TWISTOR FORMS ON KÄHLER MANIFOLDS

ANDREI MOROIANU AND UWE SEMMELMANN

Abstract. Twistor forms are a natural generalization of conformal vector fields on Riemannian manifolds. They are defined as sections in the kernel of a conformally invariant first order differential operator. We study twistor forms on compact Kähler manifolds and give a complete description up to special forms in the middle dimension. In particular, we show that they are closely related to Hamiltonian 2-forms. This provides the first examples of compact Kähler manifolds with non–parallel twistor forms in any even degree.

2000 *Mathematics Subject Classification.* Primary 53C55, 58J50

1. Introduction

Killing vector fields are important objects in Riemannian geometry. They are by definition infinitesimal isometries, i.e. their flow preserves a given metric. The existence of Killing vector fields determines the degree of symmetry of the manifold. Slightly more generally one can consider *conformal vector fields*, i.e. vector fields whose flows preserve a conformal class of metrics. The covariant derivative of a vector field can be seen as a section of the tensor product $\Lambda^1 M \otimes TM$ which is isomorphic to $\Lambda^1 M \otimes \Lambda^1 M$. This tensor product decomposes under the action of $O(n)$ as

$$\Lambda^1 M \otimes \Lambda^1 M \cong \mathbb{R} \oplus \Lambda^2 M \oplus S^2_0 M,$$

where $S^2_0 M$ is the space of trace–free symmetric 2–tensors, identified with the *Cartan product* of the two copies of $\Lambda^1 M$. A vector field X is a conformal vector field if and only if the projection on $S^2_0 M$ of ∇X vanishes.

More generally, the tensor product $\Lambda^1 M \otimes \Lambda^p M$ decomposes under the action of $O(n)$ as

$$\Lambda^1 M \otimes \Lambda^p M \cong \Lambda^{p-1} M \oplus \Lambda^{p+1} M \oplus \Lambda^{p,1} M,$$

where again $\Lambda^{p,1} M$ denotes the Cartan product. As natural generalizations of conformal vector fields, *twistor forms* are defined to be p–forms ψ such that the projection of $\nabla\psi$ onto $\Lambda^{p,1} M$ vanishes.

Coclosed twistor p–forms are called *Killing forms.* For $p = 1$ they are dual to Killing vector fields. Note that parallel forms are trivial examples of twistor forms.

The authors are members of the *European Differential Geometry Endeavour* (EDGE), Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme; both authors would like to thank the CNRS for support and the Centre de Mathématiques de l'Ecole Polytechnique for hospitality during the preparation of this work.

Killing forms, as generalization of Killing vector fields, were introduced by K. Yano in [19]. Twistor forms were introduced later on by S. Tachibana [17], for the case of 2–forms, and by T. Kashiwada [12], [16] in the general case.

The projection $\Lambda^1 M \otimes \Lambda^p M \to \Lambda^{p,1} M$ defines a first order differential operator T, which was already studied in the context of Stein–Weiss operators (c.f. [9]). As forms in the kernel of T, twistor forms are very similar to twistor spinors in spin geometry. We will give an explicit construction relating these two objects in Section 2.

The special interest for twistor forms in the physics literature stems from the fact that they can be used to define quadratic first integrals of the geodesic equation, i.e. functions which are constant along geodesics. Hence, they can be used to integrate the equation of motion, which was done for the first time by R. Penrose and M. Walker in [13]. More recently Killing forms and twistor forms have been successfully applied to define symmetries of field equations (c.f. [6], [7]).

Despite this longstanding interest in Killing forms there are only very few global results on the existence or non–existence of twistor forms on Riemannian manifolds. In this article we will give a description of twistor forms on compact Kähler manifolds and show how they are related to the Hamiltonian 2–forms which were recently introduced in [2].

The first result for twistor forms on compact Kähler manifolds was obtained in [18], where it is proved that any Killing form has to be parallel. Some years later S. Yamaguchi et al. stated in [11] that, with a few exceptions for small degrees and in small dimensions, any twistor form on a compact Kähler manifold has to be parallel. Nevertheless, it turns out that their proof contains several serious gaps.

In fact, we will show that there are examples of compact Kähler manifolds having non–parallel twistor forms in any even degree. The simplest examples can be found on the complex projective space. More precisely we will show that on a compact Kähler manifold M^{2m} any twistor p–form with $p \neq m$ can (modulo parallel forms) be written as $L^{k-1}\varphi + L^k f$, where $\varphi \in \Lambda^2 M$ and $f \in C^\infty M$ are, up to some constants depending on m and k, the primitive part and the trace of a *Hamiltonian* 2–form (here L denotes wedging with the Kähler form). Conversely, any Hamiltonian 2–form can be used to construct twistor forms. Hamiltonian 2–forms were recently studied in [2] and [3]. They appear in many different areas of Kähler geometry. In particular they arise on weakly Bochner–flat Kähler manifolds and on Kähler manifolds which are conformally Einstein.

The case of twistor m–forms is more complicated. They are either of the form $L^{k-1}\varphi$, where φ is the primitive part of a Hamiltonian 2–form, or they belong to a special class of forms, whose existence is still open for the time being.

2. Twistor forms on Riemannian manifolds

Let $(V, \langle\cdot,\cdot\rangle)$ be an n–dimensional Euclidean vector space. The tensor product $V^* \otimes \Lambda^p V^*$ has the following $O(n)$–invariant decomposition:

$$V^* \otimes \Lambda^p V^* \;\cong\; \Lambda^{p-1} V^* \oplus \Lambda^{p+1} V^* \oplus \Lambda^{p,1} V^*$$

where $\Lambda^{p,1}V^*$ is the intersection of the kernels of wedge and inner product maps, which can be identified with the Cartan product of V^* and $\Lambda^p V^*$. This decomposition immediately translates to Riemannian manifolds (M^n, g), where we have

$$T^*M \otimes \Lambda^p T^*M \;\cong\; \Lambda^{p-1}T^*M \oplus \Lambda^{p+1}T^*M \oplus \Lambda^{p,1}T^*M \tag{1}$$

with $\Lambda^{p,1}T^*M$ denoting the vector bundle corresponding to the vector space $\Lambda^{p,1}$. The covariant derivative $\nabla\psi$ of a p–form ψ is a section of $T^*M \otimes \Lambda^p T^*M$. Its projections onto the summands $\Lambda^{p+1}T^*M$ and $\Lambda^{p-1}T^*M$ are just the differential $d\psi$ and the codifferential $\delta\psi$. Its projection onto the third summand $\Lambda^{p,1}T^*M$ defines a natural first order differential operator T, called the *twistor operator*. The twistor operator $T : \Gamma(\Lambda^p T^*M) \to \Gamma(\Lambda^{p,1}T^*M) \subset \Gamma(T^*M \otimes \Lambda^p T^*M)$ is given for any vector field X by the following formula

$$[\,T\psi\,]\,(X) \;:=\; [\mathrm{pr}_{\Lambda^{p,1}}(\nabla\psi)]\,(\mathrm{X}) \;=\; \nabla_{\mathrm{X}}\,\psi \;-\; \tfrac{1}{\mathrm{p}+1}\,\mathrm{X} \lrcorner\, \mathrm{d}\psi \;+\; \tfrac{1}{\mathrm{n}-\mathrm{p}+1}\,\mathrm{X} \wedge \delta\psi \;.$$

Note that here, and in the remaining part of this article, we identify vectors and 1–forms using the metric.

The twistor operator T is a typical example of a so-called Stein–Weiss operator and it was in this context already considered by T. Branson in [9]. It particular it was shown that T^*T is elliptic, which easily follows from computing the principal symbol. Its definition is also similar to the definition of the twistor operator in spin geometry. The tensor product between the spinor bundle and the cotangent bundle decomposes under the action of the spinor group into the sum of the spinor bundle and the kernel of the Clifford multiplication. The (spinorial) twistor operator is then defined as the projection of the covariant derivative of a spinor onto the kernel of the Clifford multiplication.

Definition 2.1. *A p–form ψ is called a twistor p–form if and only if ψ is in the kernel of T, i.e. if and only if ψ satisfies*

$$\nabla_X\,\psi \;=\; \tfrac{1}{p+1}\,X \lrcorner\, d\psi \;-\; \tfrac{1}{n-p+1}\,X \wedge \delta\psi \;, \tag{2}$$

for all vector fields X. If the p–form ψ is in addition coclosed, it is called a Killing p–form. This is equivalent to $\nabla\psi \in \Gamma(\Lambda^{p+1}T^*M)$ or to $X \lrcorner \nabla_X\psi = 0$ for any vector field X.

It follows directly from the definition that the Hodge star-operator $*$ maps twistor p–forms into twistor $(n-p)$–forms. In particular, it interchanges closed and coclosed twistor forms.

In the physics literature, equation (2) defining a twistor form is often called the *Killing–Yano equation.* The terminology *conformal Killing forms* to denote twistor forms is also used. Our motivation for using the name *twistor form* is not only the similarity of its definition to that of twistor spinors in spin geometry, but also the existence of a direct relation between these objects. We recall that a *twistor spinor* on a Riemannian spin manifold is a section ψ of the spinor bundle lying in the kernel of the (spinorial) twistor operator. Equivalently, ψ satisfies for all vector fields X the equation $\nabla_X \psi \;=\; -\frac{1}{n} X \cdot D\psi$, where D denotes the Dirac operator. Given two such twistor spinors, ψ_1 and ψ_2, we can introduce k–forms ω_k, which on tangent vectors $X_1, \dots , X_k$ are defined as

$$\omega_k(X_1, \dots , X_k) := \langle (X_1 \wedge \dots \wedge X_k) \cdot \psi_1, \psi_2 \rangle \ .$$

It is well–known that for $k = 1$ the form ω_1 is dual to a conformal vector field. Moreover, if ψ_1 and ψ_2 are Killing spinors the form ω_1 is dual to a Killing vector field. More generally we have

Proposition 2.2. (cf. [14]) *Let (M^n, g) be a Riemannian spin manifold with twistor spinors ψ_1 and ψ_2. Then for any k the associated k–form ω_k is a twistor form.*

We will now give an important integrability condition which characterizes twistor forms on compact manifolds. A similar characterization was obtained in [12]. We first obtain two Weitzenböck formulas by differentiating the equation defining the twistor operator.

$$\nabla^* \nabla \psi \;=\; \tfrac{1}{p+1}\, \delta d\, \psi \;+\; \tfrac{1}{n-p+1}\, d\delta\, \psi \;+\; T^*T\, \psi \ , \tag{3}$$

$$q(R)\, \psi \;=\; \tfrac{p}{p+1}\, \delta d\, \psi \;+\; \tfrac{n-p}{n-p+1}\, d\delta\, \psi \;-\; T^*T\, \psi \ , \tag{4}$$

where $q(R)$ is the curvature expression appearing in the classical Weitzenböck formula for the Laplacian on p–forms: $\Delta \;=\; \delta d \;+\; d\delta \;=\; \nabla^*\nabla \;+\; q(R)$. It is the symmetric endomorphism of the bundle of differential forms defined by

$$q(R) \;=\; \sum e_j \,\wedge\, e_i \lrcorner\, R_{e_i, e_j}, \tag{5}$$

where $\{e_i\}$ is any local orthonormal frame and R_{e_i,e_j} denotes the curvature of the form bundle. On forms of degree one and two one has an explicit expression for the action of $q(R)$, e.g. if ξ is any 1–form, then $q(R)\, \xi = \mathrm{Ric}(\xi)$. In fact it is possible to define $q(R)$ in a more general context. For this we first rewrite equation (5) as

$$q(R) \;=\; \sum_{i<j} (e_j \,\wedge\, e_i \lrcorner \;-\; e_i \,\wedge\, e_j \lrcorner\,)\, R_{e_i,e_j} \;=\; \sum_{i<j} (e_i \wedge e_j) \bullet R(e_i \wedge e_j) \bullet$$

where the Riemannian curvature R is considered as element of $\mathrm{Sym}^2(\Lambda^2 T_pM)$ and $\bullet$ denotes the standard representation of the Lie algebra $\mathfrak{so}(T_pM) \cong \Lambda^2 T_pM$ on the space of p–forms. Note that we can replace $e_i \wedge e_j$ by any orthonormal basis of $\mathfrak{so}(T_pM)$. Let (M, g) be a Riemannian manifold with holonomy group Hol. Then the curvature tensor takes values in the Lie algebra $\mathfrak{hol}$ of the holonomy group, i.e. we can write $q(R)$ as

$$q(R) \;=\; \sum \omega_i \bullet R(\omega_i) \bullet \qquad \in\; \mathrm{Sym}^2(\mathfrak{hol})$$

where $\{\omega_i\}$ is any orthonormal basis of $\mathfrak{hol}$ and $\bullet$ denotes form representation restricted to the holonomy group. Writing the bundle endomorphism $q(R)$ in this way has two immediate consequences: we see that $q(R)$ preserves any parallel subbundle of the form bundle and it is clear that by the same definition $q(R)$ gives rise to a symmetric endomorphism on any associated vector bundle defined via a representation of the holonomy group.

Integrating the second Weitzenböck formula (4) yields the following integrability condition for twistor forms.

Proposition 2.3. *Let $(M^n,\, g)$ a compact Riemannian manifold. Then a p–form ψ is a twistor p–form, if and only if*

$$q(R)\,\psi \;=\; \tfrac{p}{p+1}\,\delta d\,\psi \;+\; \tfrac{n-p}{n-p+1}\,d\delta\,\psi\;. \tag{6}$$

For coclosed forms, Proposition 2.3 is a generalization of the well–known characterization of Killing vector fields on compact manifolds, as divergence free vector fields in the kernel of $\Delta - 2\,\mathrm{Ric}$. In the general case, it can be reformulated as

Corollary 2.4. *Let $(M^n,\, g)$ a compact Riemannian manifold with a coclosed p–form ψ. Then ψ is a Killing form if and only if*

$$\Delta\,\psi \;=\; \frac{p+1}{p}\,q(R)\,\psi\;.$$

One has similar characterizations for closed twistor forms and for twistor m–forms on 2m–dimensional manifolds.

An important property of the equation defining twistor forms is its conformal invariance (c.f. [7]). We note that the same is true for the twistor equation in spin geometry. The precise formulation for twistor forms is the following. Let ψ be a twistor form on a Riemannian manifold (M,g). Then $\widehat{\psi} := e^{(p+1)\lambda}\psi$ is a twistor p–form with respect to the conformally equivalent metric $\hat{g} := e^{2\lambda}g$. Parallel forms are obviously in the kernel of the twistor operator, hence they are twistor forms. Using the conformal invariance we see that with any parallel form ψ, also the form $\widehat{\psi} := e^{(p+1)\lambda}\,\psi$ is a twistor p–form with respect to the conformally equivalent metric $\widehat{g} := e^{2\lambda}g$. The form $\widehat{\psi}$ is in general not parallel.

The first non trivial examples of twistor forms were found on the standard sphere. Here it is easy to show that any twistor p–form is a linear combination of eigenforms for the minimal eigenvalue of the Laplace operator on closed resp. coclosed p–forms. It is shown in [14] that the number of linearly independent twistor forms on a connected Riemannian manifold is bounded from above by the corresponding number on the standard sphere. Other examples exist on Sasakian, nearly Kähler and on weak G_2–manifolds. In some sense they are all related to the existence of Killing spinors. Up to now these are more or less all known global examples of twistor forms.

In section 3 we will see that there are many new examples on compact Kähler manifolds. The examples include the complex projective space and Hirzebruch surfaces.

3. Twistor forms on Kähler manifolds

In this section we will consider twistor forms on a compact Kähler manifold $(M,\, g,\, J)$ of dimension $n = 2m$. The case of forms in the middle dimension, i.e. forms of degree m, is somewhat special and will be treated in the next section.

On a complex manifold the differential splits as $d = \partial + \bar\partial$. Moreover, one has the following real differential operator

$$d^c \;=\; i\,(\bar\partial \,-\, \partial) \;=\; \sum J e_i \wedge \nabla_{e_i} \;.$$

The formal adjoints of d and d^c are denoted by δ and δ^c. If ω is the Kähler form then Λ denotes the contraction with ω and L the wedge product with ω. Another important operator acting on forms is the natural extension of the complex structure J on $-p$–forms, defined as follows :

$$Ju := \sum J(e_i) \wedge e_i \lrcorner\, u \qquad \forall u \in \Lambda^p M.$$

These operators satisfy the following fundamental commutator relations

$$\begin{aligned} d^c \;&=\; -[\,\delta,\, L\,] \;=\; -\,[\,d,\, J\,]\,, \qquad \delta^c \;=\; [\,d,\, \Lambda\,] \;=\; -[\,\delta,\, J\,]\;, \\ d \;&=\; [\,\delta^c,\, L\,] \;=\; [\,d^c,\, J\,]\;, \qquad \delta \;=\; -[\,d^c,\, \Lambda\,] \;=\; [\,\delta^c,\, J\,]\;. \end{aligned}$$

In addition the following commutators resp. anti–commutators vanish

$$\begin{aligned} 0 \;&=\; [\,d,\, L\,] \;=\; [\,d^c,\, L\,] \;=\; [\,\delta,\, \Lambda\,] \;=\; [\,\delta^c,\, \Lambda\,] \;=\; [\,\Lambda,\, J\,] \;=\; [\,J,\, *\,]\;, \\ 0 \;&=\; \delta d^c \;+\; d^c \delta \;=\; dd^c + d^c d \;=\; \delta\delta^c \;+\; \delta^c\delta \;=\; d\delta^c \;+\; \delta^c d\;. \end{aligned}$$

Using these relations we are now able to derive several consequences of the twistor equation on Kähler manifolds. We start by computing $\delta^c u$ for a twistor p–form u.

$$\begin{aligned} \delta^c u \;&=\; -\sum J e_i \lrcorner\, \nabla_{e_i} u \;=\; -\sum J e_i \lrcorner\, \big(\tfrac{1}{p+1}\, e_i \lrcorner\, du \;-\; \tfrac{1}{n-p+1}\, e_i \wedge \delta u\big) \\ &=\; -\tfrac{2}{p+1}\,\Lambda(du) \;+\; \tfrac{1}{n-p+1}\, J(\delta u) \\ &=\; -\tfrac{2}{p+1}\, d\Lambda(u) \;+\; \tfrac{2}{p+1}\,\delta^c u \;+\; \tfrac{1}{n-p+1}\, J(\delta u) \\ &=\; -\tfrac{2}{p-1}\, d\Lambda(u) \;+\; \tfrac{p+1}{(p-1)(n-p+1)}\, J(\delta u)\;. \end{aligned} \tag{7}$$

From this we immediately obtain

$$\delta^c\, d\, u \;=\; -\, d\,\delta^c u \;=\; -\,\tfrac{p+1}{(p-1)(n-p+1)}\,\big(J(d\delta u) \;+\; d^c\delta\, u\big)\;, \tag{8}$$

$$\delta\delta^c\, u \;=\; -\,\delta^c\delta u \;=\; -\,\tfrac{2(n-p+1)}{(n-p)(p+1)}\,\Lambda(\delta d u)\;. \tag{9}$$

A similar calculation for the operator d^c leads to

$$\begin{aligned} d^c u &= -\sum e_i \wedge \nabla_{Je_i} u \;=\; -\sum e_i \wedge \left(\tfrac{1}{p+1} Je_i \lrcorner\, du \;-\; \tfrac{1}{n-p+1} Je_i \wedge \delta u\right) \\ &= \tfrac{1}{p+1} J(du) \;+\; \tfrac{2}{n-p+1} L(\delta u) \;=\; \tfrac{1}{p+1} J(du) \;+\; \tfrac{2}{n-p+1} \delta(Lu) \;+\; \tfrac{2}{n-p+1} d^c u \\ &= \tfrac{n-p+1}{(p+1)(n-p-1)} J(du) \;+\; \tfrac{2}{n-p+1} \delta(Lu)\ . \end{aligned} \tag{10}$$

We apply this to conclude

$$\delta d^c u \;=\; \tfrac{n-p+1}{(p+1)(n-p-1)} \left(J\delta du \;-\; \delta^c du \right)\ . \tag{11}$$

As a first step we will prove that for a twistor p–form u the form $\Lambda J(u)$ has to be parallel. Here we can assume $p \neq 2$, since otherwise there is nothing to show. It will be convenient to introduce for a moment the following notation

$$\begin{aligned} x &:= J\Lambda(d\delta u), \qquad & a &:= d\delta(J\Lambda u), \qquad & \alpha &:= \delta^c d(\Lambda u), \\ y &:= J\Lambda(\delta d u), \qquad & b &:= \delta d(J\Lambda u), \qquad & \beta &:= \delta d^c(\Lambda u)\ . \end{aligned}$$

Using the various commutator rules we will derive several equations relating the quantities x, y, a, b, α and β. We start with

$$\begin{aligned} x &= J\Lambda(d\delta u) \;=\; J(d\delta\Lambda u \;-\; \delta^c\delta u) \\ &= d\delta(J\Lambda u) \;+\; d\delta^c(\Lambda u) \;+\; d^c\delta(\Lambda u) \;-\; J\delta^c\delta u \\ &= a \;-\; \alpha \;-\; \beta \;-\; \tfrac{2(n-p+1)}{(n-p)(p+1)}\, y\ , \end{aligned}$$

where we also used equation (9) to replace the summand $J\delta^c\delta u$. Similarly we have

$$\begin{aligned} y &= J\Lambda(\delta d u) \;=\; J(\delta d(\Lambda u) \;-\; \delta\delta^c u) \\ &= \delta d(J\Lambda u) \;+\; \delta d^c(\Lambda u) \;+\; \delta^c d(\Lambda u) \;-\; J\delta\delta^c u \\ &= b \;+\; \beta \;+\; \alpha \;+\; \tfrac{2(n-p+1)}{(n-p)(p+1)}\, y\ . \end{aligned}$$

Adding the two equations for x resp. y, we obtain that $x + y = a + b$, i.e. we can express x and y in terms of a, b, α and β as

$$y \;=\; \epsilon_1\,(b \;+\; \alpha \;+\; \beta), \qquad\qquad \epsilon_1 = \tfrac{(n-p)(p+1)}{(n-p)(p+1) - 2(n-p+1)}\ , \tag{12}$$

$$x \;=\; a \;+\; b \;-\; y \;=\; a \;+\; (1 \;-\; \epsilon_1)\, b \;-\; \epsilon_1\,(\alpha + \beta)\ . \tag{13}$$

Note that ϵ_1 is well defined since the denominator may vanish only in the case $(n,p) = (4,2)$, which we already excluded. Next, we contract equation (8) with the Kähler form to obtain an equation for α.

$$\begin{aligned} \alpha &= \delta^c d(\Lambda u) \;=\; \Lambda(\delta^c du) \\ &= -\tfrac{p+1}{(p-1)(n-p+1)}\,(\Lambda J(d\delta u) \;+\; \Lambda(d^c\delta\, u)) \\ &= -\tfrac{p+1}{(p-1)(n-p+1)}\,(x \;-\; \beta)\;. \end{aligned}$$

Contracting equation (11) with the Kähler form we obtain a similar expression for β.

$$\begin{aligned} \beta &= \delta d^c(\Lambda u) \;=\; \Lambda(\delta d^c u) \\ &= \tfrac{n-p+1}{(p+1)(n-p-1)}\,(\,\Lambda J\delta du \;-\; \Lambda(\delta^c du)\,) \\ &= \tfrac{n-p+1}{(p+1)(n-p-1)}\,(y \;-\; \alpha)\;. \end{aligned}$$

Replacing x and y in the equations for α resp. β leads to

$$\begin{aligned} a \;+\; (1 - \epsilon_1)\,b \;-\; \epsilon_1\,(\alpha + \beta) &= \epsilon_2\,\beta \;+\; \alpha, & \epsilon_2 &= \tfrac{(p-1)(n-p+1)}{p+1} \\ \epsilon_1\,(b \;+\; \alpha \;+\; \beta) &= \epsilon_3\,\beta \;+\; \alpha & \epsilon_3 &= \tfrac{(p+1)(n-p-1)}{n-p+1}\;. \end{aligned}$$

Finally we obtain two equations only involving a, b, α and β,

$$a \;+\; (1 - \epsilon_1)\,b \;=\; (\epsilon_1 - \epsilon_2)\,\alpha \;+\; (\epsilon_1 - 1)\,\beta \tag{14}$$

$$\epsilon_1\,b \;=\; (1 - \epsilon_1)\,\alpha \;+\; (\epsilon_3 - \epsilon_1)\,\beta\;. \tag{15}$$

Considering equation (15) we note that b and β are in the image of δ, whereas α is in the image of d. Hence, since the coefficient $(1-\epsilon_1)$ is obviously different from one, we can conclude that α has to vanish. Moreover, we have the equation $\epsilon_1 b = (\epsilon_3 - \epsilon_1)\beta$. Applying the same argument in equation (14) we obtain that a has to vanish and that $(1-\epsilon_1)b = (\epsilon_1 - 1)\beta$. Combining the two equations for b and β we see that also these expressions have to vanish. Taking the scalar product of a and b with $J\Lambda u$ and integrating over M yields that $J\Lambda u$ has to be closed and coclosed, hence harmonic.

Proposition 3.1. *Let $(M,\, g,\, J)$ be a compact Kähler manifold. Then for any twistor form u the form $\Lambda J u$ is parallel.*

Proof. Let u be a twistor p–form. We have already shown that $J\Lambda u$ is harmonic, i.e. $\Delta(J\Lambda u) = 0$. Since $\Delta = \nabla^*\nabla + q(R)$, the proof of the proposition would follow from $q(R)(J\Lambda u) = 0$. For a twistor form u we have the Weitzenböck formula (4) and the argument in Section 2 shows that the curvature endomorphism $q(R)$ commutes with Λ and J. Hence, we obtain

$$q(R)(J\Lambda u) \;=\; \tfrac{p}{p+1}\,x \;+\; \tfrac{n-p}{n-p+1}\,x\;,$$

with the notation above. But with a, b, α and β also x and y have to vanish. Hence, $q(R)(J\Lambda u) = 0$ and the Weitzenböck formula for Δ and integration over M prove that $J\Lambda u$ has to be parallel. $\square$

In the case where the degree of twistor form is different from the complex dimension, this proposition has an important corollary.

Corollary 3.2. *Let $(M,\, g,\, J)$ be a compact Kähler manifold of complex dimension m. Then for any twistor p–form u, with $p \neq m$, Ju is parallel.*

Proof. We first note that the Hodge star operator $*$ commutes with the complex structure J and with the covariant derivative. Moreover, it interchanges L and Λ. If u is a twistor p–form, then $*u$ is a twistor form, too, and by Proposition 3.1, $\Lambda J * u$ is parallel. Hence, also $*\Lambda J * u$ is parallel. But $*\Lambda J * u = \pm LJu$ and it follows that LJu is parallel. Since the contraction with the Kähler form commutes with the covariant derivative we conclude that ΛLJu is parallel as well. Thus $(m-p)Ju = \Lambda LJu - L\Lambda Ju$ is parallel. □

In the remaining part of this section we will investigate twistor forms u for which Ju is parallel. The results below are thus valid for all twistor p–forms with $p \neq m$, but also for J–invariant twistor m–forms, a fact used in the next section.

Proposition 3.3. *Let $(M,\, g,\, J)$ be a compact Kähler manifold of dimension $n = 2m$. Then any twistor p–form u for which Ju is parallel satisfies the equations*

$$\delta^c u \;=\; \mu_1\, d\Lambda u, \qquad d^c u \;=\; \mu_2\, \delta L u, \qquad \delta u \;=\; -\,\mu_1\, d^c \Lambda u, \qquad du \;=\; -\,\mu_2\, \delta^c L u\ ,$$

with constants $\mu_1 := -\,\frac{2(n-p+1)}{(p-1)(n-p)-2}$ *and* $\mu_2 := \frac{2(p+1)}{p(n-p-1)-2}$.

Note that this proposition is also valid for forms in the middle dimension.

Proof. The formula for $\delta^c u$ follows from equation (7) after interchanging J and δ and using the assumption that Ju is parallel. Because of $J\delta = \delta J + \delta^c$ we obtain

$$\delta^c u\,(1 - \tfrac{p+1}{(p-1)(n-p+1)}) \;=\; -\,\tfrac{2}{p-1}\, d\Lambda(u) \;+\; \tfrac{p+1}{(p-1)(n-p+1)}\, \delta(Ju) \;=\; -\,\tfrac{2}{p-1}\, d\Lambda(u)\ .$$

Since $J\Lambda u$ is also parallel, the formula for δu follows by applying J to the above equation. Similarly we obtain the expression for $d^c u$ by using equation (10) and the relation $Jd = dJ + d^c$. This yields

$$d^c u\,(1 - \tfrac{n-p+1}{(p+1)(n-p-1)}) \;=\; \tfrac{n-p+1}{(p+1)(n-p-1)}\, d(Ju) \;+\; \tfrac{2}{n-p-1}\, \delta(Lu) \;=\; \tfrac{2}{n-p-1}\, \delta(Lu)\ .$$

Applying J to this equation yields the formula for du. □

As a first application of Proposition 3.3 we will show that the forms du and δu are eigenvectors of the operator ΛL.

Lemma 3.4. *Let u be a twistor p–form with Ju parallel. Then du and δu satisfy the equations*

$$\Lambda L(du) \;=\; \tfrac14\,(n-p-2)\,(p+2)\, du\ , \qquad\qquad \Lambda L(\delta u) \;=\; \tfrac14\,(n-p)\, p\, \delta u\ .$$

Proof. From Proposition 3.3 we have the equation $du = -\mu_2\, \delta^c L u = -\mu_2 L\delta^c u - \mu_2 u$. This implies

$$\begin{aligned}(1 + \mu_2)\, du &= -\,\mu_2 L(\delta^c u) \;=\; -\,\mu_1\,\mu_2\, Ld\Lambda u \;=\; -\,\mu_1\,\mu_2\, dL\Lambda u\\ &= -\,\mu_1\,\mu_2\, d(\Lambda L u \;-\; (m-p)\, du)\\ &= -\,\mu_1\,\mu_2\,(\delta^c L u \;+\; \Lambda L u \;-\; (m-p)\, du)\\ &= \mu_1\, du \;-\; \mu_1\,\mu_2\,\Lambda L(du) \;+\; \mu_1\,\mu_2\,(m-p)\, du\ .\end{aligned}$$

Collecting the terms with du we obtain

$$\Lambda L(du) \;=\; -\,\tfrac{1+\mu_2-\mu_1-\mu_1\mu_2(m-p)}{\mu_1\mu_2}\, du$$

where m denotes the complex dimension. Substituting the values for μ_1 and μ_2 given in Proposition 3.3, the formula for $\Lambda L(du)$ follows after a straightforward calculation.

Similarly we could prove the formula for $\Lambda L(\delta u)$ by starting from the equation $d^c u = \mu_2\, \delta L u$. Nevertheless it is easier to note that $*u$ is again a twistor form (of degree $n-p$). Hence, $4\Lambda L(d*u) = (p-2)(n-p+2)d*u$ and we have

$$(p-2)(n-p+2)*d*u \;=\; 4*\Lambda L(d*u) \;=\; 4\, L\Lambda*d*u \;=\; 4\,\Lambda L*d*u \;-\; 4\,(m-p+1)*d*u\ .$$

Hence,

$$4\,\Lambda L(\delta u) \;=\; [(p-2)(n-p+2) \;+\; 4\,(m-p+1)]\,\delta u \;=\; (n-p)\,p\,\delta u\ .$$

$\square$

In order to use the property that du resp. δu are eigenvectors of ΛL we still need the following elementary lemma.

Lemma 3.5. *Let (M^{2m}, g, J) be a Kähler manifold and let α be a p–form on M, then*

$$[\,\Lambda,\, L^s\,]\,\alpha \;=\; s\,(m-p-s+1)\,L^{s-1}\alpha\ .$$

Moreover, if α is a primitive p–form, i.e. if $\Lambda(\alpha) = 0$, then α satisfies in addition the equation

$$\Lambda^r\, L^s\,\alpha \;=\; \tfrac{s!(m-p-s+r)!}{(s-r)!(m-p-s)!}\, L^{s-r}\alpha\ ,$$

where r, s are any integers.

Proof. In the case $r = 1$ we prove the formula for the commutator of Λ and L by induction with respect to s. For $s = 1$ it is just the well–known commutator relation for Λ and L. Assume now that we know the formula for $s-1$, then

$$\begin{aligned}[\Lambda, L^s]\,\alpha &= (\Lambda\circ L^s - L^s\circ\Lambda)\,\alpha \;=\; \Lambda\circ L^{s-1}(L\alpha) \;-\; L^{s-1}(L\circ\Lambda)\,\alpha\\ &= L^{s-1}\circ\Lambda(L\alpha) \;+\; (s-1)\,(m-(p+1)-(s-1)+1)\,L^{s-2}(L\alpha)\\ &\quad -\, L^{s-1}\circ\Lambda(L\alpha) \;+\; (m-p)\,L^{s-1}\alpha\\ &= ((s-1)(m-p-s) \;+\; (m-p))\,L^{s-1}\alpha\\ &= s\,(m-p-s+1)\,L^{s-1}\alpha\ .\end{aligned}$$

The formula for $\Lambda^r L^s$ in the case $r>1$ then follows by applying the commutator relation several times. □

On a Kähler manifold, any p–form u with $p\le m$ has a unique decomposition, $u=u_0+Lu_1+\ldots+L^l u_l$, where the u_i's are primitive forms. This is usually called the *Weyl decomposition* of u. From now on we suppose that $p\le m$, otherwise we just replace u by its Hodge dual $*u$. Applying ΛL and using the Lemma 3.5 in the case $r=1$ we obtain

$$\Lambda\,L(u) \;=\; (m-p)\,u_0 \;+\; 2\,(m-p+1)\Lambda u_1 \;+\; \ldots \;+\; (l+1)\,(m-p+l)\,L^l u_l\ .$$

It is easy to see that coefficients in the above sum are all different. Indeed, $(s+1)(m-p+s)=(r+1)(m-p+r)$ if and only if $(s-r)(m-p+1+s+r)=0$, recall that we assumed that $p\le m$. Hence, a p–form u with $p\le m$ is an eigenvector of ΛL if and only $u=L^i\alpha$, for some primitive form α and in that case the corresponding eigenvalue is $(i+1)(m-p+i)$.

Lemma 3.4 thus implies that $du=L^s v$ and $\delta u=L^r w$ for some primitive forms v and w, and moreover $(2m-p-2)(p+2)=4(s+1)(m-p+s-1)$ and $p(2m-p)=4(r+1)(m-p+r+1)$, which have the unique solutions $2s=p$ and $2r=p-2$. Hence, denoting $p=2k$, we have

$$du \;=\; L^k v \qquad\qquad \delta u \;=\; L^{k-1}w$$

for some vectors v and w. Moreover, we see that the Weyl decomposition of u has the form $u=u_0+\ldots+L^k u_k$, where u_k has to be a function.

Lemma 3.6. *Let u be a $2k$–form with Ju parallel and such that $du=L^k v$ and $\delta u=L^{k-1}w$, for some vectors v and w. Then*

$$w \;=\; \tfrac{k(2m-2k+1)}{2k+1}\,Jv\cdot$$

Proof. Applying J to the equation $du=L^k v$ leads to $d^c u = Jdu = L^k(Jv)$. Thus, contracting with the Kähler form yields

$$\Lambda(d^c u) \;=\; d^c\Lambda u \;+\; \delta u \;=\; \Lambda\,L^k(Jv) \;=\; k\,(m-k)\,L^{k-1}(Jv)\ .$$

From Proposition 3.3 we have the equation $\delta u=-\mu_1 d^c\Lambda u$. Hence,

$$\delta u \;=\; (1\;-\;\tfrac{1}{\mu_1})^{-1}k(m-k)L^{k-1}(Jv)\ ,$$

which proves the lemma after substituting the value of μ_1. □

In the next step we will show that in the Weyl decomposition of u only the last two terms, i.e. $L^{k-1}u_{k-1}$ and $L^k u_k$, may be non–parallel. Indeed we have

$$\begin{aligned}
\nabla_X u &= \nabla_X u_0 + L(\nabla_X u_1) + \ldots + L^k(\nabla_X u_k) \\
&= \tfrac{1}{p+1} X \lrcorner L^k v - \tfrac{1}{n-p+1} X \wedge L^{k-1} w \\
&= L^{k-1}\left(\tfrac{k}{p+1} JX \wedge v - \tfrac{1}{n-p+1} X \wedge w\right) + L^k\left(\tfrac{1}{p+1}\langle x,\, v\rangle\right) \\
&= L^{k-1}\omega_1 + L^k \omega_2 \,,
\end{aligned}$$

for some primitive forms ω_1, ω_2. Hence, comparing the first line with the last, implies that the components $u_0, u_1, \ldots, u_{k-2}$ have to be parallel. But then we can without loss of generality assume that they are zero, i.e. for a twistor p–form u, with Ju parallel, we have

$$u = L^{k-1}u_{k-1} + L^k u_k$$

where u_{k-1} is a primitive 2–form and u_k is a function, i.e. $p = 2k$.

Our next aim is to translate the twistor equation for u into equations for u_{k-1} resp. u_k. This leads naturally to the following definition, which we will need for the rest of this paper.

Definition 3.7. *A* special *2–form on a Kähler manifold M is a primitive 2–form φ of type (1,1) satisfying the equation*

$$\nabla_X \varphi = \gamma \wedge JX - J\gamma \wedge X - \tfrac{2}{m}\gamma(X)\,\omega \,, \tag{16}$$

for some 1–form γ, which then necessarily equals $\frac{m}{2(m^2-1)}\delta^c\varphi$.

Recall that a 2–form u is of type (1,1) if and only if it is J invariant, i.e. $Ju = 0$. As a first property of such special forms we obtain

Lemma 3.8. *If φ is a special 2–form on a compact Kähler manifold M of dimension $m > 2$, then $\delta^c\varphi$ is exact.*

Proof. Taking the wedge product with X in (16) and summing over an orthonormal basis yields $d\varphi = 2\frac{m-1}{m}\gamma = \frac{1}{m+1}\delta^c\varphi$. It follows that $Ld\delta^c\varphi = 0$ and, since L is injective on 2–forms, we conclude that $\delta^c\varphi$ is closed. Hence, $\delta^c\varphi = h + df$ for some function f and a harmonic 1–form h. We have to show that h vanishes. First, note that h is in the kernel of d^c and δ^c since the manifold is Kähler. Computing its L^2-norm we obtain

$$(h,\, h) = (h,\, h + df) = (h,\, \delta^c\varphi) = (d^c h,\, \varphi) = 0 \,.$$

□

Definition 3.9. *The function f given by the lemma above will be called the* generalized trace *of the special form φ. It is only defined up to a constant.*

We can now state the main result of this section

Theorem 3.10. *Let u be a form of degree p on a compact Kähler manifold M^{2m} and suppose that $n-2 \geq p \geq 2$ and $p \neq m$. Then u is a twistor form if and only if there exists a special form φ whose generalized trace is f and a positive integer k such that $p = 2k$ and*

$$u \;=\; L^{k-1}\varphi \;-\; \tfrac{m-p}{p(m^2-1)} L^k f \;+\; \textit{parallel form.}$$

The same statement is valid for $p = m$ under the additional assumption that Ju is parallel.

Proof. Consider first the case $p \leq m$. Using the notations from Lemma 3.6, the twistor equation for u reads

$$\nabla_X u_{k-1} \;+\; L\nabla_X u_k \;=\; \tfrac{k}{p+1} JX \wedge v \;-\; \tfrac{1}{n-p+1} X \wedge w. \tag{17}$$

An equality between 2–forms is equivalent with the equality of their primitive parts and that of their traces with respect to the Kähler form. The equality of the primitive parts in (17) is equivalent to

$$\nabla_X u_{k-1} \;=\; (\tfrac{k}{p+1} JX \wedge v \;-\; \tfrac{1}{n-p+1} X \wedge w) \;-\; \tfrac{1}{m}\Lambda(\tfrac{k}{p+1} JX \wedge v \;-\; \tfrac{1}{n-p+1} X \wedge w)\,\omega\;,$$

where ω is the Kähler form. Now, $\Lambda(JX \wedge v) = -\langle v, X\rangle$ and $\Lambda(X \wedge w) - \langle Jw, X\rangle$ so we obtain

$$\begin{aligned}
\nabla_X u_{k-1} \;&=\; \tfrac{k}{p+1}\, JX \wedge v \;-\; \tfrac{1}{n-p+1}\, X \wedge w \;+\; \tfrac{1}{m}\,(\tfrac{k}{p+1}\,\langle v, X\rangle \;-\; \tfrac{1}{n-p+1}\,\langle Jw, X\rangle)\,\omega \\
&=\; \tfrac{k}{p+1}\, JX \wedge v \;-\; \tfrac{1}{n-p+1}\,\tfrac{k(n-p+1)}{p+1}\, X \wedge Jv \;+\; (\tfrac{k}{m(p+1)} \;-\; \tfrac{1}{m(n-p+1)}\,\tfrac{k(n-p+1)}{p+1})\,\langle v, X\rangle\,\omega \\
&=\; \tfrac{k}{p+1}\, JX \wedge v \;-\; \tfrac{k}{p+1}\, X \wedge Jv \;+\; \tfrac{2k}{m(p+1)}\,\langle v, X\rangle\,\omega \\
&=\; \gamma \wedge JX \;-\; J\gamma \wedge X \;-\; \tfrac{2}{m}\,\gamma(X)\,\omega\;,
\end{aligned}$$

where γ is the 1–form defined by $\gamma(X) = -\frac{k}{p+1}\langle v, X\rangle$. Contracting with JX shows that $\gamma = \frac{m}{2(m^2-1)}\delta^c u_{k-1}$, so $v = \frac{(p+1)m}{p(m^2-1)}\delta^c u_{k-1}$.

The second part of (17) consists in the equality of the traces, which is equivalent to

$$X(u_k) \;=\; -\,\tfrac{p}{m(p+1)}\,\langle v, X\rangle \;+\; \tfrac{1}{p+1}\,\langle v, X\rangle \;=\; \tfrac{m-p}{m(p+1)}\, X \lrcorner\, v\;,$$

i.e.

$$du_k \;=\; \tfrac{m-p}{m(p+1)}\, v \;=\; -\tfrac{m-p}{p(m^2-1)}\delta^c u_{k-1}.$$

We have shown that u is a twistor form if and only if u_{k-1} is a special form and u_k is $-\frac{m-p}{p(m^2-1)}$ times its generalized trace.

A priori, this only proves the theorem for $p \leq m$, but because of the invariance of the hypothesis and conclusion of the theorem with respect to the Hodge duality, the theorem is proved in full generality. □

Specializing the above result for $k=1$ yields the following characterization of twistor 2–forms also obtained in [3]. Note that the result that we obtain is more complete than that of [3] since we do not assume the invariance of the 2–form u.

Lemma 3.11. *Let (M^{2m}, g, J, ω) be a Kähler manifold. Then a 2–form u is a twistor form if and only if there exists a 1–form γ with*

$$\nabla_X u \;=\; \gamma\wedge JX \;-\; J\gamma\wedge X \;-\; \gamma(X)\,\omega\;. \tag{18}$$

The 1–form γ is necessarily equal to $\gamma = \frac{1}{2m-1}\,J\delta u$. Moreover, $\gamma = -\frac{1}{m-2}\,d\,\langle u,\omega\rangle$ provided that $m>2$.

Proof. First of all we can rewrite equation (18) as $\nabla_X u = -X \lrcorner\, (\gamma\wedge\omega) + X\wedge J\gamma$. Contracting (resp. taking the wedge product) in (18) with X and summing over an orthonormal frame $\{e_i\}$ we obtain

$$J\gamma \;=\; -\,\tfrac{1}{2m-1}\,\delta u \qquad\text{and}\qquad \gamma\wedge\omega \;=\; -\,\tfrac{1}{3}\,du\;.$$

Substituting this back into (18) yields the defining equation for a twistor 2–form, i.e.:

$$\nabla_X u \;=\; \tfrac{1}{3}\,X\lrcorner\, du \;-\; \tfrac{1}{2m-1}\,X\wedge\delta u\;.$$

Conversely, Theorem 3.10 shows that $u = u_0 + Lf$, where

$$\nabla_X u_0 \;=\; (\gamma\wedge JX \;-\; J\gamma\wedge X) \;-\; \tfrac{2}{m}\,\gamma(X)\,\omega\;,$$

and $df = -\frac{m-2}{2(m^2-1)}\delta^c u_0 = -\frac{(m-2)}{m}\gamma$, so clearly u satisfies (18). ∎

This characterization of twistor 2–forms in particular implies that for $m>2$ special forms are just the primitive parts of twistor 2–forms and vice versa. In the remaining part of this section we will describe a similar relation between twistor forms and Hamiltonian 2–forms. Using the results of [3] we can thus produce many examples of non–parallel twistor forms.

Definition 3.12. *An invariant 2–form ψ is called Hamiltonian if there is a function σ such that*

$$\nabla_X\psi \;=\; \tfrac{1}{2}\,(d\sigma\wedge JX \;-\; Jd\sigma\wedge X)$$

for any vector field X. When $m=2$ one has to require in addition that $J\mathrm{grad}(f)$ is a Killing vector field.

It follows immediately from the definition that $d\sigma = d\langle\psi,\omega\rangle$. Hence, one could without loss of generality replace σ with $\langle\psi,\omega\rangle$.

Any special 2–form φ with generalized trace f defines an affine line $\varphi + \mathbb{R}f\omega$ in the space of 2–forms modulo constant multiples of the Kähler form. Lemma 3.11 shows that this line contains a twistor 2–form and it is not difficult to see that it contains a unique closed form and also an unique coclosed form. Indeed, for some real number x, $d(\varphi + xf\omega) = 0$ is equivalent to $-\frac{1}{m+1}L\delta^c\varphi + xLdf = 0$, i.e. $x = \frac{1}{m+1}$, and $\delta(\varphi + xf\omega) = 0$ is equivalent to $\delta\varphi = xd^c f = xJdf = xJ\delta^c\varphi = -x\delta\varphi$, i.e. $x=-1$. The following proposition shows that this affine line also contains a Hamiltonian 2–form.

Proposition 3.13. *Let (M^{2m}, g, J, ω) be a Kähler manifold with a Hamiltonian 2–form ψ. Then $u := \psi - \frac{\langle\psi,\omega\rangle}{2}\,\omega$ is a twistor 2–form. Conversely, if u is a twistor 2–form and $m > 2$, then $\psi := u - \frac{\langle u,\omega\rangle}{m-2}\,\omega$ is a Hamiltonian 2–form.*

Proof. Let ψ be a Hamiltonian 2–form and let u be defined as $u := \psi - \frac{\langle\psi,\omega\rangle}{2}\,\omega$, then

$$\begin{aligned} \nabla_X u \;&=\; \nabla_X \psi \;-\; \tfrac{1}{2}\, d\langle\psi,\omega\rangle(X)\,\omega \\ &=\; \tfrac{1}{2}\,(d\sigma\wedge JX \;-\; Jd\sigma\wedge X) \;-\; \tfrac{1}{2}\, d\sigma(X)\,\omega\;. \end{aligned}$$

Thus, the invariant 2–form u satisfies the equation (18) of Lemma 3.11 with, $\gamma := \frac{1}{2}\,d\sigma$, and it follows that u is a twistor 2–form. Conversely, starting from a twistor 2–form u and defining $\psi := u - \frac{\langle u,\omega\rangle}{m-2}\,\omega$, we can use the characterization of Lemma 3.11 to obtain

$$\nabla_X\psi \;=\; \gamma\wedge JX \;-\; J\gamma\wedge X\;.$$

Since for $m > 2$ we have $\gamma = -\frac{1}{m-2} d\langle u,\omega\rangle = \frac{1}{2} d\langle\psi,\omega\rangle$, which implies that ψ has to be a Hamiltonian 2–form. □

The situation is somewhat different in dimension 4. If ψ is a Hamiltonian 2–form then $u = \psi - \frac{\langle\psi,\omega\rangle}{2}\omega = u_0$ is an anti–self–dual twistor 2–form (or equivalently, invariant and primitive). Conversely, if we start with an anti–self–dual twistor 2–form u_0 and ask for which functions f the invariant 2–form $\psi := u_0 + f\omega$ is a Hamiltonian 2–form we have

Lemma 3.14. *Let (M^4, g, J) be a Kähler manifold with an anti-self-dual twistor 2–form u_0. Then $\psi := u_0 + f\omega$ is Hamiltonian 2–form if and only if*

$$\delta u_0 \;=\; -\,3\,Jdf \tag{19}$$

and δu_0 is dual to a Killing vector field. In particular, this is the case on simply connected Einstein manifolds.

Proof. Since $f = \frac{\langle\psi,\omega\rangle}{2}$ we conclude from the definition that ψ is a Hamiltonian 2–form if and only if $\nabla_X\psi = df\wedge JX - Jdf\wedge X$ for any vector field X. On the other hand u_0 is assumed to be an invariant twistor 2–form. Hence, from Lemma 3.11 u_0 satisfies $\nabla_X u_0 = -X \lrcorner\, (\gamma\wedge\omega) - J\gamma\wedge X$, where $\gamma = \frac{1}{3}J\delta u_0$. This implies that $\psi = u_0 + f\omega$ is a Hamiltonian 2–form if and only if

$$-X \lrcorner\, (\gamma\wedge\omega) - J\gamma\wedge X \;=\; df\wedge JX - Jdf\wedge X - df(X)\omega \;=\; -X \lrcorner\, (df\wedge\omega) - Jdf\wedge X.$$

This is the case if and only if $\gamma = df$, or equivalently if $\delta u_0 \;=\; -\,3\,Jdf$. If the complex dimension is 2, the definition of Hamiltonian 2–forms made the additional requirement that $Jd\sigma$, which here is proportional to δu_0, is dual to Killing vector field. Hence, it remains to show that on simply connected Einstein manifolds the equation (19) has a solution such that δu_0 is dual to a Killing vector field. Let X be any Killing vector field on an arbitrary Kähler manifold, then

$$0 \;=\; L_X\,\omega \;=\; d\,X \lrcorner\,\omega \;+\; X \lrcorner\, d\omega \;=\; d\,J\,X\;,$$

i.e. the 1–form JX is closed. Hence, since the manifold is simply connected, there exists some function f with $JX = df$ and also $X = -Jdf$. Finally, it is easy to see that on Einstein manifolds, for any twistor 2–form u_0, the 2–form δu_0 is dual to a Killing vector field (cf. [14]). □

Hamiltonian 2–forms have the following remarkable property (c.f. [3]). If ψ is Hamiltonian, and if $\sigma_1, \ldots, \sigma_m$ are the elementary symmetric functions in the eigenvalues of ψ with respect to the Kähler form ω, then all vector fields $K_j = J\mathrm{grad}(\sigma_j)$ are Killing. Furthermore, the Poisson brackets $\{\sigma_i, \sigma_j\}$ vanish, which implies that the Killing vector fields $K_1, \ldots, K_m$ commute. If the Killing vector fields are linearly independent, then the Kähler metric is toric. But even if they are not linearly independent one has further interesting properties, which eventually lead to a complete local classification of Kähler manifolds with Hamiltonian 2–forms in [3]. The most important sources of Kähler manifolds with Hamiltonian 2–forms are weakly Bochner–flat Kähler manifolds and (in dimension greater than four) Kähler manifolds which are conformally-Einstein. A Kähler manifold is weakly Bochner–flat if its Bochner tensor is coclosed, which is the case if and only if the normalized Ricci form is a Hamiltonian 2–forms. The examples include some Hirzebruch surfaces and the complex projective spaces.

4. The middle dimension

In this section we study twistor forms of degree m on compact Kähler manifolds of real dimension $n = 2m$. This case is very special thanks to the following

Lemma 4.1. *Let $\Lambda^m M = L_1 \oplus \ldots \oplus L_r$ be a decomposition of the bundle of m–forms in parallel subbundles and $u = u_1 + \ldots + u_r$ be the corresponding decomposition of an arbitrary m–form u. Then u is a twistor form if and only if u_i is a twistor form for every i.*

Proof. In the case of m–forms on $2m$-dimensional manifolds Proposition 2.3 provides a characterization of twistor forms similar to that of Killing forms. A m–form u is a twistor form if and only if

$$\Delta u \;=\; \frac{m+1}{m}\, q(R)\, u \tag{20}$$

Given a decomposition of the form bundle $\Lambda^m M$ into parallel subbundles we know that the Laplace operator Δ as well as the symmetric endomorphism $q(R)$ preserve this decomposition. Hence, equation (20) can be projected onto the summands, i.e. (20) is satisfied for $u = u_1 + \ldots + u_r$ if and only it is satisfied for all summands u_i, which proves the lemma. □

In Section 3 we defined special 2–forms, which turned out to be the main building block for twistor p–forms with $p \neq m$. In dealing with twistor m–forms on $2m$–dimensional Kähler manifolds we have to introduce the following

Definition 4.2. *A special m–form on a $2m$–dimensional Kähler manifold is a m–form ψ of type $(1, m-1) + (m-1, 1)$ satisfying for all vector fields X the equation*

$$\nabla_X \psi \;=\; (m-1)\,\tau \wedge JX \;-\; J\tau \wedge X \;-\; (m-1)\,(X \lrcorner \tau) \wedge \omega \;, \tag{21}$$

for some 1–form τ, which then necessarily equals $\frac{1}{m^2-1}\delta^c\psi$.

Note that for $m = 2$, we retrieve the definition of special 2–forms on 4–dimensional manifolds (Definition 3.7).

Proposition 4.3. *Let u be a m–form on a compact Kähler manifold M^{2m}. Then u is a twistor form if and only if $m = 2k$ and there exists a special 2–form φ and a special m–form ψ such that*

$$u \;=\; L^{k-1}\,\varphi \;+\; \psi \;+\; \textit{parallel form.}$$

Proof. Lemma 4.1 shows that we can assume $u = L^k u_k$ for some primitive form u_k. Suppose $k \geq 1$, then it follows from Proposition 3.1 that $J\Lambda u = J\Lambda L^k u_k = 0$. Thus Lemma 3.5 implies $JL^{k-1}u_k = 0$. Since L^{k-1} is injective on Λ^{m-2k}, we get $Ju_k = 0$ and eventually $Ju = 0$. For J–invariant twistor forms the conclusion of Theorem 3.10 also holds in the case $p = m$. Hence, $u = L^{k-1}\phi + parallel\, form$, where ϕ is a special 2–form.

It remains to treat the case $k = 0$, i.e. the case where u is a primitive m–form. Contracting the twistor equation with the Kähler form yields

$$X \lrcorner \delta^c u \;+\; JX \lrcorner \delta u \;=\; 0 \tag{22}$$

for all vector fields X. After wedging with X and JX and summing over an orthonormal basis, this gives

$$J\,\delta\,u \;=\; (m-1)\,\delta^c\,u \qquad \text{and} \qquad J\,\delta^c u \;=\; -\,(m-1)\,\delta\,u\;. \tag{23}$$

Now, $*u$ is again a twistor m–form and the above argument shows that we can suppose $*u$ to be primitive. (Otherwise, $*u = L^{k-1}\phi + parallel\, form$, for a special 2–form ϕ, which in particular is self-dual, i.e. $u = L^{k-1}\phi + parallel\, form$). Applying the Hodge star operator to $\Lambda(*u) = 0$ immediately implies $Lu = 0$. This shows that $du = -L\delta^c u$, so the twistor equation becomes

$$\begin{aligned}\nabla_X\,u \;&=\; \frac{1}{m+1}\,(\,-\,X \lrcorner\, L\,\delta^c\,u \;-\; X \wedge \delta\,u\,) \qquad (24)\\ &=\; \frac{1}{m+1}\,(\,-\,JX \wedge \delta^c u \;-\; L\,(X \lrcorner\, \delta^c u) \;+\; \frac{1}{m-1}\,X \wedge J\delta^c u\,)\;,\end{aligned}$$

which is just the defining equation (21) of a special m–form with $\tau = \frac{1}{m^2-1}\delta^c u$. According to our definition of special m–forms, we still have to show that (up to parallel forms) u is of type $(m-1, 1) + (1, m-1)$. Equivalently we will show this for $\nabla_X u$, where X is any vector field. Recall that u is of type $(m-1, 1) + (1, m-1)$ if and only if $J^2u = -(m-2)^2u$. We will compute $J^2(\nabla_X u)$ using the twistor equation. Equation (23) implies that δu is of type $(m-1,0)+(0,m-1)$, i.e $J^2(\delta u) = -(m-1)^2\delta u$. Moreover, using (22) and (23) we obtain

$$\begin{aligned} J^2(X\wedge\delta u) &= (-1-(m-1)^2)\,X\wedge\delta u \;+\; 2\,JX\wedge J\delta u \\ &= (-1-(m-1)^2)\,X\wedge\delta u \;+\; 2\,(m-1)\,JX\wedge\delta^c u \end{aligned}$$

and similarly

$$\begin{aligned} J^2(X\lrcorner\, L\delta^c u) &= (-1-(m-1)^2)\,X\lrcorner\, L\delta^c u \;+\; 2\,JX\lrcorner\, JL\delta^c u \\ &= (-1-(m-1)^2)\,X\lrcorner\, L\delta^c u \;-\; 2\,(m-1)\,JX\lrcorner\, L\delta u \\ &= -(m-2)^2\,X\lrcorner\, L\delta^c u \;+\; 2\,(m-1)X\wedge\delta u \;-\; 2(m-1)\,JX\wedge\delta^c u\,. \end{aligned}$$

Adding these two equations implies $J^2(\nabla_X u) = -(m-2)^2\nabla_X u$ after using the twistor equation as written in (24). $\square$

As an application of Proposition 4.3 we see that any twistor 2–form on a 4-dimensional Kähler manifold has to be invariant and primitive, i.e. any twistor 2–form has to be a special 2–form. In this case we know from Lemma 3.14 that any Hamiltonian 2–form gives rise to a twistor 2–form and vice versa, under some additional conditions. This clarifies the situation of twistor 2–forms in dimension 4 and shows in particular that one can exhibit many examples. However, one can show that any special m–form ($m\geq 3$) on a Kähler-Einstein manifold has to be parallel. Nevertheless for the moment it remains unclear whether one can exclude these forms in general.

5. Twistor forms on the complex projective space

In this section we describe the the construction of twistor forms on the complex projective space (c.f. [3]). Let $M=\mathbb{C}P^m$ be equipped with the Fubini-Study metric and the corresponding Kähler form ω. Then the Riemannian curvature is given as

$$R_{X,Y}Z \;=\; -\,(X\wedge Y \;+\; JX\wedge JY)\,Z \;-\; 2\,\omega(X,Y)\,JZ$$

for any vector fields X,Y,Z. This implies for the Ricci curvature $\mathrm{Ric} = 2(m+1)\mathrm{id}$. Let K be any Killing vector field on $\mathbb{C}P^m$. Then there exists a function f with $\Delta f = 4(m+1)f$ and $K = J\mathrm{grad}(f)$, i.e. f is an eigenfunction of the Laplace operator for the first non–zero eigenvalue. Now, consider the 2–form $\phi := dK = dJdf = dd^c(f)$. Since K is a Killing vector field it follows:

$$\begin{aligned} \nabla_X\,\phi &= \nabla_X(dK) \;=\; 2\,\nabla_X(\nabla K) \;=\; 2\,\nabla^2_{X,\cdot}\,K \;=\; -\,2\,R(K,X) \\ &= -\,2\,(df\wedge JX \;-\; Jdf\wedge X) \;-\; 4\,df(X)\,\omega\,. \end{aligned}$$

It is clear that ϕ is an invariant 2–form and an eigenform of the Laplace operator for the minimal eigenvalue $4(m+1)$. A small modification of ϕ yields a twistor 2–form.

Indeed, defining $\hat{\phi} := \phi + 6f\omega$ one obtains

$$\nabla_X \hat{\phi} \;=\; -2\,(df \wedge JX \;-\; Jdf \wedge X) \;+\; 2\,df(X)\,\omega \;.$$

Using Lemma 3.11 for $\gamma := -2\,df$ one concludes that $\hat{\phi}$ is a twistor 2–form. It is not difficult to show that indeed any invariant twistor 2–form on $\mathbb{C}P^m$ has to be of this form. Summarizing the construction one has

Proposition 5.1 ([3]). *Let $K = J\mathrm{grad}(f)$ be any Killing vector field on the complex projective space $\mathbb{C}P^m$ then*

$$\hat{\phi} \;:=\; dd^c(f) \;+\; 6\,f\,\omega \;=\; (dd^c(f))_0 \;+\; \tfrac{2m-4}{m}\,f\,\omega$$

defines an invariant, non–parallel twistor 2–form. Moreover, in dimension four, $\hat{\phi}$ is a primitive 2–form, i.e. $\hat{\phi} = (dd^c(f))_0$.

6. Concluding Remarks

For the convenience of the reader, we resume here the main results of the paper, as well as the remaining open questions.

If u is a twistor p–form on a compact Kähler manifold M^{2m} with $2 \le p \le n-2$ and $p \neq m$, then it turns out that Ju has to be parallel (Corollary 3.2). In this case u is completely determined (modulo parallel forms) by a special 2–form and its generalized trace (Theorem 3.10). In particular, the degree of u has to be even.

If $p = m$, the J–invariant part of u can still be characterized (up to parallel forms) by special 2–forms, but the complete classification of non–parallel twistor forms can only be obtained up to the hypothetical existence of *special m–forms* of type $(m-1,1)+(1,m-1)$. At this moment we have no examples of such forms for $m > 2$, and moreover we were able to show that they don't exist on compact Kähler–Einstein manifolds. Note that for $m = 2$, the definition of special m–forms coincides with the previous one for special 2–forms on 4–manifolds.

For $m > 2$, each special 2–form φ with generalized trace f (defined by $df = \delta^c\varphi$) determines an affine line $\{\varphi_x = \varphi + xf\omega\}$ in the space of two–forms, which contains distinguished elements : φ_x is Hamiltonian for $x = \frac{1}{m^2-1}$, closed for $x = \frac{1}{m+1}$, coclosed for $x = -1$ and a twistor form for $x = -\frac{m-2}{2(m^2-1)}$.

For $m = 2$, the picture is slightly different since the generalized trace of a special 2–form (and thus the affine line above) are no longer defined. The results are as follows : every twistor 2–form is (up to parallel forms) primitive and invariant (i.e. anti–self–dual), and defines a special 2–form. This special 2–form induces a Hamiltonian 2–form if and only if its codifferential is a Killing vector field. Conversely, a Hamiltonian 2–form always defines the affine line above, and in particular its primitive part is simultaneously a special 2–form and a twistor form. There are examples of twistor 2–forms on 4–dimensional manifolds which do not come from Hamiltonian 2–forms [1]. The complete local classification of Hamiltonian 2–forms was obtained in [2] for $m = 2$ and [3] in the general case. The same references contain examples of compact Kähler manifolds with non–parallel Hamiltonian 2–forms.

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Andrei Moroianu, CMAT, École Polytechnique, UMR 7640 du CNRS, 91128 Palaiseau, France

E-mail address: am@math.polytechnique.fr

Uwe Semmelmann, Mathematisches Institut, Universität München, Theresienstr. 39, D-80333 München, Germany

E-mail address: Uwe.Semmelmann@mathematik.uni-muenchen.de