of $dd^*\psi$ is an expression only involving $\psi, d\psi$ and $d^*\psi$. With regard to equation (2.6) it suffices to consider $T(dd^*\psi)$. For this we derive Weitzenböck formulas which then also imply Proposition 5.2. The twistor operator T was defined as the composition of the covariant derivative ∇ with the projection $\mathrm{pr}_{\Lambda^{p,1}}$. Similarly we obtain operators like $Td\psi$ or $Td^*\psi$ by applying certain projections to $\nabla^2\psi$. Hence, it suffices to consider relations between such projections which then translate into Weitzenböck formulas for the corresponding differential operators. As a first projection we define

$$\begin{aligned}\mathrm{pr}_1^+ \;:\; T^*M\otimes T^*M\otimes\Lambda^pT^*M \;&\to\; T^*M\otimes\Lambda^{p+1}T^*M \;\to\; \Lambda^{p+1,1}T^*M \\ e_1\otimes e_2\otimes\psi \;&\mapsto\; e_1\otimes(e_2\wedge\psi) \;\mapsto\; \mathrm{pr}_{\Lambda^{p+1,1}}(e_1\otimes(e_2\wedge\psi)) \;.\end{aligned}$$

Let ψ be any p-form then $\nabla^2\psi$ is a section of $T^*M\otimes T^*M\otimes\Lambda^pT^*M$ and it is easy to show that $\mathrm{pr}_1^+(\nabla^2\psi) = T(d\psi)$. Next we need the map

$$\begin{aligned}\mathrm{pr}_2^+ \;:\; T^*M\otimes T^*M\otimes\Lambda^pT^*M \;&\to\; T^*M\otimes\Lambda^{p,1}T^*M \;\to\; \Lambda^{p+1,1}T^*M \\ e_1\otimes e_2\otimes\psi \;&\mapsto\; e_1\otimes\mathrm{pr}_{\Lambda^{p,1}}(e_2\otimes\psi) \mapsto \mathrm{pr}_{\Lambda^{p+1,1}}(e_1\wedge\mathrm{pr}_{\Lambda^{p,1}}(e_2\otimes\psi)).\end{aligned}$$

In this case there appears a new first order differential operator, which we denote by θ^+. It maps sections of $\Lambda^{p,1}T^*M$ into sections of $\Lambda^{p+1,1}T^*M$ and is defined as $\mathrm{pr}\circ\nabla$, where pr is the projection $T^*M \otimes \Lambda^{p,1}T^*M \to \Lambda^{p+1,1}T^*M$ defined above (as the second map in the definition of pr_2^+). We have

$$\mathrm{pr}_2^+ (\nabla^2 \psi) \;=\; \theta^+ T(\psi) \; .$$

Then we need a third projection which will produce the curvature term. We define it as

$$\pi^+ \;:\; T^*M \otimes T^*M \otimes \Lambda^p T^*M \quad\to\quad \Lambda^2 T^*M \otimes \Lambda^p T^*M \quad\to\quad \Lambda^{p+1,1}T^*M$$

$$e_1 \otimes e_2 \otimes \psi \;\mapsto\; (e_1 \wedge e_2) \otimes \psi \;\mapsto\; \mathrm{pr}_{\Lambda^{p+1,1}} \big(\sum e_i \lrcorner (e_1 \wedge e_2) \otimes (e_1 \wedge \psi)\big) \; .$$

Let ψ be any p-form, then $\nabla^2 \psi$ is a section of $T^*M \otimes T^*M \otimes \Lambda^p T^*M$ and the first map in the definition of π^+ maps this section to the curvature $R(\cdot, \cdot)\psi$. Computing the result of the second map we obtain

$$\pi^+ (\nabla^2 \psi) \;=\; -\tfrac{1}{p} R^+(\cdot)\, \psi \;-\; \tfrac{1}{p(n-p)} \cdot \wedge\, q(R)\, \psi \; .$$

Having defined these three projections it is an elementary calculation to prove that they satisfy the following linear relation

$$(p+1)\, \pi^+ \;+\; \mathrm{pr}_1^+ \;=\; \tfrac{p+1}{p}\, \mathrm{pr}_2^+ \; .$$

To obtain a twistor Weitzenböck formula we only have to apply this relation to $\nabla^2\psi$ and to substitute the expressions for the three different projections of $\nabla^2\psi$. The result is

Lemma 5.3 *Let ψ be any p-form then:*

$$T(d\psi) \;=\; \tfrac{p+1}{p}\, \theta^+(T\psi) \;+\; \tfrac{p+1}{p}\, R^+(\cdot)\, \psi \;+\; \tfrac{p+1}{p(n-p)} \cdot \wedge\, q(R)\, \psi \; .$$

By defining similar projections or by applying the Hodge star operator to the equation of Lemma 5.3, with ψ replaced by $*\psi$, we obtain a corresponding formula for $T(d^*\psi)$. Here appears an operator θ^-, which is defined as θ^+, only with the wedge product replaced by the contraction. In this case the result is

Lemma 5.4 *Let ψ be any p-form then:*

$$T(d^*\psi) \;=\; \tfrac{n-p+1}{n-p}\, \theta^-(T\psi) \;-\; \tfrac{n-p+1}{n-p}\, R^-(\cdot)\, \psi \;-\; \tfrac{n-p+1}{p(n-p)} \cdot \lrcorner\, q(R)\, \psi \; .$$

If ψ is a conformal Killing form then $T\psi = 0$ and the summands with θ^+ resp. θ^- vanish. Substituting the expressions for $Td\psi$ resp. $Td^*\psi$ into equation (2.6) proves Proposition 5.2.

Finally we have to show that the covariant derivative of $dd^*\psi$ for a conformal Killing p-form ψ can be obtained from $d\psi$ resp. $d^*\psi$ by applying certain bundle homomorphisms. We replace in the formula of Proposition 5.3 the p-form ψ with $d^*\psi$ and p with $p-1$ to obtain

$$T(dd^*\psi) \;=\; \tfrac{p}{p-1}\, \theta^+(Td^*\psi) \;+\; \tfrac{p}{p-1}\, R^+(\cdot)\, d^*\psi \;+\; \tfrac{p}{(p-1)(n-p+1)} \cdot \wedge\, q(R)\, d^*\psi \; .$$

It remains to investigate the summand with $\theta^+(Td^*\psi)$. Since ψ is a conformal Killing form we can use Lemma 5.4 to replace $Td^*\psi$, i.e. we have

$$\begin{aligned} \theta^+(Td^*\psi) \;&=\; c_1 \sum \theta^+(e_i \otimes R^-(e_i)\, \psi) \;+\; c_2 \sum \theta^+(e_i \otimes e_i \lrcorner\, q(R)\psi) \\ &=\; c_1 \sum \mathrm{pr}(e_j \otimes e_i \otimes \nabla_{e_j}(R^-(e_i)\, \psi)) \;+\; c_2 \sum \mathrm{pr}(e_j \otimes e_i \otimes e_i \lrcorner\, \nabla_{e_j}(q(R)\psi)) \; , \end{aligned}$$

where the constants c_1 and c_2 are given by Lemma 5.4 and where we do the calculation in a point with $\nabla e_i = 0$. Computing the covariant derivative of $R^-(\cdot)\psi$ and $q(R)\psi$ easily leads to

Lemma 5.5 *Let ψ be any differential form then for any vector fields X, Y*

$$
\begin{array}{lrcl}
(i) & \nabla_X(R^-(Y)\psi) & = & R^-(\nabla_X Y)\psi \;+\; R^-(Y)\nabla_X\psi \;+\; (\nabla_X R)^-(Y)\psi \\
(ii) & \nabla_X(q(R)\psi) & = & q(\nabla_X R)\psi \;+\; q(R)\nabla_X\psi
\end{array}
$$

Using this lemma we can substitute the summands $\nabla_{e_j}(R^-(e_i)\,\psi)$ and $\nabla_{e_j}(q(R)\psi))$ in the formula for $\theta^+(Td^*\psi)$ and see that it indeed only involves the covariant derivative of ψ, which for the conformal Killing form ψ is an expression in $d\psi$ and $d^*\psi$. Summarizing the calculations we see that the covariant derivative of the four sections $\psi, d\psi, d^*\psi$ and $dd^*\psi$ can be obtained from these sections by applying certain bundle homomorphisms. Hence we can collect the covariant derivatives to define a Killing connection (as explained above), which then concludes the proof of the dimension bound.

6 Further results

In this section we state (without proof) several further results on conformal Killing forms. We start with compact Kähler manifolds, where it is easy to show that any Killing form has to be parallel (c.f. [27]). More generally we proved in [19]

Theorem 6.1 *On a compact Kähler manifold M^{2m} any conformal Killing form ψ has to be of the form*

$$\psi \;=\; L^{k-1}\phi \;+\; L^k f \;+\; \psi_0 \;,$$

where L denotes the wedging with the Kähler form, ϕ is a special 2-form with associated function f and ψ_0 is any parallel form. Conversely any special 2-forms defines in this way conformal Killing forms on M in any even degree.

Special 2-forms are defined as primitive (1,1)-forms satisfying an additional differential equation. In particular, it follows for a special 2-form ϕ that $Jd^*\phi$ is exact, thus defining the function f (up to constants). Special 2-forms are closely related to Hamiltonian 2-forms, which were studied and locally classified in [1]. In particular, if $m > 2$ then any special 2-form is the primitive part of a Hamiltonian 2-form and vice versa. Starting from the differential of eigenfunctions for the minimal eigenvalue of the Laplacian on the complex projective space one easily can construct special 2-forms. Hence, the complex projective space admits conformal Killing forms in any even degree. Besides the complex projective spaces there are several other examples of compact Kähler manifolds with conformal Killing forms. This is in contrast to results in [16]. However, it turns out that the proofs in [16] contain serious gaps.

Let (M^n, g) be a Riemannian manifold such that the holonomy group of M is a proper subgroup of $O(n)$. In this situation the bundle of forms decomposes into a sum of parallel subbundles, which are preserved by the Laplace operator Δ and the curvature endomorphism $q(R)$. For any form ψ we have the corresponding holonomy decomposition

$\psi = \sum \psi_i$, where the forms ψ_i are the projections of ψ onto the parallel subbundles. We would like to use the characterization of Killing forms given in Corollary 2.5 to conclude that a form ψ, with holonomy decomposition $\psi = \sum \psi_i$, is a Killing form if and only if all components ψ_i are Killing forms. This not true in general since the components ψ_i of a coclosed form ψ need not to be coclosed. However, the statement is true for Killing m-forms on a 2m-dimensional manifold and for Killing forms on manifolds with G_2- resp. $Spin_7$-holonomy. In the case of compact manifolds with holonomy G_2 (and similarly for manifolds with holonomy $Spin_7$, we can derive the following result.

Theorem 6.2 *Let (M^7, g) be a compact manifold with holonomy G_2. Then any Killing form and any $*$–Killing form is parallel. Moreover, any conformal Killing p–form, with $p \neq 3, 4$, is parallel.*

First of all we note that on a compact Ricci-flat manifold any conformal vector field has to be a Killing vector field and any Killing vector field has to be parallel. This follows from results of M. Obata in [21] and Corollary 2.5. Hence, on a compact manifold with holonomy G_2 or $Spin_7$ any conformal Killing 1-form has to be parallel. Moreover, it is easy to show that on an Einstein manifold any conformal Killing form ψ is either coclosed, i.e. Killing, or $d^*\psi$ is a non-trivial Killing vector field. Thus we obtain that any conformal Killing p–form, with $p \neq 3, 4$, is either closed or coclosed, with a similar statement for $Spin_7$-manifolds. In the end it remains to consider Killing resp. $*$-Killing forms lying in one of the parallel subbundles of the 2- resp. 3-form bundle of a G_2-manifold. Using additional twistor operators and explicit formulas for the projections onto the parallel subbundles it is easy to derive a contradiction to the norm estimate of Lemma 2.3, which proves that any Killing resp. $*$-Killing form has to be parallel.

A special case of a manifold with restricted holonomy is a Riemannian product $M = M_1 \times M_2$. In this case the holonomy decomposition of the form bundle coincides with the decomposition $\Lambda^p(T^*M) = \sum_{r=0}^{p} \Lambda^r(T^*M_1) \otimes \Lambda^{p-r}(T^*M_2)$ and it is easy to verify that a form ψ is a Killing form if and only if all its components ψ_i are Killing forms. More generally we can show (c.f. [20]) the following

Theorem 6.3 *Every conformal Killing form on a Riemannian product $M = M_1 \times M_2$ is a sum of forms of the following types: parallel forms, pull–backs of Killing forms on M_1 or M_2, and wedge products of the volume form of M_1 (or M_2) with the pull–back of a $*$-Killing form on M_2 (resp. M_1).*

Finally we want to give a new description of a curvature condition, which already appears in [17]. Using the notation of Section 5 the condition can be reformulated as

Proposition 6.4 *Let (M^n, g) be a Riemannian manifold with a conformal Killing p–form ψ, then for any vector fields X, Y the following equation is satisfied:*

$$\begin{aligned} R(X,Y)\,\psi \;&=\; \tfrac{1}{p(n-p)}\,(Y \wedge X \lrcorner \;-\; X \wedge Y \lrcorner\,)\, q(R)\,\psi \\ &\quad - \tfrac{1}{p}\big(X \lrcorner\, R^+(Y) \;-\; Y \lrcorner\, R^+(X)\big)\,\psi \;-\; \tfrac{1}{n-p}\big(X \wedge R^-(Y) \;-\; Y \wedge R^-(X)\big)\,\psi\ . \end{aligned}$$

The proof of this proposition is simple local calculation, which is contained in the computation of the components of the Killing connection. Considering $R(\cdot,\cdot)\,\psi$ as a section of $\Lambda^p(T^*M)\otimes\Lambda^2(T^*M)$, we can write the above curvature condition in a much shorter form. Indeed, we have a decomposition of the tensor product $\Lambda^p(T^*M)\otimes\Lambda^2(T^*M)$ corresponding to the following isomorphism of $O(n)$–representations:

$$\Lambda^pV^*\otimes\Lambda^2V^*\;\cong\;\Lambda^pV^*\oplus\Lambda^{p+1,1}V^*\oplus\Lambda^{p-1,1}V^*\oplus\Lambda^{p+2}V^*\oplus\Lambda^{p-2}V^*\oplus\Lambda^{p,2}V^*\;. \tag{6.5}$$

Here $\Lambda^{p,2}V^*$ is defined as the irreducible representation which has as highest weight the sum of the highest weights of Λ^pV^* and Λ^2V^*. It is easy to find explicit expressions for the projections onto the six summands on the right hand side of (6.5), denoted as pr_{Λ^p}, $\mathrm{pr}_{\Lambda^{p\pm1,1}}$, $\mathrm{pr}_{\Lambda^{p\pm2}}$ and $\mathrm{pr}_{\Lambda^{p,2}}$. It then follows that the projections of $R\,(\cdot,\cdot)\,\psi$ onto the summands $\Lambda^{p\pm2}T^*M$ vanish because of the Bianchi identity and that the projection of $R\,(\cdot,\cdot)\,\psi$ onto Λ^pT^*M is precisely $q(R)\psi$. Moreover, it is also not difficult to show that the curvature relation can be written as

Corollary 6.5 *Let $(M^n,\,g)$ be a Riemannian manifold with a conformal Killing form ψ. Then*

$$\mathrm{pr}_{\Lambda^{p,2}}(\,R\,(\cdot,\cdot)\,\psi\,)\;\;=\;\;0\;. \tag{6.6}$$

If ψ is coclosed, then the additional equation $\mathrm{pr}_{\Lambda^{p-1,1}}(\,R\,(\cdot,\cdot)\,\psi\,)=0$ is satisfied. Similarly, if ψ is closed then the additional equation $\mathrm{pr}_{\Lambda^{p+1,1}}(\,R\,(\cdot,\cdot)\,\psi\,)=0$ holds.

We note that it is possible to give an alternative proof of the curvature condition of Proposition 6.4, using the Killing connection. Indeed, since a conformal Killing form is parallel with respect to the Killing connection, it follows that the curvature of the Killing connection applied to a conformal Killing form has to vanish. This yields four equations corresponding to the four components of the bundle $\mathcal{E}^p(M)$. The first of these equations turns out to be equivalent to the curvature condition of Proposition 6.4.

A Proof of Proposition 2.2

In this appendix we will give the proof of Proposition 2.2. Let ψ be a *twistor spinor*, i.e. a spinor satisfying for all vector fields X the equation $\nabla_X\psi\;=\;-\frac{1}{n}\,X\cdot D\psi$, with Clifford multiplication $\cdot$ and Dirac operator D. Given two such twistor spinors, ψ_1 and ψ_2, we introduced k-forms ω_k, defined on tangent vectors $X_1,\ldots,X_k$ by

$$\omega_k(X_1,\ldots,X_k):=\langle(X_1\wedge\ldots\wedge X_k)\cdot\psi_1,\psi_2\rangle\;.$$

In order to prove that the ω_k's are indeed twistor forms we compute first the covariant derivative $(\nabla_{X_0}\omega_k)(X_1,\ldots,X_k)$. Without loss of generality we will do the calculation for a point $p\in M$ and with vector fields X_i satisfying $\nabla_{X_i}X_j=0$ in p. We obtain

$$\begin{aligned}
(\nabla_{X_0}\omega_k)(X_1,\ldots,X_k) \;&=\; \nabla_{X_0}(\omega_k(X_1,\ldots,X_k))\\
&=\; \langle[X_1\wedge\ldots\wedge X_k]\cdot\nabla_{X_0}(\psi_1),\psi_2\rangle \;+\; \langle[X_1\wedge\ldots\wedge X_k]\cdot\psi_1,\nabla_{X_0}\psi_2\rangle\\
&=\; -\tfrac{1}{n}\,\langle[X_1\wedge\ldots\wedge X_k]\cdot X_0\cdot D\psi_1,\psi_2\rangle \;-\; \tfrac{1}{n}\,\langle[X_1\wedge\ldots\wedge X_k]\cdot\psi_1,X_0\cdot D\psi_2\rangle\\
&=\; -\tfrac{1}{n}\,\langle[X_1\wedge\ldots\wedge X_k]\cdot X_0\cdot D\psi_1,\psi_2\rangle \;-\; \tfrac{1}{n}\,\epsilon\,\langle\psi_1,(X_1\wedge\ldots\wedge X_k)\cdot X_0\cdot D\psi_2\rangle,
\end{aligned}$$

where $\epsilon = (-1)^{\frac{k(k+1)}{2}}$. Using the formula $\omega \cdot X = (-1)^k (X \wedge \omega + X \lrcorner \omega)$, valid for any k-form ω and any vector field X, we can further reformulate the expression for $\nabla_{X_0}\omega_k$. Setting $X_0 = X_1$ and summing over an orthonormal basis $\{e_i\}$ we find

$$\begin{aligned} d^*\omega_k(X_2, \ldots, X_k) &= -\sum (\nabla e_i\, \omega_k)(e_i, X_2, \ldots, X_k) \\ &= \tfrac{n-k+1}{n} (-1)^k \; (\langle [X_2 \wedge \ldots \wedge X_k] \cdot D\psi_1, \psi_2 \rangle \;+\; \epsilon\, \langle \psi_1, [X_2 \wedge \ldots \wedge X_k] \cdot D\psi_2 \rangle)\,. \end{aligned}$$

Hence,

$$\begin{aligned} (X_0 \,\wedge\, d^*\omega_k)(X_1, \ldots, X_k) \;&=\; \tfrac{n-k+1}{n} (-1)^k \, \langle (X_0 \lrcorner [X_1 \wedge \ldots \wedge X_k] \cdot D\psi_1, \psi_2 \rangle \\ &\quad + \epsilon\, \tfrac{n-k+1}{n} (-1)^k \, \langle \psi_1, X_0 \lrcorner [X_1 \wedge \ldots \wedge X_k] \cdot D\psi_2 \rangle . \end{aligned}$$

A similar calculation for $d\omega_k$ yields

$$\begin{aligned} d\omega_k(X_0, \ldots, X_k) &= \sum (-1)^i (\nabla_{X_i}\omega_k)(X_0, \ldots \hat{X}_i \ldots, X_k) \\ &= \tfrac{k+1}{n} (-1)^{k+1} (\langle [X_0 \wedge \ldots \wedge X_k] \cdot D\phi_1, \phi_2 \rangle \;+\; \epsilon \langle \phi_1, [X_0 \wedge \ldots \wedge X_k] \cdot D\phi_2 \rangle) \end{aligned}$$

Here we used the simple fact that $\sum (-1)^i \, X_i \lrcorner \, [X_0 \wedge \ldots \hat{X}_i \ldots \wedge X_k] = 0$. Comparing these expressions for $\nabla_{X_0}\omega_k$, $X_0 \lrcorner \, d\omega_k$ and $X_0 \wedge d^*\omega_k$ we immediately conclude that ω_k is a twistor form, i.e. it satisfies the equation

$$\nabla_{X_0}\omega_k \;=\; \frac{1}{k+1}\, X_0 \lrcorner \, d\omega_k \;-\; \frac{1}{n-k+1}\, X_0 \wedge d^*\omega_k \;.$$

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