KILLING FORMS ON QUATERNION–KÄHLER MANIFOLDS

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Abstract. We show that every Killing p–form on a compact quaternion–Kähler manifold has to be parallel for $p \geq 2$.

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1. Introduction

In this paper we continue the study of twistor forms on compact Riemannian manifolds with non–generic holonomy initiated in [2] and [3]. While the cases studied in the previous articles concern Kähler, G_2– and $Spin_7$–manifolds, we now turn our attention to the quaternion–Kähler situation, which is the last one in the Berger list of irreducible non–locally symmetric Riemannian structures.

Recall that twistor (resp. Killing) 1–forms are duals of conformal (resp. Killing) vector fields. Twistor p–forms are natural generalizations of twistor 1–forms, defined by the property that the projection of their covariant derivative on the Cartan product of the cotangent bundle and the p–form bundle vanishes, and Killing p–forms have the further property of being coclosed.

The main result of this paper is the fact that every Killing p–form ($p \geq 2$) on a compact quaternion–Kähler manifold is automatically parallel (Theorem 6.2). The techniques used in the proof are both representation–theoretic and analytic. We first compute some Casimir operators for the group $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ which give explicit formulas for natural algebraic operators defined on the exterior bundle of quaternion–Kähler manifolds. We then introduce natural differential operators (similar to d^c and δ^c in Kähler geometry) on every quaternion–Kähler manifold, compute commutator relations between them, and apply Weitzenböck–type formulas in order to show that every Killing form has to be closed, and hence parallel.

The structure of the paper is the following. In Section 2 we recall general facts about Killing and twistor forms, in Section 3 we describe the decomposition of the exterior bundle of a quaternion–Kähler manifold (analog to the LePage decomposition on Kähler manifolds), and in Section 4 we introduce natural algebraic operators on the exterior

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bundle and study their behavior with respect to this decomposition. The next section deals with natural first order differential operators, and in Section 6 we prove the main result. Some basics on Casimir operators are explained in the Appendix.

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2. Twistor forms on Riemannian manifolds

Let $(V, \langle\cdot,\cdot\rangle)$ be an n–dimensional Euclidean vector space. The tensor product $V^*\otimes \Lambda^p V^*$ has the following $O(n)$–invariant decomposition:

$$V^*\otimes\Lambda^pV^* \;\cong\; \Lambda^{p-1}V^*\oplus\Lambda^{p+1}V^*\oplus\mathcal{T}^{p,1}V^*$$

where $\mathcal{T}^{p,1}V^*$ is the intersection of the kernels of wedge and inner product maps, which can be identified with the Cartan product of V^* and Λ^pV^*. This decomposition immediately translates to Riemannian manifolds (M^n, g), where we have

$$T^*M\otimes\Lambda^pT^*M \;\cong\; \Lambda^{p-1}T^*M\oplus\Lambda^{p+1}T^*M\oplus\mathcal{T}^{p,1}M \tag{1}$$

with $\mathcal{T}^{p,1}M$ denoting the vector bundle corresponding to the vector space $\mathcal{T}^{p,1}V^*$. The covariant derivative $\nabla\psi$ of a p–form ψ is a section of $T^*M\otimes\Lambda^pT^*M$. Its projections onto the summands $\Lambda^{p+1}T^*M$ and $\Lambda^{p-1}T^*M$ are just the differential $d\psi$ and the codifferential $\delta\psi$. Its projection onto the third summand $\mathcal{T}^{p,1}M$ defines a natural first order differential operator T, called the *twistor operator*. The twistor operator $T : \Gamma(\Lambda^pT^*M)\to\Gamma(\mathcal{T}^{p,1}M)\subset\Gamma(T^*M\otimes\Lambda^pT^*M)$ is given for any vector field X by the following formula

$$[T\psi](X) \;:=\; [\mathrm{pr}_{\mathcal{T}^{p,1}M}(\nabla\psi)](X) \;=\; \nabla_X\psi \;-\; \tfrac{1}{p+1}X\lrcorner\, d\psi \;+\; \tfrac{1}{n-p+1}X\wedge\delta\psi\;.$$

Note that here, and in the remaining part of this article, we identify vectors and 1–forms using the metric.

Definition 2.1. *Differential forms in the kernel of the twistor operator are called* twistor forms *(or* conformal Killing forms *by some authors).*

A p–form ψ is a twistor p–form, if and only if it satisfies the equation

$$\nabla_X\psi \;=\; \tfrac{1}{p+1}X\lrcorner\, d\psi \;-\; \tfrac{1}{n-p+1}X\wedge\delta\psi\;, \tag{2}$$

for all vector fields X. In the physics literature this equation is also called *Killing–Yano equation.* In this article we are interested in twistor forms which are in addition coclosed.

Definition 2.2. *A p–form ψ is called a Killing p–form if and only if ψ is coclosed and in the kernel of T, i.e. if and only if ψ satisfies*

$$\nabla_X\psi \;=\; \tfrac{1}{p+1}X\lrcorner\, d\psi \tag{3}$$

for all vector fields X.

Equivalently Killing p–forms may be described as p–forms ψ for which $\nabla\psi$ is a $(p+1)$–form, or by the condition that $X \lrcorner \nabla_X \psi = 0$ for all vector fields X. Equation (3) is a natural generalization of the defining equation for Killing vector fields, i.e. Killing 1–forms are dual to Killing vector fields.

It is easy to see that T^*T is an elliptic operator. Hence the space of twistor forms is finite dimensional on compact manifolds. It actually turns out that this space is finite dimensional on any connected manifold. The upper bound of the dimension is given by the dimension of the space of twistor forms on the standard sphere (cf. [3]), which coincides with the eigenspace of the Laplacian on p–forms corresponding to the smallest eigenvalue. In particular, Killing forms on the standard sphere are precisely the coclosed minimal eigenforms.

The only other known examples of compact manifolds admitting Killing forms in degree greater than one are Sasakian manifolds, nearly Kähler manifolds, weak G_2–manifolds and products of these manifolds (cf. [3]).

On compact manifolds one can characterize Killing vector fields as divergence–free vector fields in the kernel of $\Delta - 2\mathrm{Ric}$. A similar characterization of arbitrary Killing forms may be given (see for instance [3]):

Proposition 2.3. *Let (M^n, g) be a compact Riemannian manifold with a coclosed p–form ψ. Then ψ is a Killing form if and only if*

$$\Delta \psi \;=\; \frac{p+1}{p}\, q(R)\, \psi \;,$$

where $q(R)$ is defined as the curvature term appearing in the Weitzenböck formula $\Delta = \nabla^*\nabla + q(R)$, for the Laplace operator Δ acting on p–forms.

As a corollary to this characterization we immediately obtain an interesting property of Killing forms on manifolds admitting a parallel form.

Lemma 2.4. *If Ω is a parallel k–form and ψ is a Killing p–form on a compact manifold M, then the contraction $\Omega \lrcorner u$ of ψ with Ω is a parallel $(p-k)$–form.*

Proof. First of all we note that $\Omega \lrcorner u$ is again a Killing form. Indeed we have

$$X \lrcorner \nabla_X(\Omega \lrcorner \psi) = X \lrcorner \Omega \lrcorner \nabla_X \psi = (-1)^k\, \Omega \lrcorner X \lrcorner \nabla_X \psi = 0 \;.$$

From Proposition 2.3 it follows that $\Delta\psi = \frac{p+1}{p}\, q(R)\, \psi$. Since the contraction with a parallel form commutes with the Laplacian and with $q(R)$ we obtain

$$\Delta(\Omega \lrcorner \psi) \;=\; \frac{p+1}{p}\, q(R)\, (\Omega \lrcorner \psi) \;.$$

But since $\Omega \lrcorner \psi$ is a Killing $(p-k)$–form, Proposition 2.3 also implies that

$$\Delta(\Omega \lrcorner \psi) \;=\; \frac{p-k+1}{p-k}\, q(R)\, (\Omega \lrcorner \psi) \;.$$

Comparing these two equations for $\Delta(\Omega \lrcorner\, \psi)$ yields that the Killing form $\Omega \lrcorner\, \psi$ is harmonic. Hence, since M is compact, ψ is closed and coclosed and thus parallel. $\square$

In the following we need further information on the curvature term $q(R)$. First of all it is a symmetric endomorphism of the bundle of differential forms defined by

$$q(R) \;=\; \sum e_j \wedge e_i \lrcorner\, R_{e_i,e_j}, \tag{4}$$

where $\{e_i\}$ is any local orthonormal frame and R_{e_i,e_j} denotes the curvature of the form bundle. On forms of degree one and two one has an explicit expression for the action of $q(R)$, e.g. if ξ is any 1–form, then $q(R)\,\xi = \mathrm{Ric}(\xi)$. In fact it is possible to define $q(R)$ in a more general context. For this we first rewrite equation (4) as

$$q(R) \;=\; \sum_{i<j} (e_j \wedge e_i \lrcorner \;-\; e_i \wedge e_j \lrcorner\,)\, R_{e_i,e_j} \;=\; \sum_{i<j} (e_i \wedge e_j) \bullet R(e_i \wedge e_j)\bullet$$

where the Riemannian curvature R is considered as element of $\mathrm{Sym}^2(\Lambda^2 T_pM)$ and $\bullet$ denotes the standard representation of the Lie algebra $\mathfrak{so}(T_pM) \cong \Lambda^2 T_pM$ on the space of p–forms. Note that we can replace $e_i \wedge e_j$ by any orthonormal basis of $\mathfrak{so}(T_pM)$. Let (M, g) be a Riemannian manifold with holonomy group Hol. Then the curvature tensor takes values in the Lie algebra $\mathfrak{hol}$ of the holonomy group, i.e. we can write $q(R)$ as

$$q(R) \;=\; \sum \omega_i \bullet R(\omega_i)\bullet \qquad \in\; \mathrm{Sym}^2(\mathfrak{hol})$$

where $\{\omega_i\}$ is any orthonormal basis of $\mathfrak{hol}$ and $\bullet$ denotes the form representation restricted to the holonomy group. Writing the bundle endomorphism $q(R)$ in this way has two immediate consequences: we see that $q(R)$ preserves any parallel subbundle of the form bundle and it is clear that by the same definition $q(R)$ gives rise to a symmetric endomorphism on any associated vector bundle defined via a representation of the holonomy group.

3. Quaternion-Kähler manifolds

Let (M^{4m}, g) be a *quaternion-Kähler manifold* defined by the holonomy reduction to $\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1) \subset \mathrm{SO}(4m)$. Any representation of $\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1)$ gives rise to a vector bundle over M. The tensor product of a $\mathrm{Sp}(1)$ representation and a $\mathrm{Sp}(m)$ representation defines a vector bundle if and only if it factors through the projection $\mathrm{Sp}(m) \times \mathrm{Sp}(1) \to \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$. In particular the standard representations H resp. E of $\mathrm{Sp}(1)$ resp. $\mathrm{Sp}(m)$ induce only locally defined vector bundles, whereas $\mathrm{Sym}^2 H$ or $E \otimes H \cong T^*M^{\mathbb{C}}$ are globally defined. In the following we will often make no difference between a vector bundle and its defining representation.

More explicitly the quaternion-Kähler reduction may be described by three locally defined almost complex structures J_α which satisfy the quaternion relations and span a rank three subbundle of $\mathrm{End}(TM)$ preserved by the Levi–Civita connection. The complexification of this subbundle is isomorphic to $\mathrm{Sym}^2 H$ and in any point $x \in M$ the subspace in $\mathrm{End}(T_xM)$ spanned by the three almost complex structures is isomorphic to the Lie algebra $\mathfrak{sp}(1)$. Since J_α is a skew-symmetric endomorphism one may realize $\mathfrak{sp}(1)$ also as a subspace of $\Lambda^2(T^*_xM)$. Under this identification J_α is mapped to the 2-form $\omega_\alpha = \frac{1}{2}\sum e_i \wedge J_\alpha e_i$, where $\{e_i\}$ is any orthonormal basis of T_xM.

We still need some information on the decomposition of the form bundle of a quaternion Kähler manifold. Of course this decomposition corresponds to the decomposition of $\Lambda^p(H\otimes E)$ under the action of the group $\mathrm{Sp}(1)\cdot\mathrm{Sp}(m)$. Details for this can be found in [5]. First of all it is easy to see that all possible irreducible summands are of the form $\mathrm{Sym}^kH\otimes\Lambda^{a,b}_0E$, where $\Lambda^{a,b}_0E\subset\Lambda^a_0E\otimes\Lambda^b_0E$ denotes the Cartan product of the two irreducible $\mathrm{Sp}(m)$–representations Λ^a_0E and Λ^b_0E. E.g. the decomposition of the 2– and 3–forms is given as

$$\Lambda^2(H\otimes E) \;=\; \Lambda^{1,1}_0E \;\oplus\; \mathrm{Sym}^2H\otimes[\Lambda^2_0E\oplus\mathbb{C}]\ ,$$

$$\Lambda^3(H\otimes E) \;=\; H\otimes[\Lambda^{2,1}_0E\oplus E] \;\oplus\; \mathrm{Sym}^3H\otimes[\Lambda^3_0E\oplus E]\ .$$

Analyzing the form representation in more detail it is actually possible to determine for which numbers (k,a,b) the summand $\mathrm{Sym}^kH\otimes\Lambda^{a,b}_0E$ appears in the decomposition of the space of p–forms. In this article we will only need the following weaker information.

Lemma 3.1 ([5]). *Let $\mathrm{Sym}^kH\otimes\Lambda^{a,b}_0E$ be an irreducible summand appearing in the decomposition of $\Lambda^p(H\otimes E)$. Then the numbers (k,a,b) satisfy the conditions*

$$\begin{array}{ll} (i) & 0\le b\le a\le m\ ,\\ (ii) & k\le p-2b\le p\ ,\\ (iii) & a+b\le p\ .\\ (iv) & 2a\le\min\{p+k,m-p+k\} \end{array}$$

Moreover, the numbers k,p and $a+b$ have the same parity.

It is rather difficult to describe the action of the differential d and the codifferential δ on the irreducible subbundles $\mathrm{Sym}^kH\otimes\Lambda^{a,b}_0E$ of the form bundle. However, it is much easier to get the following information.

Lemma 3.2. *For any p–form u which is a section in $\mathrm{Sym}^kH\otimes\Lambda^{a,b}_0E$ the forms du and δu are sections in a sum of bundles of the type $\mathrm{Sym}^{k'}H\otimes\Lambda^{a',b'}_0E$ where the numbers (k',a',b') satisfy the conditions*

$$\begin{array}{ll} (i) & |k-k'|\;=\;1\ ,\\ (ii) & |a-a'|\;\le\;1\ ,\\ (iii) & |b-b'|\;\le\;1\ , \end{array}$$

and the conditions of Lemma 3.1.

Proof. The differential d resp. the codifferential δu are projections of ∇u onto $\Lambda^{p+1}T^*M$ resp. $\Lambda^{p-1}T^*M$ considered as subbundles of the tensor product $\Lambda^{p+1}T^*M \otimes T^*M$. Hence, the conditions of the lemma follow from the decomposition of $(\mathrm{Sym}^k H \otimes \Lambda_0^{a,b}E) \otimes (H \otimes E)$ into irreducible summands. The Clebsch–Gordan formula for Sp(1) implies that $\mathrm{Sym}^k H \otimes H \cong \mathrm{Sym}^{k+1}H \oplus \mathrm{Sym}^{k-1}H$, which proves the condition on k. Similarly, the decomposition of $\Lambda_0^{a,b}E \otimes E$ implies conditions (ii) and (iii) (cf. [4]). □

4. Decomposition of the space of forms in irreducible components

On a Kähler manifold one defines three endomorphisms of the form bundle: the wedge product and the contraction with the Kähler form and their commutator. Similarly we may associate with any almost complex structure J_α, given by the quaternion–Kähler reduction, three locally defined endomorphisms of the form bundle. In the quaternion Kähler case the corresponding operators L_α, Λ_α and J_α are defined by

$$L_\alpha := \frac{1}{2}\sum_i e_i \wedge J_\alpha(e_i) \wedge \qquad \Lambda_\alpha := -\frac{1}{2}\sum_i e_i \lrcorner\, J_\alpha(e_i) \lrcorner \qquad J_\alpha := -\sum_i e_i \wedge J_\alpha(e_i) \lrcorner$$

where $\{e_i\}$ is a local orthonormal base of the tangent bundle. It is straightforward to check the following relations:

$$[X\wedge, \Lambda_\alpha] = -J_\alpha(X) \lrcorner \qquad [X\wedge, J_\alpha] = -J_\alpha(X)\wedge \tag{5}$$

$$[X \lrcorner\,, L_\alpha] = J_\alpha(X) \wedge \qquad [X \lrcorner\,, J_\alpha] = -J_\alpha(X) \lrcorner \tag{6}$$

Composing the local endomorphism L_α, Λ_α and J_α defined above we obtain globally defined endomorphisms of the form bundle:

$$L := \sum_\alpha L_\alpha \circ L_\alpha, \qquad L^- := \sum_\alpha L_\alpha \circ J_\alpha, \qquad J := \sum_\alpha J_\alpha \circ J_\alpha,$$

$$\Lambda := \sum_\alpha \Lambda_\alpha \circ \Lambda_\alpha, \qquad \Lambda^+ := \sum_\alpha \Lambda_\alpha \circ J_\alpha, \qquad C := \sum_\alpha L_\alpha \circ \Lambda_\alpha.$$

It is straightforward to check that J and C are self–adjoint, while L and L^- are adjoints of Λ and Λ^+ respectively. Moreover it is important to note that J and C are comuting endomorphisms. The commutators of these operators with the inner and wedge product with vectors are given by the following

Lemma 4.1. *The following relations hold:*

$$
\begin{array}{lcll}
[X\wedge, \Lambda] & = & -2\sum_{i,\alpha}\Lambda_\alpha \circ J_\alpha(X)\,\lrcorner & [X\,\lrcorner\,, L] = 2\sum_{i,\alpha} L_\alpha \circ J_\alpha(X)\wedge \\
[X\wedge, L^-] & = & -\sum_{i,\alpha} L_\alpha \circ J_\alpha(X)\wedge & [X\,\lrcorner\,, \Lambda^+] = \sum_{i,\alpha}\Lambda_\alpha \circ J_\alpha(X)\,\lrcorner \\
[X\wedge, \Lambda^+] & = & -3X\,\lrcorner\, - \sum_{i,\alpha}(\Lambda_\alpha \circ J_\alpha(X)\wedge + J_\alpha \circ J_\alpha(X)\,\lrcorner\,) & \\
[X\,\lrcorner\,, L^-] & = & 3X\wedge + \sum_{i,\alpha}(J_\alpha \circ J_\alpha(X)\wedge - L_\alpha \circ J_\alpha(X)\,\lrcorner\,) & \\
[X\wedge, J] & = & -3X\wedge - 2\sum_{i,\alpha} J_\alpha \circ J_\alpha(X)\wedge & \\
[X\,\lrcorner\,, J] & = & -3X\,\lrcorner\, - 2\sum_{i,\alpha} J_\alpha \circ J_\alpha(X)\,\lrcorner & \\
[X\wedge, C] & = & \sum_{i,\alpha} L_\alpha \circ J_\alpha(X)\wedge & [X\,\lrcorner\,, C] = 3X\,\lrcorner\, + \sum_{i,\alpha}\Lambda_\alpha \circ J_\alpha(X)\wedge
\end{array}
$$

These relations easily follow from (5) and (6). We leave the necessary verifications to the reader.

Fixing an arbitrary point of the manifolds we may consider the bundle endomorphisms J and C as linear maps on the space of p–forms $\Lambda^p(H\otimes E)$. Obviously J and C are invariant under the action of the holonomy group $\mathrm{Sp}(1)\cdot\mathrm{Sp}(m)$. Hence, both maps restrict to equivariant maps on the irreducible subrepresentations $\mathrm{Sym}^k H\otimes\Lambda_0^{a,b}E$. By the Schur Lemma, these restricted maps have to be certain multiples of the identity. In the remaining part of this section we will show how to compute the action of J and C on the irreducible components of the form representation.

Lemma 4.2. *Let $\mathrm{Sym}^k H\otimes\Lambda_0^{a,b}E$ be an irreducible summand of the space of complex p–forms $\Lambda^p(H\otimes E)$. Then the bundle endomorphism J acts on it as*

$$J \;=\; -\,k(k+2)\,\mathrm{id}\ .$$

Proof. We will consider $J=\sum J_\alpha^2$ as a linear map acting on an irreducible subspace $\mathrm{Sym}^k H\otimes\Lambda_0^{a,b}E$ of the space of p–forms $\Lambda^p(H\otimes E)$. Let ω_α be the 2–form corresponding to J_α, i.e. $\omega_\alpha=\frac{1}{2}\sum e_i\wedge J_\alpha e_i$. It immediately follows that $J_\alpha=\omega_\alpha\bullet$ as endomorphism on the space of p–forms. Here $\bullet$ denotes the the standard representation of the Lie algebra $\mathfrak{so}(T_pM)\cong\Lambda^2T_pM$ on the space of p–forms, which is defined as

$$(X\wedge Y)\bullet = Y\wedge X\,\lrcorner\, - X\wedge Y\,\lrcorner\ .$$

Let $\mathfrak{g}$ be a semi-simple Lie algebra equipped with an invariant scalar product g. Then the Casimir operator Cas^g_π, acting on a representation π of $\mathfrak{g}$, is defined as $\mathrm{Cas}^g_\pi=\sum\pi(X_i)\,\pi(X_i)$, where $\{X_i\}$ is an g-orthonormal basis of $\mathfrak{g}$. More information about Casimir operators can be found in the appendix.

Realizing $\mathfrak{sp}(1)$ as a subspace of the space of 2-forms we obtain a scalar product on $\mathfrak{sp}(1)$ by restricting the standard scalar product on 2-forms. The corresponding Casimir operator of $\mathfrak{sp}(1)$ is denoted by $\mathrm{Cas}^{\Lambda^2}_\pi$. With respect to this standard scalar product the 2-forms ω_α are orthogonal and of length $2m$. Hence, we have for the representation

$\pi = \mathrm{Sym}^k H \otimes \Lambda_0^{a,b} E$ the expression

$$J \;=\; 2m \sum_\alpha \frac{\omega_\alpha}{\sqrt{2m}} \bullet \frac{\omega_\alpha}{\sqrt{2m}} \bullet \;=\; 2m \;\mathrm{Cas}_\pi^{\Lambda^2} \;.$$

Note that $\mathrm{Sp}(1) \subset \mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$ acts only on the $\mathrm{Sym}^k H$-factor.

In the appendix we show the relation $\mathrm{Cas}_\pi^{\Lambda^2} = \frac{8}{2m} \mathrm{Cas}_\pi^{g_B}$, where $\mathrm{Cas}_\pi^{g_B}$ denotes the $\mathrm{Sp}(1)$-Casimir operator defined with respect to the scalar product induced by the Killing form. Moreover, we show that $\mathrm{Cas}_\pi^{g_B}$ acts on $\mathrm{Sym}^k H$ as $-\frac{1}{8}\, k(k+2)\,\mathrm{id}$. Thus it follows that

$$J \;=\; 2m \;\mathrm{Cas}_\pi^{\Lambda^2} \;=\; 8\,\mathrm{Cas}_\pi^{g_B} \;=\; -\,k\,(k+2)\,\mathrm{id}\;.$$

We may check this formula in the case $k = 1$. Here J acts as a sum of the squares of the three almost complex structures J_α. Hence, $J = -3\,\mathrm{id}$ on $TM^{\mathbb{C}} \cong H \otimes E$, which agrees with our formula. □

Lemma 4.3. *Let* $\mathrm{Sym}^k H \otimes \Lambda_0^{a,b} E$ *be an irreducible summand of the space of* p*–forms* $\Lambda^p(T^*M^{\mathbb{C}}) = \Lambda^p(H \otimes E)$*. Then the bundle endomorphism* C *acts as*

$$C \;=\; \frac{1}{4}\,(p(n-p+6) \,-\, k(k+2) \,-\, 4b \,+\, 2a^2 \,+\, 2b^2 \,-\, (a+b)(n+4))\,\mathrm{id}\;,$$

where $n = 4m$ *is the real dimension of* M*.*

Proof. Starting directly from the definition of C we obtain in a first step the formula

$$\begin{aligned}
4\,C \;&=\; 4 \sum \Lambda_\alpha \circ L_\alpha \;=\; -\sum e_i \wedge J_\alpha(e_i) \,\wedge\, e_j \lrcorner\, J_\alpha(e_j) \lrcorner \\
&=\; \sum (e_j \wedge e_i \lrcorner\,)\,(J_\alpha e_j \wedge J_\alpha e_i \lrcorner\,) \,+\, 3 \sum e_j \wedge e_j \lrcorner \\
&=\; \frac{1}{2} \sum (e_i \wedge e_j) \bullet (J_\alpha e_i \wedge J_\alpha e_j) \bullet \,+\, \sum (e_j \wedge e_i \lrcorner\,)\,(J_\alpha e_i \wedge J_\alpha e_j \lrcorner\,) \,+\, 3p\,\mathrm{id} \\
&=\; \sum_{i<j} (e_i \wedge e_j) \bullet (J_\alpha e_i \wedge J_\alpha e_j) \bullet \,+ J \,+\, 6p\,\mathrm{id}\;.
\end{aligned}$$

We now want to express the first summand in terms of the $\mathrm{Sp}(m)$–Casimir operator $\mathrm{Cas}_\pi^{\Lambda^2}$. Hence we have to rewrite the first summand using a basis of $\mathfrak{sp}(m) \subset \mathfrak{so}(4m)$. A projection map

$$\mathrm{pr} : \Lambda^2(H \otimes E) \;\longrightarrow\; \mathrm{Sym}^2 E \subset \Lambda^2(H \otimes E)$$

can be defined by

$$\mathrm{pr}(X \wedge Y) \;=\; \frac{1}{4}\,(X \wedge Y \;+\; \sum J_\alpha X \wedge J_\alpha Y\,)\;.$$

Note that pr indeed satisfies the condition $\mathrm{pr}^2 = \mathrm{pr}$. Substituting pr into the formula for $4C$ we obtain

$$4\,C \;=\; 4\sum_{i<j}(e_i\wedge e_j)\bullet \mathrm{pr}(e_i\wedge e_j)\bullet \;-\; \sum_{i<j}(e_i\wedge e_j)\bullet(e_i\wedge e_j)\bullet \;+J\;+\;6p\,\mathrm{id}$$

The second summand is just the $\mathrm{SO}(n)$–Casimir operator acting on p–forms and a short calculation shows that it is equal to $-p(n-p)\,\mathrm{id}$. Moreover, we can replace the ortho–normal basis $e_i\wedge e_j$ with any basis of Λ^2T^*M, which is adapted to the decomposition of $\Lambda^2(H\otimes E)$ as $\mathrm{Sp}(1)\cdot\mathrm{Sp}(m)$-representation and which is ortho-normal with respect to the standard scalar product of Λ^2. Hence, it remains only the sum over an ortho–normal basis $\{\omega_i\}$ of the summand corresponding to Sym^2E and

$$C \;=\; \mathrm{Cas}^{\Lambda^2}_{\pi} \;+\; \frac{1}{4}\,p(n-p)\,\mathrm{id} \;+\; \frac{1}{4}\,J \;+\; \frac{6}{4}p\,\mathrm{id}\;,$$

where $\mathrm{Cas}^{\Lambda^2}_{\pi}$ denotes the $\mathrm{Sp}(m)$–Casimir operator defined with respect to the standard scalar product of Λ^2 and acting on the representation $\pi = \mathrm{Sym}^kH\otimes\Lambda^{a,b}_0\,E$. Note that the action on the Sym^kH-factor is trivial.

In the appendix we show that $\mathrm{Cas}^{\Lambda^2}_{\pi} = 2(m+1)\,\mathrm{Cas}^{g_B}_{\pi}$, where $\mathrm{Cas}^{g_B}_{\pi}$ is the Casimir operator defined with respect to the scalar product induced by the Killing form. Moreover we calculate the action of $\mathrm{Cas}^{g_B}_{\pi}$ on the $\mathrm{Sp}(m)$-representation $\pi = \Lambda^{a,b}_0E$, which implies for the Casimir operator in the Λ^2-normalization the formula

$$\mathrm{Cas}^{\Lambda^2}_{\pi} \;=\; -\,\frac{1}{2}\,(2b-a^2-b^2+2(a+b)(m+1))\;.$$

Substituting this into our last expression for C concludes the proof of the lemma. □

5. Natural first order operators on quaternion–Kähler manifolds

In the previous section we have introduced several natural algebraic operators acting on forms. We now turn our attention towards differential operators and define natural extensions of the exterior derivative and codifferential on quaternion–Kähler manifolds. We will only introduce those of the operators which will be useful to the study of our particular problem. The general theory of these operators will (hopefully) be developed in a forthcoming paper.

Let as before J_α be a local basis of almost complex structures and define

$$d^+ \;:=\; \sum_{i,\alpha} L_\alpha J_\alpha(e_i)\wedge\nabla_{e_i},\quad d^- := \sum_{i,\alpha}\Lambda_\alpha J_\alpha(e_i)\wedge\nabla_{e_i},\quad d^c := \sum_{i,\alpha} J_\alpha J_\alpha(e_i)\wedge\nabla_{e_i},$$

$$\delta^+ \;:=\; -\sum_{i,\alpha} L_\alpha J_\alpha(e_i)\lrcorner\,\nabla_{e_i},\quad \delta^- := -\sum_{i,\alpha}\Lambda_\alpha J_\alpha(e_i)\lrcorner\,\nabla_{e_i},\quad \delta^c := -\sum_{i,\alpha} J_\alpha J_\alpha(e_i)\lrcorner\,\nabla_{e_i}.$$

Lemma 5.1. *The following relations hold:*

$$
\begin{aligned}
[\,d,\Lambda] &= \quad 2\delta^- & [\,\delta, L] &= -2d^+\\
[\,d,L^-] &= -d^+ & [\,\delta, L^-] &= -\delta^+ - d^c - 3d\\
[\,d,\Lambda^+] &= -d^- + \delta^c + 3\delta & [\,\delta, \Lambda^+] &= \quad \delta^-\\
[\,d,J] &= -2d^c - 3d & [\,\delta, J] &= -2\delta^c - 3\delta\\
[\,d,C] &= \quad \delta^+ & [\,\delta, C] &= -d^- + 3\delta
\end{aligned}
$$

Proof. All algebraic operators $L,\ \Lambda,\ L^-,\ \Lambda^+,\ J,$ and C appearing in the lemma are parallel. The result thus follows directly from Lemma 4.1. $\square$

6. Killing forms

Let u be a Killing p–form on a compact quaternion–Kähler manifold M^n. By definition, u satisfies the equation

$$\nabla_X u \;=\; \frac{1}{p+1} X \lrcorner\, du, \qquad \forall X \in TM. \tag{7}$$

The goal of this section is to prove that u has to be parallel if $p \geq 2$. This is, of course, equivalent to $du = 0$.

In (7) we perform three operations: 1. take the interior or wedge product with $J_\alpha(X)$ for some α; 2. apply one of the operators L_α, Λ_α or J_α to both terms; 3. sum over $\alpha = 1, 2, 3$ and an orthonormal basis $X = e_i$. This yields the following six equations:

$$
\begin{aligned}
d^+u &= \tfrac{1}{p+1} L^- du & d^c u &= \tfrac{1}{p+1} J du & d^- u &= \tfrac{1}{p+1}\Lambda^+ du\\
\delta^+ u &= -\tfrac{2}{p+1} C du & \delta^c u &= -\tfrac{2}{p+1}\Lambda^+ du & \delta^- u &= -\tfrac{2}{p+1}\Lambda du
\end{aligned}
$$

From Lemma 5.1 we then obtain

$$
\begin{aligned}
pd^+u &= d(L^-u) & (p-1)d^c u &= d(Ju + 3u) & pd^-u &= d(\Lambda^+ u) - \delta^c u\\
(p-1)\delta^+ u &= -2d(Cu) & p\delta^c u &= -2d(\Lambda^+ u) - d^- u & (p-1)\delta^- u &= -2d(\Lambda u)
\end{aligned}
$$

As $p > 1$, the third and fifth equations together show that d^-u and $\delta^c u$ are exact forms. Thus, all 6 natural first order differential operators $d^\pm, d^c, \delta^\pm, \delta^c$ map u to an exact form. On the other hand, the right hand side equations in Lemma 5.1 show that the images of u through $d^+, \delta^-, \delta^c, d^-$ and $\delta^+ + d^c + 3d$ are coexact (in the image of δ). Since M is compact, a form which is simultaneously exact and coexact must vanish. We deduce that

$$d^+u,\ \delta^-u,\ \delta^c u,\ d^-u,\ \text{and}\ \delta^+u + d^c u + 3du$$

all vanish. Using Lemma 5.1 again we get from the equations above:

$$\begin{aligned} d(Ju+3u) &= (p-1)d^c u = -\frac{p-1}{2}([d,J]u+3du), \\ -2d(Cu) &= (p-1)\delta^+ u = (p-1)[d,C]u. \end{aligned}$$

Together with $\delta^+ u + d^c u + 3du = 0$, this gives the following system:

$$\begin{cases} (p+1)d(Ju+3u) = (p-1)Jdu \\ (p+1)d(Cu) = (p-1)Cdu \\ -2Cdu + Jdu + 3(p+1)du = 0 \end{cases} \tag{8}$$

Any Killing p–form u decomposes according to the decomposition of $\Lambda^p(H \otimes E)$ into irreducible summands of the type $\mathrm{Sym}^k H \otimes \Lambda_0^{a,b} E$. A priori it is not clear whether the irreducible components of u are again Killing forms. However, the following lemma shows that u can be written as a sum of Killing forms.

Lemma 6.1. *The forms Ju and Cu are again Killing forms on M.*

Proof. As $d^- u = \delta^c u = \delta u = 0$, Lemma 5.1 shows that Ju and Cu are coclosed. On the other hand, C and J are parallel operators, so they commute with the curvature operator $q(R)$, with $\nabla^*\nabla$ and hence with the Laplace operator Δ. The characterization of Killing forms given in Proposition 2.3 then immediately implies the result. □

The operators C and J are thus self–adjoint linear operators acting on the finite dimensional space of Killing forms. Moreover they commute, so can be simultaneously diagonalized. We thus may assume that u is an eigenvector for both operators: $Ju = ju$ and $Cu = cu$ for some real constants j and c. Lemma 4.3 in Section 5 shows that $j = -k(k+2)$ for some integer k with $0 \leq k \leq p$. The first equation in the system (8) reads $Jdu = -\frac{p+1}{p-1}(k+3)(k-1)du$. Thus du is an eigenvector of J.

From now on we suppose that $du \neq 0$ and show that this leads to a contradiction. From Lemma 3.2 we have that the eigenvalue of du with respect to J can be either $-(k+1)(k+3)$ or $-(k-1)(k+1)$. In the first case we get $\frac{p+1}{p-1}(k+3)(k-1) = (k+1)(k+3)$, so $k \neq 1$ and $\frac{p+1}{p-1} = \frac{k+1}{k-1}$, hence $k = p$, while in the second case the only solution is $k = 1$.

Case 1. Suppose that $Jdu = -(k+1)(k+3)du$, so $k = p$. Denote by

$$P(k,a,b,p) := \frac{1}{4}\left(p(n-p+6) - k(k+2) - 4b + 2a^2 + 2b^2 - (a+b)(n+4)\right)$$

the eigenvalue of C on the subspace $\mathrm{Sym}^k H \otimes \Lambda_0^{a,b} E$ of $\Lambda^p M$. Let $v \in \mathrm{Sym}^k H \otimes \Lambda_0^{a,b} E \subset \Lambda^p$ be a component of u such that $dv \neq 0$ and let $w \in \mathrm{Sym}^{k'} H \otimes \Lambda_0^{a',b'} E \subset \Lambda^{p+1}$ be a non–zero component of dv. The two last equations of the system (8) become:

$$\begin{cases} (p+1)d(Cu) = (p-1)Cdu \\ -2Cdu + Jdu + 3(p+1)du = 0 \end{cases} \Longrightarrow \begin{cases} cdu = -\frac{p(p-1)}{2}du \\ Cdu = -\frac{p(p+1)}{2}du \end{cases}$$

We deduce $P(p,a,b,p) = c = -\frac{p(p-1)}{2}$ and $P(k',a',b',p+1) = -\frac{p(p+1)}{2}$. Now $k' = p+1$ because $Jdu = -(k+1)(k+3)du$, which implies $b = b' = 0$ from the inequalities of Lemma 3.1. As $P(p,a,0,p) = \frac{1}{4}(p-a)(n+4-2p-2a)$, we obtain the system

$$\begin{cases} (p-a)(n+4-2p-2a) = -2p(p-1) \\ (p+1-a')(n+2-2p-2a') = -2p(p+1) \end{cases}$$

From Lemma 3.2 we know that $a' = a \pm 1$. If $a' = a+1$, subtracting the two equations above yields $4(p-a) = 4p$ hence $a = 0$, so the first equation becomes $n+4 = 2$ which is impossible. If $a' = a-1$, the same subtraction yields

$$-2(n+4-2p-2a) = 4p,$$

so $n+4 = 2a$. From the first equation we get $(p-a) = (p-1)$, hence $a = 1$ and we find again $n+4 = 2$, a contradiction.

Case 2. Suppose that $Jdu = -(k-1)(k+1)du$, so $k = 1$. In particular $Jdu = 0$ and $Ju = -3u$, so (8) becomes

$$\begin{cases} Cdu = \frac{p+1}{p-1}d(Cu) \\ Cdu = \frac{3(p+1)}{2}du \end{cases} \tag{9}$$

As before let $v \in \mathrm{Sym}^k H \otimes \Lambda_0^{a,b} E \subset \Lambda^p$ be a component of u and $w \in \mathrm{Sym}^{k'} H \otimes \Lambda_0^{a',b'} \subset \Lambda^{p+1}$ be a non–zero component of dv. As u and du are eigenvectors for C we conclude that the same is true for v and w. Moreover, since $Jdu = 0$ we get $k' = 0$. Thus (9) reads

$$\begin{cases} P(0,a',b',p+1) = \frac{p+1}{p-1}P(1,a,b,p) \\ P(0,a',b',p+1) = \frac{3(p+1)}{2} \end{cases} \Longleftrightarrow \begin{cases} P(1,a,b,p) = \frac{3(p-1)}{2} \\ P(0,a',b',p+1) = \frac{3(p+1)}{2} \end{cases} \tag{10}$$

Subtracting these two equations yields

$$n - 2p - 4 - 4b' + 4b + 2a'^2 - 2a^2 + 2b'^2 - 2b^2 + (n+4)(a+b-a'-b') = 0. \tag{11}$$

Now, we have seen that (a',b') is one of the four neighbors of (a,b) in $\mathbb{Z}^2$. We have four sub–cases:

a) $(a',b') = (a+1,b)$. Then (11) gives $p = 2a-3$, which contradicts the inequality $a \leq \frac{p+k}{2}$ ((iv) of Lemma 3.1).

b) $(a',b') = (a-1,b)$. Then (11) reads $n-p = 2a-1$, so from the inequality $a \leq \frac{p+k}{2} = \frac{n}{2} - a + 1$ we obtain $a \leq \frac{n}{4}$, hence $p \geq \frac{n}{2}+1$. Now, it is well–known (see [1]) that Λ is injective on q–forms for every $q \geq \frac{n}{2}+2$. The previous calculations give $\Lambda(du) = d(\Lambda u) - 2\delta^- u = 0$ (since Λu is parallel), so $du = 0$ and u is thus parallel in this case.

c) $(a', b') = (a, b+1)$. From (11) we get $p = 2b - 5$, so from Lemma 3.1 we get $b \le a \le \frac{p+k}{2} = b - 2$, which is impossible.

d) $(a', b') = (a, b-1)$. Using (11) again we obtain $n - p = 2b - 3$ and Lemma 3.1 yields $b \le a \le \frac{n-p+k}{2} = b - 1$, a contradiction.

We have proved the

Theorem 6.2. *Every Killing p–form $(p \ge 2)$ on a compact quaternion–Kähler manifold is parallel.*

Appendix A. The computation of the Casimir eigenvalues

Let G be a compact semi–simple Lie group, with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Furthermore let g be any invariant scalar product on $\mathfrak{g}$, e.g. $g = g_B := -B_{\mathfrak{g}}$ where $B_{\mathfrak{g}}$ is the *Killing form* defined as $B_{\mathfrak{g}}(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. For simple Lie groups G the Killing form is some multiple of the *trace form* B_0, which is for subalgebras $\mathfrak{g} \subset \mathfrak{g}l(n, \mathbb{C})$ defined as $B_0(X, Y) := \mathrm{Re}\,\mathrm{tr}(X \circ Y)$. For the group $G = \mathrm{Sp}(m)$ we have $B_{\mathfrak{sp}(m)} = (2m+2)\, B_0$.

Let $\pi : G \to \mathrm{Aut}(V)$ be a representation of G on the complex vector space V. If $\{X_i\}$ is a basis of $\mathfrak{g}$, orthonormal with respect to the invariant scalar product g, the *Casimir operator* $\mathrm{Cas}^g_\pi \in \mathrm{End}(V)$ is defined as

$$\mathrm{Cas}^g_\pi \;:=\; \sum_i \pi_*(X_i) \circ \pi_*(X_i) \ ,$$

where $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$ denotes the differential of the representation π. It is easy to see that the definition of Cas^g_π does not depend from the chosen g-orthonormal basis $\{X_i\}$. Moreover, Cas^g_π is an endomorphism of V commuting with all endomorphisms of the form $\pi_*(X)$, where X is any vector in $\mathfrak{g}$. More precisely, one defines the *Casimir element* $C := \sum X_i^2$ as a vector in the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. It then turns out that C is in the center of $\mathcal{U}\mathfrak{g}$.

If the representation π is irreducible, the Schur Lemma implies that Cas^g_π is some multiple of the identity. In fact it is possible to express this multiple in terms of the highest weight of π.

Lemma A.1. *Let $\lambda \in \mathfrak{t}^*$ be the highest weight of the irreducible representation $\pi : G \to \mathrm{Aut}(V)$ and let ρ be the half sum of the positive roots of $\mathfrak{g}$ relative to a fixed Weyl chamber of $\mathfrak{t}$. The Casimir operator is given as $\mathrm{Cas}^{g_B}_\pi = -\, c_\pi\, \mathrm{id}_V$ with*

$$c_\pi \;=\; \|\lambda + \rho\|^2 \;-\; \|\rho\|^2 \;=\; (\lambda,\, \lambda) \;+\; (\lambda,\, 2\rho) \ ,$$

where $(\cdot, \cdot)$ denotes the scalar product on $\mathfrak{t}^*$ induced by the Killing form B.

As an application we want to compute the Casimir eigenvalues, i.e. the scalars c_π for several irreducible $\mathrm{Sp}(m)$–representations. With respect to the standard realization of the Cartan algebra $\mathfrak{t} \cong \mathbb{R}^m$ of $\mathfrak{sp}(m)$, the weights λ can be written as vectors $\lambda =$

$(\lambda_1, \dots, \lambda_m) = \sum \lambda_i \, \varepsilon_i$, where $\{\varepsilon_i\}$ is dual to the standard basis in $\mathbb{R}^m$. We are interested in the following representations of $\mathrm{Sp}(m)$:

$$
\begin{array}{lll}
\pi \;=\; \mathrm{Sym}^k E & \text{with highest weight} & \lambda \;=\; k\,\varepsilon_1 \;=\; (k, 0, \dots, 0) \\
\pi \;=\; \Lambda^a_0 E & \text{with highest weight} & \lambda \;=\; \sum_{i=1}^a \varepsilon_i \;=\; (\underbrace{1, \dots, 1}_{a}, 0, \dots, 0) \\
\pi \;=\; \Lambda^{a,b}_0 E & \text{with highest weight} & \lambda \;=\; \sum_{i=1}^a \varepsilon_i \,+\, \sum_{i=1}^b \varepsilon_i \\
 & & \;=\; (\underbrace{2, \dots, 2}_{b}, \underbrace{1, \dots, 1}_{a-b}, 0, \dots, 0)
\end{array}
$$

Under the identification $\mathfrak{t} \cong \mathbb{R}^m$, the trace form B_0 corresponds to twice the standard scalar product on $\mathbb{R}^m$. Hence, in the formula of Lemma A.1 we can replace $(\cdot, \cdot)$ by $\frac{1}{4(m+1)}$ times the standard scalar product on $\mathbb{R}^m$. Moreover the half-sum of positive roots is given as the vector $\rho = (m, m-1, \dots, 1)$. Using these remarks we obtain the following Casimir eigenvalues:

$$
\begin{array}{ll}
\pi \;=\; \mathrm{Sym}^k E & c_\pi \;=\; \frac{1}{4(m+1)}\, k(k+2m) \\
\pi \;=\; \Lambda^a_0 E & c_\pi \;=\; \frac{1}{4(m+1)}\, a(2-a+2m) \\
\pi \;=\; \Lambda^{a,b}_0 E & c_\pi \;=\; \frac{1}{4(m+1)}\, (2b - a^2 - b^2 + 2(a+b)(m+1))
\end{array}
$$

In particular we obtain for $m = 1$ the Casimir eigenvalues for $\mathrm{Sp}(1)$, e.g. the Casimir operator on $\mathrm{Sym}^k H$ is given as $-\frac{1}{8} k(k+2)\mathrm{id}$. Moreover, the Casimir eigenvalue for $\pi = \Lambda^a_0 E$ is of course the special case $b = 0$ for $\pi = \Lambda^{a,b}_0 E$. Note that the Casimir eigenvalue (with respect to the Killing form) of the adjoint representation is always one. Using our formula we can check this for $\mathfrak{sp}(m) \cong \mathrm{Sym}^2 E$.

In the remaining part of this section we want to make some remarks concerning the Casimir normalization, i.e. we will give a formula comparing the Casimir operators corresponding to different scalar products.

Let $\mathfrak{g}$ be the Lie algebra of a compact simple Lie group and let V be a real vector space with a isotypical representation of $\mathfrak{g}$, i.e. V is a sum of isomorphic irreducible $\mathfrak{g}$-representations. Thus, the Casimir operator Cas^g_V acts on V as $\mathrm{Cas}^g_V = -c^g_V \mathrm{id}$ for some number c^g_V. Moreover we assume that V is equipped with a $\mathfrak{g}$–invariant scalar product $\langle \cdot, \cdot \rangle$, i.e. $\mathfrak{g} \subset \mathfrak{so}(V) \cong \Lambda^2 V$. Restricting the induced scalar product onto $\Lambda^2 V$ we obtain a natural scalar product $\langle \cdot, \cdot \rangle_{\Lambda^2}$ on $\mathfrak{g}$. Note that,

$$
\langle \alpha, \beta \rangle_{\Lambda^2} \;=\; -\frac{1}{2}\, \mathrm{tr}_V(\alpha \circ \beta) \;=\; \frac{1}{2}\, \langle \alpha, \beta \rangle_{\mathrm{End} V} \; .
$$

Lemma A.2. *Let* $\mathrm{Cas}^{\Lambda^2}_\pi$ *be the* $\mathfrak{g}$*-Casimir operator defined with respect to the scalar product* $\langle \cdot, \cdot \rangle_{\Lambda^2}$ *restricted to* $\mathfrak{g}$ *and let* Cas^g_π *be the* $\mathfrak{g}$*-Casimir operator corresponding to* *any other invariant scalar product g. Then for any irreducible $\mathfrak{g}$-representation π it follows that*

$$\mathrm{Cas}_\pi^{\Lambda^2} \;=\; 2\,\frac{\dim \mathfrak{g}}{\dim V}\,\frac{1}{c_V^g}\;\mathrm{Cas}_\pi^g\;.$$

Proof. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{\Lambda^2}$ and let $\{e_i\}$ be an orthonormal basis of V. Then $\mathrm{Cas}_V^{\Lambda^2}(v) = -c_V^{\Lambda^2}\, v = \sum_i X_i^2(v)$ and we get

$$-\dim V\, c_V^{\Lambda^2} \;=\; \sum_{i,j} \langle X_i^2(e_j), e_j\rangle = -\sum_{i,j}\langle X_i(e_j), X_i(e_j)\rangle = -2\sum |X_i|^2_{\Lambda^2} = -2\dim \mathfrak{g}$$

which proves the lemma in the case $\pi = V$. Since $\mathfrak{g}$ is a simple Lie algebra it follows that two Casimir operators defined with respect to different scalar products differ only by a factor independent from the irreducible representation π. Hence

$$\frac{c_\pi^{\Lambda^2}}{c_\pi^g} \;=\; \frac{c_V^{\Lambda^2}}{c_V^g}$$

and the statement of the lemma follows from the special case $\pi = V$. □

As a first application we consider the case $\mathfrak{g} = \mathfrak{sp}(m)$ with $V = \mathbb{R}^{4m} \cong E$ and $\dim \mathfrak{sp}(m) = m(2m+1)$. Since $V \cong E = \mathrm{Sym}^1 E$ the formulas above imply that $c_V^{g_B} = \frac{2m+1}{4(m+1)}$. Hence,

$$\mathrm{Cas}_\pi^{\Lambda^2} \;=\; 2\,\frac{m(2m+1)}{4m}\,\frac{4(m+1)}{2m+1}\;\; c_\pi^{g_B} \;=\; 2\,(m+1)\;\mathrm{Cas}_\pi^{g_B}\;.$$

As a second application we want to derive a similar formula for the Sp(1)-Casimir operators. Here we take V to be the real subspace of $H \otimes E$. Hence $V \cong \mathbb{R}^{4m}$ with the standard representation of Sp(1), acting trivially on the E-factor. The general formula then implies

$$\mathrm{Cas}_\pi^{\Lambda^2} \;=\; 2\,\frac{3}{4m}\,\frac{8}{3}\;c_\pi^{g_B} \;=\; \frac{8}{2m}\;\mathrm{Cas}_\pi^{g_B}\;.$$

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