THE HERMITIAN LAPLACE OPERATOR ON NEARLY KÄHLER MANIFOLDS

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Abstract. The moduli space $\mathcal{NK}$ of infinitesimal deformations of a nearly Kähler structure on a compact 6-dimensional manifold is described by a certain eigenspace of the Laplace operator acting on co-closed primitive $(1,1)$ forms (c.f. [10]). Using the Hermitian Laplace operator and some representation theory, we compute the space $\mathcal{NK}$ on all 6-dimensional homogeneous nearly Kähler manifolds. It turns out that the nearly Kähler structure is rigid except for the flag manifold $F(1,2) = \mathrm{SU}_3/T^2$, which carries an 8-dimensional moduli space of infinitesimal nearly Kähler deformations, modeled on the Lie algebra $\mathfrak{su}_3$ of the isometry group.

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1. Introduction

Nearly Kähler manifolds were introduced in the 70's by A. Gray [8] in the context of weak holonomy. More recently, 6-dimensional nearly Kähler manifolds turned out to be related to a multitude of topics among which we mention: Spin manifolds with Killing spinors (Grunewald), SU_3-structures, geometries with torsion (Cleyton, Swann), stable forms (Hitchin), or super-symmetric models in theoretical physics (Friedrich, Ivanov).

Up to now, the only sources of compact examples are the naturally reductive 3-symmetric spaces, classified by Gray and Wolf [13], and the twistor spaces over positive quaternion-Kähler manifolds, equipped with the non-integrable almost complex structure. Based on previous work by R. Cleyton and A. Swann [6], P.-A. Nagy has shown in 2002 that every simply connected nearly Kähler manifold is a Riemannian product of factors which are either of one of these two types, or 6-dimensional [12]. Moreover, J.-B. Butruille has shown [5] that every homogeneous 6-dimensional nearly Kähler manifold is a 3-symmetric space G/K, more precisely isometric with $S^6 = G_2/\mathrm{SU}_3$, $S^3\times S^3 = \mathrm{SU}_2\times\mathrm{SU}_2\times\mathrm{SU}_2/\mathrm{SU}_2$, $\mathbb{CP}^3 = \mathrm{SO}_5/\mathrm{U}_2\times S^1$ or $F(1,2) = \mathrm{SU}_3/T^2$, all endowed with the metric defined by the Killing form of G.

A method of finding new examples is to take some homogeneous nearly Kähler manifold and try to deform its structure. In [10] we have studied the deformation problem for 6-dimensional nearly Kähler manifolds (M^6, g) and proved that if M is compact,

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and has normalized scalar curvature $\mathrm{scal}_g = 30$, then the space $\mathcal{NK}$ of infinitesimal deformations of the nearly Kähler structure is isomorphic to the eigenspace for the eigenvalue 12 of the restriction of the Laplace operator Δ^g to the space of *co-closed* primitive $(1,1)$-forms $\Lambda_0^{(1,1)}M$.

It is thus natural to investigate the Laplace operator on the known 3-symmetric examples (besides the sphere S^6, whose space of nearly Kähler structures is well-understood, and isomorphic to $\mathrm{SO}_7/G_2 \cong \mathbb{R}\mathrm{P}^7$, see [7] or [5, Prop. 7.2]). Recall that the spectrum of the Laplace operator on symmetric spaces can be computed in terms of Casimir eigenvalues using the Peter-Weyl formalism. It turns out that a similar method can be applied in order to compute the spectrum of a modified Laplace operator $\bar{\Delta}$ (called the Hermitian Laplace operator) on 3-symmetric spaces. This operator is SU_3-equivariant and coincides with the usual Laplace operator on co-closed primitive $(1,1)$-forms. The space of infinitesimal nearly Kähler deformations is thus identified with the space of co-closed forms in $\Omega_0^{(1,1)}(12) := \{\alpha \in \mathcal{C}^\infty(\Lambda_0^{(1,1)}M) \mid \bar{\Delta}\alpha = 12\alpha\}$. Our main result is that the nearly Kähler structure is rigid on $S^3 \times S^3$ and $\mathbb{C}\mathrm{P}^3$, and that the space of infinitesimal nearly Kähler deformations of the flag manifold $F(1,2)$ is eight-dimensional.

The paper is organized as follows. After some preliminaries on nearly Kähler manifolds, we give two general procedures for constructing elements in $\Omega_0^{(1,1)}(12)$ out of Killing vector fields or eigenfunctions of the Laplace operator for the eigenvalue 12 (Corollary 4.5 and Proposition 4.11). We show that these elements can not be co-closed, thus obtaining an upper bound for the dimension of the space of infinitesimal nearly Kähler deformations (Proposition 4.12). We then compute this upper bound explicitly on the 3-symmetric examples and find that it vanishes for $S^3 \times S^3$ and $\mathbb{C}\mathrm{P}^3$, which therefore have no infinitesimal nearly Kähler deformation. This upper bound is equal to 8 on the flag manifold $F(1,2) = \mathrm{SU}_3/T^2$ and in the last section we construct an explicit isomorphism between the Lie algebra of the isometry group $\mathfrak{su}_3$ and the space of infinitesimal nearly Kähler deformations on $F(1,2)$.

In addition, our explicit computations (in Section 5) of the spectrum of the Hermitian Laplace operator on the 3-symmetric spaces, together with the results in [11] show that every infinitesimal Einstein deformation on a 3-symmetric space is automatically an infinitesimal nearly Kähler deformation.

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2. Preliminaries on nearly Kähler manifolds

An almost Hermitian manifold (M^{2m}, g, J) is called *nearly Kähler* if

$$(\nabla_X J)(X) = 0, \qquad \forall\ X \in TM, \tag{1}$$

where ∇ denotes the Levi-Civita connection of g. The canonical Hermitian connection $\bar\nabla$, defined by

$$\bar\nabla_X Y := \nabla_X Y - \tfrac{1}{2} J(\nabla_X J)Y, \qquad \forall\, X \in TM,\ Y \in \chi(M) \tag{2}$$

is a U_m connection on M (*i.e.* $\bar\nabla g = 0$ and $\bar\nabla J = 0$) with torsion $\bar T_X Y = -J(\nabla_X J)Y$. A fundamental observation, which goes back to Gray, is the fact that $\bar\nabla \bar T = 0$ on every nearly Kähler manifold (see [2]).

We denote the Kähler form of M by $\omega := g(J.,.)$. The tensor $\Psi^+ := \nabla\omega$ is totally skew-symmetric and of type $(3,0)+(0,3)$ by (1). From now on we assume that the dimension of M is $2m = 6$ and that the nearly Kähler structure is strict, *i.e.* (M,g,J) is not Kähler. It is well-known that M is Einstein in this case. We will always normalize the scalar curvature of M to $\mathrm{scal} = 30$, in which case we also have $|\Psi^+|^2 = 4$ point-wise. The form Ψ^+ can be seen as the real part of a $\bar\nabla$-parallel complex volume form $\Psi^+ + i\Psi^-$ on M, where $\Psi^- = *\Psi^+$ is the Hodge dual of Ψ^+. Thus M carries a SU_3 structure whose minimal connection (cf. [6]) is exactly $\bar\nabla$. Notice that Hitchin has shown that a SU_3 structure (ω, Ψ^+, Ψ^-) is nearly Kähler if and only if the following exterior system holds:

$$\begin{cases} d\omega = 3\Psi^+ \\ d\Psi^- = -2\omega\wedge\omega \end{cases} . \tag{3}$$

Let $A \in \Lambda^1 M \otimes \mathrm{End} M$ denote the tensor $A_X := J(\nabla_X J) = -\Psi^+_{JX}$, where Ψ^+_Y denotes the endomorphism associated to $Y \lrcorner \Psi^+$ via the metric. Since for every unit vector X, A_X defines a complex structure on the 4-dimensional space $X^\perp \cap (JX)^\perp$, we easily get in a local orthonormal basis $\{e_i\}$ the formulas

$$|A_X|^2 = 2|X|^2, \qquad \forall\, X \in TM. \tag{4}$$

$$A_{e_i} A_{e_i}(X) = -4X, \qquad \forall\, X \in TM, \tag{5}$$

where here and henceforth, we use Einstein's summation convention on repeating subscripts. The following algebraic relations are satisfied for every SU_3 structure (ω, Ψ^+) on TM (notice that we identify vectors and 1-forms via the metric):

$$A_X e_i \wedge e_i \lrcorner \Psi^+ = -2X\wedge\omega, \qquad \forall\, X \in TM. \tag{6}$$

$$X \lrcorner \Psi^- = -JX \lrcorner \Psi^+, \qquad \forall\, X \in TM, \tag{7}$$

$$(X \lrcorner \Psi^+) \wedge \Psi^+ = X \wedge \omega^2, \qquad \forall\, X \in TM. \tag{8}$$

$$(JX \lrcorner \Psi^+) \wedge \omega = X \wedge \Psi^+, \qquad \forall\, X \in TM. \tag{9}$$

The Hodge operator satisfies $*^2 = (-1)^p$ on $\Lambda^p M$ and moreover

$$*(X \wedge \Psi^+) = JX \lrcorner \Psi^+, \qquad \forall\, X \in TM. \tag{10}$$

$$*(\phi\wedge\omega) = -\phi, \qquad \forall\, \phi \in \Lambda_0^{(1,1)} M. \tag{11}$$

$$*(JX \wedge \omega^2) = -2X, \qquad \forall\, X \in TM. \tag{12}$$

From now on we assume that (M, g) is compact 6-dimensional not isometric to the round sphere (S^6, can). It is well-known that every Killing vector field ξ on M is an automorphism of the whole nearly Kähler structure (see [10]). In particular,

$$L_\xi \omega = 0, \qquad L_\xi \Psi^+ = 0, \qquad L_\xi \Psi^- = 0. \tag{13}$$

Let now R and $\bar{R}$ denote the curvature tensors of ∇ and $\bar{\nabla}$. Then the formula (c.f. [1])

$$\begin{aligned} R_{WXYZ} &= \bar{R}_{WXYZ} - \tfrac{1}{4} g(Y, W) g(X, Z) + \tfrac{1}{4} g(X, Y) g(Z, W) \\ &\quad + \tfrac{3}{4} g(Y, JW) g(JX, Z) - \tfrac{3}{4} g(Y, JX) g(JW, Z) - \tfrac{1}{2} g(X, JW) g(JY, Z) \end{aligned}$$

may be rewritten as

$$R_{XY} = -X \wedge Y + R^{CY}_{XY}$$

and

$$\bar{R}_{XY} = -\tfrac{3}{4} \left(X \wedge Y + JX \wedge JY - \tfrac{2}{3} \omega(X, Y) J \right) + R^{CY}_{XY}$$

where R^{CY}_{XY} is a curvature tensor of Calabi-Yau type.

We will recall the definition of the curvature endomorphism $q(R)$ (c.f. [10]). Let EM be the vector bundle associated to the bundle of orthonormal frames via a representation $\pi : \mathrm{SO}(n) \to \mathrm{Aut}(E)$. The Levi-Civita connection of M induces a connection on EM, whose curvature satisfies $R^{EM}_{XY} = \pi_*(R_{XY}) = \pi_*(R(X \wedge Y))$, where we denote with π_* the differential of π and identify the Lie algebra of $SO(n)$, i.e. the skew-symmetric endomorphisms, with Λ^2. In order to keep notations as simple as possible, we introduce the notation $\pi_*(A) = A_*$. The curvature endomorphism $q(R) \in \mathrm{End}(EM)$ is defined as

$$q(R) = \tfrac{1}{2} (e_i \wedge e_j)_* R(e_i \wedge e_j)_* \tag{14}$$

for any local orthonormal frame $\{e_i\}$. In particular, $q(R) = \mathrm{Ric}$ on TM. By the same formula we may define for any curvature tensor S, or more generally any endomorphism S of $\Lambda^2 TM$, a bundle morphism $q(S)$. In any point $q : R \mapsto q(R)$ defines an equivariant map from the space of algebraic curvature tensors to the space of endomorphisms of E. Since a Calabi-Yau algebraic curvature tensor has vanishing Ricci curvature, $q(R^{CY}) = 0$ holds on TM. Let R^0_{XY} be defined by $R^0_{XY} = X \wedge Y + JX \wedge JY - \frac{2}{3} \omega(X, Y) J$. Then a direct calculation gives

$$q(R^0) = \tfrac{1}{2} \sum (e_i \wedge e_j)_* (e_i \wedge e_j)_* + \tfrac{1}{2} \sum (e_i \wedge e_j)_* (Je_i \wedge Je_j)_* - \tfrac{2}{3} \omega_* \omega_*.$$

We apply this formula on TM. The first summand is exactly the $\mathrm{SO}(n)$-Casimir, which acts as $-5\mathrm{id}$. The third summand is easily seen to be $\frac{2}{3}\mathrm{id}$, whereas the second summand acts as $-\mathrm{id}$ (c.f. [11]). Altogether we obtain $q(R^0) = -\frac{16}{3}\mathrm{id}$, which gives the following expression for $q(\bar{R})$ acting on TM:

$$q(\bar{R})|_{TM} = 4\, \mathrm{id}_{TM}. \tag{15}$$

3. The Hermitian Laplace operator

In the next two sections (M^6, g, J) will be a compact nearly Kähler manifold with scalar curvature normalized to $\mathrm{scal}_g = 30$. We denote as usual by Δ the Laplace operator $\Delta = d^*d + dd^* = \nabla^*\nabla + q(R)$ on differential forms. We introduce the *Hermitian Laplace operator*

$$\bar{\Delta} = \bar{\nabla}^*\bar{\nabla} + q(\bar{R}), \tag{16}$$

which can be defined on any associated bundle EM. In [11] we have computed the difference of the operators Δ and $\bar{\Delta}$ on a primitive $(1,1)$-form ϕ:

$$(\Delta - \bar{\Delta})\phi = (Jd^*\phi) \lrcorner \Psi^+. \tag{17}$$

In particular, Δ and $\bar{\Delta}$ coincide on co-closed primitive $(1,1)$-forms. We now compute the difference $\Delta - \bar{\Delta}$ on 1-forms. Using the calculation in [11] (or directly from (15)) we have $q(R) - q(\bar{R}) = \mathrm{id}$ on TM. It remains to compute the operator $P = \nabla^*\nabla - \bar{\nabla}^*\bar{\nabla}$ on TM. A direct calculation using (5) gives for every 1-form θ

$$\begin{aligned} P(\theta) &= -\tfrac{1}{4}A_{e_i}A_{e_i}\theta - A_{e_i}\bar{\nabla}_{e_i}\theta = \theta - A_{e_i}\bar{\nabla}_{e_i}\theta = \theta + \tfrac{1}{2}A_{e_i}A_{e_i}\theta - A_{e_i}\nabla_{e_i}\theta \\ &= -\theta - A_{e_i}\nabla_{e_i}\theta. \end{aligned}$$

In order to compute the last term, we introduce the metric adjoint $\alpha : \Lambda^2 M \to TM$ of the bundle homomorphism $X \in TM \mapsto X \lrcorner \Psi^+ \in \Lambda^2 M$. It is easy to check that $\alpha(X \lrcorner \Psi^+) = 2X$ (c.f. [10]). Keeping in mind that A is totally skew-symmetric, we compute for an arbitrary vector $X \in TM$

$$\begin{aligned} \langle A_{e_i}(\nabla_{e_i}\theta), X\rangle &= \langle A_X e_i, \nabla_{e_i}\theta\rangle = \langle A_X, e_i \wedge \nabla_{e_i}\theta\rangle = \langle A_X, d\theta\rangle \\ &= -\langle \Psi^+_{JX}, d\theta\rangle = -\langle JX, \alpha(d\theta)\rangle = \langle J\alpha(d\theta), X\rangle, \end{aligned}$$

whence $A_{e_i}(\nabla_{e_i}\theta) = J\alpha(d\theta)$. Summarizing our calculations we have proved the following

Proposition 3.1. *Let (M^6, g, J) be a nearly Kähler manifold with scalar curvature normalized to $\mathrm{scal}_g = 30$. Then for any 1-form θ it holds that*

$$(\Delta - \bar{\Delta})\theta = -J\alpha(d\theta).$$

The next result is a formula for the commutator of J and $\alpha \circ d$ on 1-forms.

Lemma 3.2. *For all 1-forms θ, the following formula holds:*

$$\alpha(d\theta) = 4J\theta + J\alpha(dJ\theta).$$

Proof. Differentiating the identity $\theta \wedge \Psi^+ = J\theta \wedge \Psi^-$ gives $d\theta \wedge \Psi^+ = dJ\theta \wedge \Psi^- + 2J\theta \wedge \omega^2$. With respect to the SU_3-invariant decomposition $\Lambda^2 M = \Lambda^{(1,1)}M \oplus \Lambda^{(2,0)+(0,2)}M$, we can write $d\theta = (d\theta)^{(1,1)} + \frac{1}{2}\alpha(d\theta) \lrcorner \Psi^+$ and $dJ\theta = (dJ\theta)^{(1,1)} + \frac{1}{2}\alpha(dJ\theta) \lrcorner \Psi^+$. Since the wedge product of forms of type $(1,1)$ and $(3,0)$ vanishes we derive the equation

$$\tfrac{1}{2}(\alpha(d\theta) \lrcorner \Psi^+) \wedge \Psi^+ = \tfrac{1}{2}(\alpha(dJ\theta) \lrcorner \Psi^+) \wedge \Psi^- + 2J\theta \wedge \omega^2.$$

Using (8) and (9) we obtain

$$\tfrac{1}{2}\alpha(d\theta) \wedge \omega^2 = \tfrac{1}{2}J\alpha(dJ\theta) \wedge \omega^2 + 2J\theta \wedge \omega^2.$$

Taking the Hodge dual of this equation and using (12) gives $J\alpha(d\theta) = -\alpha(dJ\theta) - 4\theta$, which proves the lemma. $\square$

Finally we note two interesting consequences of Proposition 3.1 and Lemma 3.2.

Corollary 3.3. *For any closed 1-form θ it holds that*

$$(\Delta - \bar\Delta)\theta = 0, \qquad (\Delta - \bar\Delta)J\theta = 4J\theta.$$

Proof. For a closed 1-form θ Lemma 3.1 directly implies that Δ and $\bar\Delta$ coincide on θ. For the second equation we use Proposition 3.1 together with Lemma 3.2 to conclude

$$(\Delta - \bar\Delta)J\theta = -J\alpha(dJ\theta) = 4J\theta - \alpha(d\theta) = 4J\theta$$

since θ is closed. This completes the proof of the corollary. $\square$

4. Special $\bar\Delta$-eigenforms on nearly Kähler manifolds

In this section we assume moreover that (M, g) is not isometric to the standard sphere (S^6, can). In the first part of this section we will show how to construct $\bar\Delta$-eigenforms on M starting from Killing vector fields.

Let ξ be a non-trivial Killing vector field on (M, g), which in particular implies $d^*\xi = 0$ and $\Delta\xi = 2\mathrm{Ric}(\xi) = 10\xi$. As an immediate consequence of the Cartan formula and (13) we obtain

$$dJ\xi = L_\xi\omega - \xi\lrcorner d\omega = -3\xi\lrcorner\psi^+ \tag{18}$$

so by (4), the square norm of $dJ\xi$ (as a 2-form) is

$$|dJ\xi|^2 = 18|\xi|^2. \tag{19}$$

In [9] we showed already that the vector field $J\xi$ is co-closed if ξ is a Killing vector field and has unit length. However it turns out that this also holds more generally.

Proposition 4.1. *Let ξ be a Killing vector field on M. Then $d^*J\xi = 0$.*

Proof. Let dv denote the volume form of (M, g). We start with computing the L^2-norm of $d^*J\xi$.

$$\begin{aligned} \|d^*J\xi\|^2_{L^2} &= \int_M \langle d^*J\xi, d^*J\xi\rangle dv = \int_M [\langle \Delta J\xi, J\xi\rangle - \langle d^*dJ\xi, J\xi\rangle] dv \\ &= \int_M [\langle \nabla^*\nabla J\xi, J\xi\rangle + 5|J\xi|^2 - |dJ\xi|^2] dv \\ &= \int_M [|\nabla J\xi|^2 + 5|\xi|^2 - |dJ\xi|^2] dv = \int_M [|\nabla J\xi|^2 - 13|\xi|^2] dv \end{aligned}$$

Here we used the well-known Bochner formula for 1-forms, i.e. $\Delta\theta = \nabla^*\nabla\theta + \mathrm{Ric}(\theta)$, with $\mathrm{Ric}(\theta) = 5\theta$ in our case. Next we consider the decomposition of $\nabla J\xi$ into its symmetric and skew-symmetric parts $2\nabla J\xi = dJ\xi + L_{J\xi}g$, which together with (19) leads to

$$|\nabla J\xi|^2 = \tfrac14(|dJ\xi|^2 + |L_{J\xi}g|^2) = 9|\xi|^2 + \tfrac14|L_{J\xi}g|^2. \tag{20}$$

(Recall that the endomorphism square norm of a 2-form is twice its square norm as a form). In order to compute the last norm, we express $L_{J\xi}g$ as follows:

$$\begin{aligned} L_{J\xi}g(X,Y) &= g(\nabla_X J\xi, Y) + g(X, \nabla_Y J\xi) \\ &= g(J\nabla_X \xi, Y) + g(X, J\nabla_Y \xi) + \Psi^+(X,\xi,Y) + \Psi^+(Y,\xi,X) \\ &= -g(\nabla_X \xi, JY) - g(JX, \nabla_Y \xi) = -d\xi^{(1,1)}(X, JY), \end{aligned}$$

whence

$$\|L_{J\xi}g\|^2_{L^2} = 2\|d\xi^{(1,1)}\|^2_{L^2}. \tag{21}$$

On the other hand, as an application of Lemma 3.2 together with Equation (18) we get $\alpha(d\xi) = 4J\xi + J\alpha(dJ\xi) = -2J\xi$, so

$$d\xi^{(2,0)} = -J\xi \lrcorner \Psi^+. \tag{22}$$

Moreover, $\Delta\xi = 10\xi$ since ξ is a Killing vector field, which yields

$$\|d\xi^{(1,1)}\|^2_{L^2} = \|d\xi\|^2_{L^2} - \|d\xi^{(2,0)}\|^2_{L^2} = 10\|\xi\|^2_{L^2} - 2\|\xi\|^2_{L^2} = 8\|\xi\|^2_{L^2}.$$

This last equation, together with (20) and (21) gives $\|\nabla J\xi\|^2_{L^2} = 13\|\xi\|^2_{L^2}$. Substituting this into the first equation proves that $d^*J\xi$ has vanishing L^2-norm and thus that $J\xi$ is co-closed. □

Proposition 4.2. *Let ξ be a Killing vector field on M. Then*

$$\Delta\xi = 10\xi, \qquad \textit{and} \qquad \Delta J\xi = 18J\xi.$$

In particular, $J\xi$ can never be a Killing vector field.

Proof. The first equation holds for every Killing vector field on an Einstein manifold with $\mathrm{Ric} = 5\mathrm{id}$. From (18) we know $dJ\xi = -3\xi\lrcorner\Psi^+$. Hence the second assertion follows from:

$$d^*dJ\xi = -*d*dJ\xi \overset{(10)}{=} -3*d(J\xi\wedge\Psi^+) = 9*(\xi\wedge\omega^2) \overset{(12)}{=} 18J\xi.$$

□

Since the differential d commutes with the Laplace operator Δ, every Killing vector field ξ defines two Δ-eigenforms of degree 2:

$$\Delta dJ\xi = 18dJ\xi \qquad \text{and} \qquad \Delta d\xi = 10.d\xi$$

As a direct consequence of Proposition 4.2, together with formulas (18), (22), and Proposition 3.1 we get:

Corollary 4.3. *Every Killing vector field on M satisfies*

$$\bar\Delta\xi = 12\xi, \qquad \bar\Delta J\xi = 12J\xi.$$

Our next goal is to show that the (1,1)-part of $d\xi$ is a $\bar\Delta$ -eigenform. By (22) we have

$$d\xi = \phi - J\xi\lrcorner\Psi^+, \tag{23}$$

for some $(1,1)$-form ϕ. Using Proposition 4.1, we can write in a local orthonormal basis $\{e_i\}$:

$$\langle d\xi, \omega\rangle = \tfrac{1}{2}\langle d\xi, e_i \wedge Je_i\rangle = \langle \nabla_{e_i}\xi, Je_i\rangle = d^*J\xi = 0,$$

thus showing that ϕ is primitive. The differential of ϕ can be computed from the Cartan formula:

$$d\phi \stackrel{(23)}{=} d(J\xi \lrcorner \Psi^+ + d\xi) \stackrel{(7)}{=} -d(\xi\lrcorner\Psi^-) = -L_\xi\Psi^- + \xi\lrcorner d\Psi^- \stackrel{(13)}{=} -2\xi\lrcorner\omega^2 = -4J\xi\wedge\omega. \quad (24)$$

From here we obtain

$$*d\phi = -4*(J\xi\wedge\omega) = 4\xi\wedge\omega,$$

whence

$$\begin{aligned} d*d\phi &= 4d\xi\wedge\omega - 12\xi\wedge\Psi^+ \stackrel{(23)}{=} 4\phi\wedge\omega - 4(J\xi\lrcorner\Psi^+)\wedge\omega - 12\xi\wedge\Psi^+ \\ &\stackrel{(9)}{=} 4\phi\wedge\omega - 16\xi\wedge\Psi^+. \end{aligned}$$

Using (10) and (11), we thus get

$$d^*d\phi = -*d*d\phi = 4\phi + 16J\xi\lrcorner\Psi^+.$$

On the other hand,

$$d^*\phi = -*d*\phi \stackrel{(11)}{=} *d(\phi\wedge\omega) \stackrel{(24)}{=} X(-4J\xi\wedge\omega^2 + 3\phi\wedge\Psi^+) \stackrel{(12)}{=} 8\xi$$

and finally

$$dd^*\phi = 8d\xi = 8\phi - 8J\xi\lrcorner\Psi^+.$$

The calculations above thus prove the following proposition

Proposition 4.4. *Let (M^6, g, J) be a compact nearly Kähler manifold with scalar curvature $\mathrm{scal}_g = 30$, not isometric to the standard sphere. Let ξ be a Killing vector field on M and let ϕ be the $(1,1)$-part of $d\xi$. Then ϕ is primitive, i.e. $\phi = (d\xi)_0^{(1,1)}$. Moreover $d^*\phi = 8\xi$ and $\Delta\phi = 12\phi + 8J\xi\lrcorner\Psi^+$.*

Corollary 4.5. *The primitive $(1,1)$-form φ satisfies*

$$\bar\Delta\phi = 12\phi.$$

Proof. From (17) and the proposition above we get

$$\bar\Delta\phi = \Delta\phi - (\Delta - \bar\Delta)\phi = 12\phi + 8J\xi\lrcorner\Psi^+ - (Jd^*\phi)\lrcorner\Psi^+ = 12\phi.$$

□

In the second part of this section we will present another way of obtaining primitive $\bar\Delta$-eigenforms of type $(1,1)$, starting from eigenfunctions of the Laplace operator. Let f be such an eigenfunction, i.e. $\Delta f = \lambda f$. We consider the primitive $(1,1)$-form $\eta := (dJdf)_0^{(1,1)}$.

Lemma 4.6. *The form η is explicitly given by*

$$\eta = dJdf + 2df\lrcorner\Psi^+ + \tfrac{\lambda}{3}f\omega.$$

Proof. According to the decomposition of $\Lambda^2 M$ into irreducible SU_3-summands, we can write

$$dJdf = \eta + \gamma\lrcorner\Psi^+ + h\omega$$

for some vector field γ and function h. From Lemma 3.2 we get $2\gamma = \alpha(dJdf) = -4df$. In order to compute h, we write

$$6h\, dv = h\omega\wedge\omega^2 = dJdf\wedge\omega^2 = d(Jdf\wedge\omega^2) \overset{(12)}{=} 2d*df = 2\lambda f\, dv.$$

□

We will now compute the Laplacian of the three summands of η separately. First, we have $\Delta df = \lambda df$ and Corollary 3.3 yields $\bar\Delta df = \lambda df$. Since $\bar\Delta$ commutes with J, we also have $\bar\Delta Jdf = \lambda Jdf$ and from the second equation in Corollary 3.3 we obtain

$$\Delta Jdf = \bar\Delta Jdf + (\Delta - \bar\Delta)Jdf = (\lambda+4)Jdf.$$

Hence, $dJdf$ is a Δ-eigenform for the eigenvalue $\lambda+4$.

Lemma 4.7. *The co-differential of the* $(1,1)$*-form* η *is given by*

$$d^*\eta = \left(\tfrac{2\lambda}{3} - 4\right) Jdf.$$

Proof. Notice that $d^*(f\omega) = -df\lrcorner\omega$ and that $d^*Jdf = -*d*Jdf = -\frac{1}{2}*d(df\wedge\omega^2) = 0$, since $d\omega^2 = 0$. Using this we obtain

$$\begin{aligned} d^*\eta &= \Delta Jdf + 2d^*(df\lrcorner\Psi^+) - \tfrac{\lambda}{3}df\lrcorner\omega = (\lambda+4)Jdf - 2*d(df\wedge\Psi^-) - \tfrac{\lambda}{3}Jdf \\ &= (\lambda + 4 - \tfrac{\lambda}{3})Jdf - 4*(df\wedge\omega^2) \overset{(12)}{=} (\tfrac{2\lambda}{3} - 4)Jdf. \end{aligned}$$

□

In order to compute Δ of the second summand of η we need three additional formulas

Lemma 4.8.

$$\bar\Delta(X\lrcorner\Psi^+) = (\bar\Delta X)\lrcorner\Psi^+.$$

Proof. Recall that $\bar\Delta = \bar\nabla^*\bar\nabla + q(\bar R)$. Since Ψ^+ is $\bar\nabla$-parallel we immediately obtain

$$\bar\nabla^*\bar\nabla(X\lrcorner\Psi^+) = -\bar\nabla_{e_i}\bar\nabla_{e_i}(X\lrcorner\Psi^+) = -(\bar\nabla_{e_i}\bar\nabla_{e_i}X)\lrcorner\Psi^+.$$

The map $A\mapsto A_*\Psi^+$ is a SU_3-equivariant map from Λ^2 to Λ^3. But since Λ^3 does not contain the representation $\Lambda^{(1,1)}_0$ as an irreducible summand, it follows that $A_*\Psi^+ = 0$ for any skew-symmetric endomorphism A corresponding to some primitive $(1,1)$-form. Hence we conclude

$$q(\bar R)(X\lrcorner\Psi^+) = \omega_{i*}\bar R(\omega_i)_*(X\lrcorner\Psi^+) = (\omega_{i*}\bar R(\omega_i)_*X)\lrcorner\Psi^+ = (q(\bar R)X)\lrcorner\Psi^+,$$

where, since the holonomy of $\bar\nabla$ is included in SU_3, the sum goes over some orthonormal basis $\{\omega_i\}$ of $\Lambda^{(1,1)}_0 M$. Combining these two formulas we obtain $\bar\Delta(X\lrcorner\Psi^+) = (\bar\Delta X)\lrcorner\Psi^+$.

□

Lemma 4.9.

$$(\Delta - \bar{\Delta})(df \lrcorner \Psi^+) = 6(df \lrcorner \Psi^+) - \tfrac{4\lambda}{3} f\omega - 2\eta.$$

Proof. From Proposition 3.4 in [11] we have

$$\begin{aligned}(\Delta - \bar{\Delta})(df \lrcorner \Psi^+) &= (\nabla^*\nabla - \bar{\nabla}^*\bar{\nabla})(df \lrcorner \Psi^+) + (q(R) - q(\bar{R}))(df \lrcorner \Psi^+)\\ &= (\nabla^*\nabla - \bar{\nabla}^*\bar{\nabla})(df \lrcorner \Psi^+) + 4 df \lrcorner \Psi^+.\end{aligned}$$

The first part of the right hand side reads

$$(\nabla^*\nabla - \bar{\nabla}^*\bar{\nabla})(df \lrcorner \Psi^+) = -\tfrac{1}{4} A_{e_i *} A_{e_i *} df \lrcorner \Psi^+ - A_{e_i *} \bar{\nabla}_{e_i}(df \lrcorner \Psi^+). \tag{25}$$

From (5) we get

$$\begin{aligned}A_{e_i *} A_{e_i *} df \lrcorner \Psi^+ &= A_{e_i *}(A_{e_i} e_k \wedge \Psi^+(df, e_k, \cdot))\\ &= A_{e_i} A_{e_i} e_k \wedge \Psi^+(df, e_k, \cdot) + A_{e_i} e_k \wedge A_{e_i} \Psi^+(df, e_k, \cdot)\\ &= -4 e_k \wedge e_k \lrcorner \Psi^+_{df} + A_{e_i} e_k \wedge A_{e_i} e_j \Psi^+(df, e_k, e_j) = -8\Psi^+_{df},\end{aligned}$$

where we used the vanishing of the expression $E = A_{e_i} e_k \wedge A_{e_i} e_j \Psi^+(df, e_k, e_j)$:

$$\begin{aligned}E &= A_{Je_i} e_k \wedge A_{Je_i} e_j \Psi^+(df, e_k, e_j) = A_{e_i} J e_k \wedge A_{e_i} J e_j \Psi^+(df, e_k, e_j)\\ &= A_{e_i} e_k \wedge A_{e_i} e_j \Psi^+(df, Je_k, Je_j) = -E.\end{aligned}$$

It remains to compute the second term in (25). We notice that by Schur's lemma, every SU_3-equivariant map from the space of symmetric tensors $Sym^2 M$ to TM vanishes, so in particular (since ∇df is symmetric), one has $A_{e_i} \nabla_{e_i} df = 0$. We then compute

$$\begin{aligned}A_{e_i *} \bar{\nabla}_{e_i} \Psi^+_{df} &= A_{e_i *}((\bar{\nabla}_{e_i} df) \lrcorner \Psi^+) = (A_{e_i} \bar{\nabla}_{e_i} df) \lrcorner \Psi^+ + (\bar{\nabla}_{e_i} df) \lrcorner A_{e_i *} \Psi^+\\ &\overset{(6)}{=} (A_{e_i} \nabla_{e_i} df) \lrcorner \Psi^+ - \tfrac{1}{2}(A_{e_i} A_{e_i} df) \lrcorner \Psi^+ - 2(\bar{\nabla}_{e_i} df) \lrcorner (e_i \wedge \omega)\\ &= 2\Psi^+_{df} + 2 d^* df \omega + \langle A_{e_i} df, e_i \rangle \omega + 2 e_i \wedge J \bar{\nabla}_{e_i} df\\ &= 2\Psi^+_{df} + 2\lambda f\omega + 2 e_i \wedge \bar{\nabla}_{e_i} J df = 2\Psi^+_{df} + 2\lambda f\omega + 2 dJdf - e_i \wedge A_{e_i} J df\\ &= 2\Psi^+_{df} + 2\lambda f\omega + 2 dJdf + 2A_{Jdf} = 4\Psi^+_{df} + 2\lambda f\omega + 2 dJdf.\end{aligned}$$

Plugging back what we obtained into (25) yields

$$(\nabla^*\nabla - \bar{\nabla}^*\bar{\nabla})(df \lrcorner \Psi^+) = -(2\Psi^+_{df} + 2\lambda f\omega + 2 dJdf),$$

which together with Lemma 4.6 and the first equation prove the desired formula. □

Lemma 4.10.

$$\Delta f\omega = (\lambda + 12) f\omega - 2(df \lrcorner \Psi^+).$$

Proof. Since $d^*(f\omega) = -df \lrcorner \omega = -Jdf$ we have $dd^*(f\omega) = -dJdf$. For the second summand of $\Delta(f\omega)$ we first compute $d(f\omega) = df \wedge \omega + 3f\Psi^+$. Since $d^*\Psi^+ = \frac{1}{3} d^* d\omega = 4\omega$, we get $d^*f\Psi^+ = -df\lrcorner\Psi^+ + fd^*\Psi^+ = -df\lrcorner\Psi^+ + 4f\omega$. Moreover

$$\begin{aligned} d^*(df\wedge\omega) &= -*d(Jdf\wedge\omega) = -*(dJdf\wedge\omega - 3Jdf\wedge\Psi^+) \\ &= -*([\eta - 2df\lrcorner\Psi^+ - \tfrac{\lambda}{3}f\omega]\wedge\omega) + 3*(Jdf\wedge\Psi^+) \\ &= \eta + 2*((df\lrcorner\Psi^+)\wedge\omega) + \tfrac{2\lambda}{3}f\omega - 3df\lrcorner\Psi^+ \\ &= \eta + 2df\lrcorner\Psi^+ + \tfrac{2\lambda}{3}f\omega - 3df\lrcorner\Psi^+. \end{aligned}$$

Recalling that $\eta = dJdf + 2df\lrcorner\Psi^+ + \frac{\lambda}{3}f\omega$, we obtain

$$\Delta f\omega = -dJdf - 3df\lrcorner\Psi^+ + 12f\omega + \eta - df\lrcorner\Psi^+ + \tfrac{2\lambda}{3}f\omega = (\lambda+12)f\omega - 2df\lrcorner\Psi^+.$$

□

Applying these three lemmas we conclude

$$\Delta(df\lrcorner\Psi^+) = \bar\Delta(df\lrcorner\Psi^+) + (\Delta - \bar\Delta)(df\lrcorner\Psi^+) = (\lambda+6)(df\lrcorner\Psi^+) - \tfrac{4\lambda}{3}f\omega - 2\eta$$

and thus

$$\begin{aligned} \Delta\eta &= (\lambda+4)dJdf + (2\lambda+12)(df\lrcorner\Psi^+) - \tfrac{8\lambda}{3}f\omega - 4\eta + \tfrac{\lambda}{3}(\lambda+12)f\omega - \tfrac{2\lambda}{3}(df\lrcorner\Psi^+) \\ &= \lambda\eta + \left(4 - \tfrac{2\lambda}{3}\right)(df\lrcorner\Psi^+). \end{aligned}$$

Finally we have once again to apply the formula for the difference of Δ and $\bar\Delta$ on primitive $(1,1)$-forms. We obtain

$$\bar\Delta\eta = \Delta\eta - Jd^*\eta\lrcorner\Psi^+ = \Delta\eta + \left(\tfrac{2\lambda}{3} - 4\right)(df\lrcorner\Psi^+) = \lambda\eta.$$

Summarizing our calculations we obtain the following result.

Proposition 4.11. *Let f be an Δ-eigenfunction with $\Delta f = \lambda f$ Then the primitive $(1,1)$-form $\eta := (dJdf)_0^{(1,1)}$ satisfies*

$$\bar\Delta\eta = \lambda\eta \qquad \textit{and} \qquad d^*\eta = \left(\tfrac{2\lambda}{3} - 4\right)Jdf.$$

Let $\Omega^0(12) \subset \mathcal{C}^\infty(M)$ be the $\bar\Delta$-eigenspace for the eigenvalue 12 (notice that $\bar\Delta = \Delta$ on functions) and let $\Omega_0^{(1,1)}(12)$ denote the space of primitive $(1,1)$-eigenforms of $\bar\Delta$ corresponding to the eigenvalue 12. Summarizing Corollary 4.5 and Proposition 4.11, we have constructed a linear mapping

$$\Phi : i(M) \to \Omega_0^{(1,1)}(12), \qquad \Phi(\xi) := d\xi_0^{(1,1)}$$

from the space of Killing vector fields into $\Omega_0^{(1,1)}(12)$ and a linear mapping

$$\Psi : \Omega^0(12) \to \Omega_0^{(1,1)}(12), \qquad \Psi(f) := (dJdf)_0^{(1,1)}.$$

Let moreover $\mathcal{NK} \subset \Omega_0^{(1,1)}(12)$ denote the space of nearly Kähler deformations, which by [10] is just the space of co-closed forms in $\Omega_0^{(1,1)}(12)$.

Proposition 4.12. *The linear mappings Φ and Ψ defined above are injective and the sum $\mathrm{Im}(\Phi) + \mathrm{Im}(\Psi) + \mathcal{NK} \subset \Omega_0^{(1,1)}(12)$ is a direct sum. In particular,*

$$\dim(\mathcal{NK}) \le \dim(\Omega_0^{(1,1)}(12)) - \dim(i(M)) - \dim(\Omega^0(12)). \tag{26}$$

Proof. It is enough to show that if $\xi \in i(M)$, $f \in \Omega^0(12)$ and $\alpha \in \mathcal{NK}$ satisfy

$$d\xi_0^{(1,1)} + (dJdf)_0^{(1,1)} + \alpha = 0, \tag{27}$$

then $\xi = 0$ and $f = 0$. We apply d^* to (27). Using Propositions 4.4 and 4.11 to express the co-differentials of the first two terms we get

$$8\xi + 8Jdf = 0. \tag{28}$$

Since $J\xi$ is co-closed (Proposition 4.1), formula (28) implies $0 = d^*J\xi = d^*df = 12f$, i.e. $f = 0$. Plugging back into (28) yields $\xi = 0$ too. □

5. The homogeneous Laplace operator on reductive homogeneous spaces

5.1. **The Peter-Weyl formalism.** Let $M = G/K$ be a homogeneous space with compact Lie groups $K \subset G$ and let $\pi : K \to \mathrm{Aut}(E)$ be a representation of K. We denote by $EM := G \times_\pi E$ be the associated vector bundle over M. The Peter-Weyl theorem and the Frobenius reciprocity yield the following isomorphism of G-representations:

$$L^2(EM) \cong \bigoplus_{\gamma \in \hat{G}} V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, E), \tag{29}$$

where $\hat{G}$ is the set of (non-isomorphic) irreducible G-representations. If not otherwise stated we will consider only complex representations. Recall that the space of smooth sections $\mathcal{C}^\infty(EM)$ can be identified with the space $\mathcal{C}^\infty(G;E)^K$ of K-invariant E-valued functions, i.e. functions $f : G \to E$ with $f(gk) = \pi(k)^{-1} f(g)$. This space is made into a G-representation by the left-regular representation ℓ, defined by by $(\ell(g)f)(a) = f(g^{-1}a)$. Let $v \in V_\gamma$ and $A \in \mathrm{Hom}_K(V_\gamma, E)$ then the invariant E-valued function corresponding to $v \otimes A$ is defined by $g \mapsto A(g^{-1}v)$. In particular, each summand in the Hilbert space direct sum (29) is a subset of $\mathcal{C}^\infty(EM) \subset L^2(EM)$.

Let $\mathfrak{g}$ be the Lie algebra of G. We denote by B the *Killing form* of $\mathfrak{g}$, $B(X,Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. The Killing form is non-degenerated and negative definite if G is compact and semi-simple, which will be the case in all examples below.

If $\pi : G \to \mathrm{Aut}(E)$ is a G-representation, the *Casimir operator* of (G, π) acts on E by the formula

$$\mathrm{Cas}_\pi^G = \sum (\pi_* X_i)^2, \tag{30}$$

where $\{X_i\}$ is a $(-B)$-orthonormal basis of $\mathfrak{g}$ and $\pi_* : \mathfrak{g} \to \mathrm{End}(E)$ denotes the differential of the representation π.

Remark 5.1. *Notice that the Casimir operator is divided by k if one use the scalar product $-kB$ instead of $-B$.*

If G is simple, the adjoint representation ad on the complexification $\mathfrak{g}^\mathbb{C}$ is irreducible, so, by Schur's Lemma, its Casimir operator acts as a scalar. Taking the trace in (30) for $\pi = \mathrm{ad}$ yields the useful formula $\mathrm{Cas}_{\mathrm{ad}}^G = -1$.

Let V_γ be an irreducible G-representation of highest weight γ. By Freudenthal's formula the Casimir operator acts on V_γ by scalar multiplication with $\|\rho\|^2 - \|\rho+\gamma\|^2$, where ρ denotes the half-sum of the positive roots and $\|\cdot\|$ is the norm induced by $-B$ on the dual of the Lie algebra of the maximal torus of G. Notice that these scalars are always non-positive. Indeed $\|\rho\|^2 - \|\rho+\gamma\|^2 = -\langle\gamma,\gamma+2\rho\rangle_B$ and $\langle\gamma,\rho\rangle \geq 0$, since γ is a dominant weight, i.e. it is in the the closure of the fixed Weyl chamber, whereas ρ is the half-sum of positive weights and thus by definition has a non-negative scalar product with γ.

5.2. **The homogeneous Laplace operator.** We denote by $\bar\nabla$ the canonical homogeneous connection on $M = G/K$. It coincides with the Levi-Civita connection only in the case that G/K is a symmetric space. A crucial observation is that the canonical homogeneous connection coincides with the canonical Hermitian connection on naturally reductive 3-symmetric spaces (see below). We define the curvature endomorphism $q(\bar R) \in \mathrm{End}(EM)$ as in (14) and introduce as in (16) the second order operator $\bar\Delta_\pi = \bar\nabla^*\bar\nabla + q(\bar R)$ acting on sections of the associated bundle $EM := G\times_\pi E$.

Lemma 5.2. *Let G be a compact semi-simple Lie group, $K\subset G$ a compact subgroup, and let $M = G/K$ the naturally reductive homogeneous space equipped with the Riemannian metric induced by $-B$. For every K-representation π on E, let $EM := G\times_\pi E$ be the associated vector bundle over M. Then the endomorphism $q(\bar R)$ acts fibre-wise on EM as $q(\bar R) = -\mathrm{Cas}^K_\pi$. Moreover the differential operator $\bar\Delta$ acts on the space of sections of EM, considered as G-representation via the left-regular representation, as $\bar\Delta = -\mathrm{Cas}^G_\ell$.*

Proof. Consider the $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. For any vector $X\in\mathfrak{g}$ we write $X = X^{\mathfrak{k}} + X^{\mathfrak{p}}$, with $X^{\mathfrak{k}}\in\mathfrak{k}$ and $X^{\mathfrak{p}}\in\mathfrak{p}$. The canonical homogeneous connection is the left-invariant connection in the principal K-fibre bundle $G\to G/K$ corresponding to the projection $X\mapsto X^{\mathfrak{k}}$. It follows that one can do for the canonical homogeneous connection on G/K the same identifications as for the Levi Civita connection on Riemannian symmetric spaces.

In particular, the covariant derivative of a section $\phi\in\Gamma(EM)$ with respect to some $X\in\mathfrak{p}$ translates into the derivative $X(\hat\phi)$ of the the corresponding function $\hat\phi\in\mathcal{C}^\infty(G;E)^K$, which is minus the differential of the left-regular representation $X(\hat\phi) = -\ell_*(X)\hat\phi$. Hence, if $\{e_\mu\}$ is an orthonormal basis in $\mathfrak{p}$, the rough Laplacian $\bar\nabla^*\bar\nabla$ translates into the sum $-\ell_*(e_\mu)\ell_*(e_\mu) = (-\mathrm{Cas}^G_\ell + \mathrm{Cas}^K_\ell)$. Since $\bar\Delta = \bar\nabla^*\bar\nabla + q(\bar R)$ it remains to show that $q(\bar R) = -\mathrm{Cas}^K_\ell = -\mathrm{Cas}^K_\pi$ in order to complete the proof of the lemma.

We claim that the differential $i_* : \mathfrak{k}\to\mathfrak{so}(\mathfrak{p})\cong\Lambda^2\mathfrak{p}$ of the isotropy representation $i : K\to\mathrm{SO}(\mathfrak{p})$ is given by $i_*(A) = -\frac12 e_\mu\wedge[A,e_\mu]$ for any $A\in\mathfrak{k}$. Indeed

$$(\tfrac12 e_\mu\wedge[A,e_\mu])_*X = -\tfrac12 B(e_\mu,X)[A,e_\mu] + \tfrac12 B([A,e_\mu],X)e_\mu = -[A,X].$$

Next we recall that for $X,Y\in\mathfrak{p}$ the curvature $\bar R_{X,Y}$ of the canonical connection acts by $-\pi_*([X,Y]^{\mathfrak{k}})$ on every associated vector bundle EM, defined by the representation π.

Hence the curvature operator $\bar{R}$ can be written for any $X, Y \in \mathfrak{p}$ as

$$\bar{R}(X \wedge Y) = \tfrac{1}{2} e_\mu \wedge \bar{R}_{X,Y} e_\mu = -\tfrac{1}{2} e_\mu \wedge [[X,Y]^{\mathfrak{k}}, e_\mu] = i_*([X,Y]^{\mathfrak{k}}).$$

Let $P_{\mathrm{SO}(\mathfrak{p})} = G \times_i \mathrm{SO}(\mathfrak{p})$ be the bundle of orthonormal frames of $M = G/K$. Then any $\mathrm{SO}(\mathfrak{p})$-representation $\tilde{\pi}$ defines a K-representation by $\pi = \tilde{\pi} \circ i$. Moreover any vector bundle EM associated to $P_{\mathrm{SO}(\mathfrak{p})}$ via $\tilde{\pi}$ can be written as a vector bundle associated via π to the K-principle bundle $G \to G/K$, i.e.

$$EM = P_{\mathrm{SO}(\mathfrak{p})} \times_{\tilde{\pi}} E = G \times_\pi E$$

Let $\{f_\alpha\}$ be an orthonormal basis of $\mathfrak{k}$. Then by the definition of $q(\bar{R})$ we have

$$\begin{aligned}
q(\bar{R}) &= \tfrac{1}{2}\tilde{\pi}_*(e_\mu \wedge e_\nu)\, \tilde{\pi}_*(\bar{R}(e_\mu \wedge e_\nu)) = \tfrac{1}{2}\tilde{\pi}_*(e_\mu \wedge e_\nu)\, \pi_*([e_\mu, e_\nu]^{\mathfrak{k}}) \\
&= -\tfrac{1}{2} B([e_\mu, e_\nu], f_\alpha) \tilde{\pi}_*(e_\mu \wedge e_\nu)\, \pi_*(f_\alpha) = -\tfrac{1}{2} B(e_\nu, [f_\alpha, e_\mu]) \tilde{\pi}_*(e_\mu \wedge e_\nu)\, \pi_*(f_\alpha) \\
&= \tfrac{1}{2}\tilde{\pi}_*(e_\mu \wedge [f_\alpha, e_\mu])\, \pi_*(f_\alpha) = -\pi_*(f_\alpha)\, \pi_*(f_\alpha) \\
&= -\mathrm{Cas}^K_\pi .
\end{aligned}$$

We have shown that $q(\bar{R}) \in \mathrm{End}(EM)$ acts fibre-wise as $-\mathrm{Cas}^K_\pi$. Let $Z \in \mathfrak{k}$ and $f \in C^\infty(G; E)^K$, then the K-invariance of f implies $\pi_*(Z) f = -Z(f) = \ell_*(Z) f$ and also $\mathrm{Cas}^K_\pi = \mathrm{Cas}^K_\ell$, which concludes the proof of the lemma. ☐

It follows from this lemma that the spectrum of $\bar{\Delta}$ on sections of EM is the set of numbers $\lambda_\gamma = \|\rho + \gamma\|^2 - \|\rho\|^2$, where γ is the highest weight of an irreducible G-representation V_γ such that $\mathrm{Hom}_K(V_\gamma, E) \neq 0$, i.e. such that the decomposition of V_γ, considered as K-representation, contains components of the K-representation E.

5.3. **Nearly Kähler deformations and Laplace eigenvalues.** Let (M, g, J) be a compact simply connected 6-dimensional nearly Kähler manifold not isometric to the round sphere, with scalar curvature normalized to $\mathrm{scal}_g = 30$. Recall the following result from [10]:

Theorem 5.3. *The Laplace operator Δ coincides with the Hermitian Laplace operator $\bar{\Delta}$ on co-closed primitive $(1,1)$-forms. The space $\mathcal{NK}$ of infinitesimal deformations of the nearly Kähler structure of M is isomorphic to the eigenspace for the eigenvalue 12 of the restriction of Δ (or $\bar{\Delta}$) to the space of* co-closed *primitive $(1,1)$-forms on M.*

Assume from now on that M is a 6-dimensional naturally reductive 3-symmetric space G/K in the list of Gray and Wolf, i.e. $\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2/\mathrm{SU}_2$, $\mathrm{SO}_5/\mathrm{U}_2$ or SU_3/T^2 . As was noticed before, the canonical homogeneous and the canonical Hermitian connection coincide, since for the later can be shown that is torsion and its curvature are parallel, a property, which by the Ambrose-Singer-Theorem characterizes the canonical homogeneous connection (c.f. [5]). In order to determine the space $\mathcal{NK}$ on M we thus need to apply the previous calculations to compute the $\bar{\Delta}$-eigenspace for the eigenvalue 12 on primitive $(1,1)$-forms and decide which of these eigenforms are co-closed.

According to Lemma 5.2 and the decomposition (29) we have to carry out three steps: first to determine the K-representation $\Lambda_0^{1,1}\mathfrak{p}$ defining the bundle $\Lambda_0^{1,1}TM$, then to compute the Casimir eigenvalues with the Freudenthal formula, which gives all possible $\bar{\Delta}$-eigenvalues and finally to check whether the G-representation V_γ realizing the eigenvalue 12 satisfies $\mathrm{Hom}_K(V_\gamma, \Lambda_0^{1,1}\mathfrak{p}) \neq \{0\}$ and thus really appears as eigenspace.

Before going on, we make the following useful observation

Lemma 5.4. *Let $(G/K, g)$ be a 6-dimensional homogeneous strict nearly Kähler manifold of scalar curvature $\mathrm{scal}_g = 30$. Then the homogeneous metric g is induced from $-\frac{1}{12}B$, where B is the Killing form of G.*

Proof. Let G/K be a 6-dimensional homogeneous strict nearly Kähler manifold. Then the metric is induced from a multiple of the Killing form, i.e. G/K is a normal homogeneous space with $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The scalar curvature of the metric h induced by $-B$ may be computed as (c.f. [3])

$$\mathrm{scal}_h = \tfrac{3}{2} - 3\mathrm{Cas}_\lambda^K$$

where $\lambda : K \to \mathfrak{so}(\mathfrak{p})$ is the isotropy representation. From Lemma 5.2 we know that $\mathrm{Cas}_\lambda^K = -q(\bar{R})$, which on the tangent bundle was computed in Lemma 15 as $q(\bar{R}) = \frac{2\mathrm{scal}_h}{15}\,\mathrm{id}$. Hence we obtain the equation $\mathrm{scal}_h = \frac{3}{2} + \frac{2}{5}\mathrm{scal}_h$ and it follows $\mathrm{scal}_h = \frac{5}{2}$, i.e. the metric g corresponding to $-\frac{1}{12}B$ has scalar curvature $\mathrm{scal}_g = 30$. □

5.4. **The $\bar{\Delta}$-spectrum on $S^3 \times S^3$.** Let $K = \mathrm{SU}_2$ with Lie algebra $\mathfrak{k} = \mathfrak{su}_2$ and $G = K \times K \times K$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k}$. We consider the 6-dimensional manifold $M = G/K$, where K is diagonally embedded. The tangent space at $o = eK$ can be identified with

$$\mathfrak{p} = \{(X, Y, Z) \in \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} \,|\, X + Y + Z = 0\}.$$

Let B be the Killing form of $\mathfrak{k}$ and define $B_0 = -\frac{1}{12}B$. Then it follows from Lemma 5.4 that the invariant scalar product

$$B_0((X, Y, Z), (X, Y, Z)) = B_0(X, X) + B_0(Y, Y) + B_0(Z, Z)$$

defines a normal metric, which is the homogeneous nearly Kähler metric g of scalar curvature $\mathrm{scal}_g = 30$.

The canonical almost complex structure on the 3-symmetric space M, corresponding to the 3rd order G-automorphism σ, with $\sigma(k_1, k_2, k_3) = (k_2, k_3, k_1)$, is defined as

$$J(X, Y, Z) = \tfrac{2}{\sqrt{3}}(Z, X, Y) + \tfrac{1}{\sqrt{3}}(X, Y, Z).$$

The $(1,0)$-subspace $\mathfrak{p}^{1,0}$ of $\mathfrak{p}^{\mathbb{C}}$ defined by J is isomorphic to the complexified adjoint representation of SU_2 on $\mathfrak{su}_2^{\mathbb{C}}$. Let $E = \mathbb{C}^2$ denote the standard representation of SU_2 (notice that $E \cong \bar{E}$ because every $\mathrm{SU}_2 \cong \mathrm{Sp}_1$ representation is quaternionic).

Lemma 5.5. *The SU_2-representation defining the bundle $\Lambda_0^{(1,1)}TM$ splits into the irreducible summands $\mathrm{Sym}^4 E$ and $\mathrm{Sym}^2 E$.*

Proof. The defining SU_2-representation of $\Lambda^{(1,1)}TM$ is $\mathfrak{p}^{1,0} \otimes \mathfrak{p}^{0,1} \cong \mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E \cong \mathrm{Sym}^4 E \oplus \mathrm{Sym}^2 E \oplus \mathrm{Sym}^0 E$ from the Clebsch-Gordan formula. Since we are interested in primitive $(1,1)$-forms, we still have to delete the trivial summand $\mathrm{Sym}^0 E \cong \mathbb{C}$. □

Since $G = \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2$, every irreducible G-representation is isomorphic to one of the representations $V_{a,b,c} = \mathrm{Sym}^a E \otimes \mathrm{Sym}^b E \otimes \mathrm{Sym}^c E$. The Casimir operator of the SU_2-representation $\mathrm{Sym}^k E$ (with respect to B) is $-\frac{1}{8}k(k+2)$ and the Casimir operator of G is the sum of the three SU_2-Casimir operators. Hence all possible $\bar{\Delta}$-eigenvalues with respect to the metric B_0 are of the form

$$\tfrac{3}{2}(a(a+2) + b(b+2) + c(c+2)). \tag{31}$$

for non-negative integers a, b, c. It is easy to check that the eigenvalue 12 is obtained only for (a,b,c) equal to $(2,0,0)$, $(0,2,0)$ or $(0,0,2)$. The restrictions to SU_2 (diagonally embedded in G) of the three corresponding G-representations are all equal to the SU_2-representation $\mathrm{Sym}^2 E$, thus $\dim \mathrm{Hom}_{\mathrm{SU}_2}(V_{2,0,0}, \Lambda_0^{(1,1)}\mathfrak{p}) = 1$, and similarly for the two other summands. Hence the eigenspace of $\bar{\Delta}$ on primitive $(1,1)$-forms for the eigenvalue 12 is isomorphic to $V_{2,0,0} \oplus V_{0,2,0} \oplus V_{0,0,2}$ and its dimension, i.e. the multiplicity of the eigenvalue 12, is equal to 9.

Since the isometry group of the nearly Kähler manifold $M = \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2/\mathrm{SU}_2$ has dimension 9, the inequality (26) yields

$$\dim(\mathcal{NK}) \leq \dim(\Omega_0^{(1,1)}(12)) - \dim(i(M)) - \dim(\Omega^0(12)) = -\dim(\Omega^0(12)) \leq 0.$$

We thus have obtained the following

Theorem 5.6. *The homogeneous nearly Kähler structure on $S^3 \times S^3$ does not admit any infinitesimal nearly Kähler deformations.*

Finally we remark that there are also no infinitesimal Einstein deformations neither. In [11] we showed that the space of infinitesimal Einstein deformations of a nearly Kähler metric g, with normalized scalar curvature $\mathrm{scal}_g = 30$, is isomorphic to the direct sum of $\bar{\Delta}$-eigenspaces of primitive co-closed $(1,1)$-forms for the eigenvalues $2, 6$ and 12. It is clear from (31) that neither 2 nor 6 can be realized as $\bar{\Delta}$-eigenvalues.

Corollary 5.7. *The homogeneous nearly Kähler metric on $S^3 \times S^3$ does not admit any infinitesimal Einstein deformations.*

5.5. **The $\bar{\Delta}$-spectrum on $\mathbb{C}P^3$.** In this section we consider the complex projective space $\mathbb{C}P^3 = \mathrm{SO}_5/\mathrm{U}_2$, where U_2 is embedded by $\mathrm{U}_2 \subset \mathrm{SO}_4 \subset \mathrm{SO}_5$. Let $G = \mathrm{SO}_5$ with Lie algebra $\mathfrak{g}$ and $K = \mathrm{U}_2$ with Lie algebra $\mathfrak{k}$. We denote the Killing form of G with B. Then we have the B-orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ can be identified with the tangent space in $o = eK$. The space $\mathfrak{p}$ splits as $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$, where $\mathfrak{m}$ resp. $\mathfrak{n}$ can be identified with the horizontal resp. vertical tangent space at o of the twistor space fibration $\mathrm{SO}_5/\mathrm{U}_2 \to \mathrm{SO}_5/\mathrm{SO}_4 = S^4$. We know from Lemma 5.4 that $B_0 = -\frac{1}{12}B$ defines the homogeneous nearly Kähler metric g of scalar curvature $\mathrm{scal}_g = 30$.

Let $\{\varepsilon_1, \varepsilon_2\}$ denote the canonical basis of $\mathbb{R}^2$. Then the positive roots of SO_5 are $\alpha_1 = \varepsilon_1$, $\alpha_2 = \varepsilon_2$, $\alpha_3 = \varepsilon_1 + \varepsilon_2$, $\alpha_4 = \varepsilon_1 - \varepsilon_2$, with $\rho = \frac{3}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2$. Let $\mathfrak{g}^\alpha \subset \mathfrak{g}^{\mathbb{C}}$ be the root space corresponding to the root α. Then

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{-\alpha_1} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{-\alpha_2}, \qquad \mathfrak{n}^{\mathbb{C}} = \mathfrak{g}^{\alpha_3} \oplus \mathfrak{g}^{-\alpha_3}.$$

The invariant almost complex structure J may be defined by specifying the $(1,0)$-subspace $\mathfrak{p}^{1,0}$ of $\mathfrak{p}^{\mathbb{C}}$:

$$\mathfrak{p}^{1,0} = \{X - iJX \mid X \in \mathfrak{p}\} = \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{-\alpha_3},$$

It follows that J is not integrable, since the restricted root system $\{\alpha_1, \alpha_2, -\alpha_3\}$ is not closed under addition (cf. [4]). We note that replacing $-\alpha_3$ by α_3 yields an integrable almost complex structure. This corresponds to the well-known fact that on the twistor space the non integrable almost complex structure J is transformed into the the integrable one by replacing J with $-J$ on the vertical tangent space.

Let $\mathbb{C}_k$ denote the U_1-representation on $\mathbb{C}$ defined by $(z, v) \mapsto z^k v$, for $v \in \mathbb{C}$ and $z \in \mathrm{U}_1 \cong \mathbb{C}^*$. Then, since $\mathrm{U}_2 = (\mathrm{SU}_2 \times \mathrm{U}_1)/\mathbb{Z}_2$, any irreducible U_2-representation is of the form $E_{a,b} = \mathrm{Sym}^a E \otimes \mathbb{C}_b$, with $a \in \mathbb{N}$, $b \in \mathbb{Z}$ and $a \equiv b \mod 2$. As usual let $E = \mathbb{C}^2$ denote the standard representation of SU_2.

With this notation we obtain the following decomposition of $\mathfrak{p}^{1,0}$ considered as U_2-representation

$$\mathfrak{p}^{1,0} \cong E_{0,-2} \oplus E_{1,1} \qquad \text{with} \qquad E_{0,-2} \cong \mathfrak{g}^{-\alpha_3} \quad \text{and} \quad E_{1,1} \cong \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{\alpha_2}. \tag{32}$$

Since $\mathfrak{p}^{0,1}$ is obtained from $\mathfrak{p}^{1,0}$ by conjugation we have $\mathfrak{p}^{0,1} \cong E_{0,2} \oplus E_{1,-1}$. The defining U_2-representation of $\Lambda^{(1,1)}TM$ is $\mathfrak{p}^{1,0} \otimes \mathfrak{p}^{0,1}$, which obviously decomposes into 5 irreducible summands, among which, two are isomorphic to the trivial representation $E_{0,0}$. Considering only primitive $(1,1)$-forms we still have to delete one of the trivial summands and obtain

Lemma 5.8. *The U_2-representation defining the bundle $\Lambda_0^{(1,1)}TM$ has the following decomposition into irreducible summands*

$$\Lambda_0^{(1,1)}\mathfrak{p} = E_{0,0} \oplus E_{1,3} \oplus E_{1,-3} \oplus E_{2,0}.$$

Let $V_{a,b}$ be an irreducible SO_5-representation of highest weight $\gamma = (a,b)$ with $a, b \in \mathbb{N}$ and $a \geq b \geq 0$, e.g. $V_{1,0} = \Lambda^1$ and $V_{1,1} = \Lambda^2$. The scalar product induced by the Killing form B on the dual $\mathfrak{t}^* \cong \mathbb{R}^2$ of the maximal torus of SO_5 is $-\frac{1}{6}$ times the Euclidean scalar product. By the Freudenthal formula we thus get

$$\mathrm{Cas}_{V_{a,b}} = \langle \gamma, \gamma + 2\rho \rangle_B = -\tfrac{1}{6}(a(a+3) + b(b+1)). \tag{33}$$

Notice that we have $V_{1,1} = \mathfrak{so}_5^{\mathbb{C}}$ and $\mathrm{Cas}_{V_{1,1}} = -1$, which is consistent with $\mathrm{Cas}^G_{\mathrm{ad}} = -1$.

It follows (c.f. Remark 5.1) that all possible $\bar{\Delta}$-eigenvalues with respect to the metric induced by B_0 are of the form $2(a(a+3) + b(b+1))$. The eigenvalue 12 is realized if and only if $(a,b) = (1,1)$. We still have to decide whether the SO_5-representation $V_{1,1}$ actually appears in the decomposition (29) of $L^2(\Lambda_0^{1,1}TM)$. However this follows from

Lemma 5.9. *The* SO_5*-representation* $V_{1,1}$ *restricted to* $\mathrm{U}_2 \subset \mathrm{SO}_5$ *has the following decomposition as* U_2*-representation:*

$$V_{1,1} \cong (E_{0,0} \oplus E_{2,0}) \oplus (E_{0,-2} \oplus E_{1,1} \oplus E_{0,2} \oplus E_{1,-1})$$

and in particular

$$\dim \mathrm{Hom}_{\mathrm{U}_2}(V_{1,1}, \Lambda_0^{1,1}\mathfrak{p}^{\mathbb{C}}) = 2 \qquad \text{and} \qquad \dim \mathrm{Hom}_{\mathrm{U}_2}(V_{1,1}, \mathbb{C}) = 1.$$

Proof. We know already that $V_{1,1} = \mathfrak{so}_5^{\mathbb{C}}$ is the complexified adjoint representation and that $\mathfrak{so}_5^{\mathbb{C}} = \mathfrak{u}_2^{\mathbb{C}} \oplus (\mathfrak{p}^{1,0} \oplus \mathfrak{p}^{0,1})$. The decomposition of the last two summands is contained in (32). Hence it remains to explicit the adjoint representation of U_2 on $\mathfrak{u}_2^{\mathbb{C}}$. It is clear that its restriction to U_1 acts trivially, whereas its restriction to SU_2 decomposes into $\mathbb{C} \oplus \mathfrak{su}_2^{\mathbb{C}}$, i.e. $\mathfrak{u}_2^{\mathbb{C}} \cong E_{0,0} \oplus E_{2,0}$. □

The eigenspace of $\bar{\Delta}$ on primitive $(1,1)$-forms for the eigenvalue 12 is thus isomorphic to the sum of two copies of $V_{1,1}$, i.e. the eigenvalue 12 has multiplicity $2 \cdot 10 = 20$.

It is now easy to calculate the smallest eigenvalue and the corresponding eigenspace of the Laplace operator Δ on non-constant functions. We do this for $\bar{\Delta}$, which coincides with Δ on functions. Then we have to replace $\Lambda_0^{(1,1)}\mathfrak{p}$ in the calculations above with the trivial representation $\mathbb{C}$ and to look for SO_5-representations $V_{a,b}$ containing the zero weight. It follows from Lemma 5.9 and (33) that the Δ-eigenspace on functions $\Omega^0(12)$ is isomorphic to $V_{1,1}$ and is thus 10-dimensional. Since the dimension of the isometry group of the nearly Kähler manifold $\mathrm{SO}(5)/\mathrm{U}_2$ is 10, the inequality (26) shows that

$$\dim(\mathcal{NK}) \le \dim(\Lambda_0^{(1,1)}(12)) - \dim(i(M)) - \dim(\Lambda^0(12)) = 20 - 10 - 10 = 0,$$

so there are no infinitesimal nearly Kähler deformations in this case neither.

Finally, we remark like before that there are also no other infinitesimal Einstein deformations, since by (33), the eigenvalues 2 and 6 do not occur in the spectrum of $\bar{\Delta}$ on $\Lambda_0^{(1,1)}M$. Summarizing, we have obtained the following:

Theorem 5.10. *The homogeneous nearly Kähler structure on* $\mathbb{C}\mathrm{P}^3 = \mathrm{SO}_5/\mathrm{U}_2$ *does not admit any infinitesimal nearly Kähler or Einstein deformations.*

5.6. **The $\bar{\Delta}$-spectrum on the flag manifold $F(1,2)$.** In this section we consider the flag manifold $M = SU_3/T^2$, where $T^2 \subset \mathrm{SU}_3$ is the maximal torus. Let $\mathfrak{g} = \mathfrak{su}_3$ and let $\mathfrak{k} = \mathfrak{t}$, the Lie algebra of T^2. We have the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}.$$

Denoting by E_{ij}, S_{ij} are "real and imaginary" part of the projection of the vector $X_{ij} \in \mathfrak{gl}_3$ (equal to 1 on ith row and jth column and 0 elsewhere) onto $\mathfrak{su}_3$:

$$E_{ij} = X_{ij} - X_{ji} \qquad S_{ij} = i(X_{ij} + X_{ji}),$$

the subspaces $\mathfrak{m}$ and $\mathfrak{n}$ are explicitly given by

$$\mathfrak{m} = \mathrm{span}\{E_{12}, S_{12}, E_{13}, S_{13}\} = \mathrm{span}\{e_1, e_2, e_3, e_4\},$$

$$\mathfrak{n} = \mathrm{span}\{E_{23}, S_{23}\} = \mathrm{span}\{e_5, e_6\}.$$

The dual of the Lie algebra $\mathfrak{t}$ of the maximal torus T^2 can be identified with

$$\mathfrak{t}^* \cong \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \mid \lambda_1 + \lambda_2 + \lambda_3 = 0\}.$$

If $\{\varepsilon_i\}$ denotes the canonical basis in $\mathbb{R}^3$ then the set of positive roots is given as $\phi^+ = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j \mid 1 \le i < j \le 3\}$ and the half-sum of the positive roots is $\rho = \varepsilon_1 - \varepsilon_3$

Let B denote the Killing form of SU_3. By Lemma 5.4, $B_0 = -\frac{1}{12}B$ defines the homogeneous nearly Kähler metric g of scalar curvature $\mathrm{scal}_g = 30$.

The almost complex structure J is explicitly defined on $\mathfrak{p}$ by

$$J(e_1) = e_2, \qquad J(e_3) = -e_4, \qquad J(e_5) = e_6.$$

Alternatively we may define the $(1,0)$-subspace of $\mathfrak{p}^{\mathbb{C}}$:

$$\mathfrak{p}^{1,0} = \mathfrak{g}^{\alpha_{12}} \oplus \mathfrak{g}^{\alpha_{31}} \oplus \mathfrak{g}^{\alpha_{23}} = \mathrm{span}\{X_{12}, X_{31}, X_{23}\},$$

where $\mathfrak{g}^{\alpha}$ is the root space for α. It follows that J is not integrable, since the restricted root system $\{\alpha_{12}, \alpha_{31}, \alpha_{23}\}$ is not closed under addition (c.f. [4]).

Let $E = \mathbb{C}^3$ be the standard representation of SU_3 with conjugate representation $\bar{E}$. Any irreducible representations of SU_3 is isomorphic to one of the representations

$$V_{k,l} := (\mathrm{Sym}^k E \otimes \mathrm{Sym}^l \bar{E})_0,$$

where the right hand side denotes the kernel of the contraction map

$$\mathrm{Sym}^k E \otimes \mathrm{Sym}^l \bar{E} \to \mathrm{Sym}^{k-1} E \otimes \mathrm{Sym}^{l-1} \bar{E},$$

i.e. $V_{k,l}$ is the Cartan summand in $\mathrm{Sym}^k E \otimes \mathrm{Sym}^l \bar{E}$. The weights of $\mathrm{Sym}^k E$ are

$$a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3, \qquad \text{with } a, b, c \ge 0,\ a + b + c = k.$$

If v_1, v_2, v_3 are the weight vectors of E, then these weights correspond to the weight vectors $v_1^a \cdot v_2^b \cdot v_3^c$ in $\mathrm{Sym}^k E$. Since the weights of $\mathrm{Sym}^l \bar{E}$ are just minus the weights of $\mathrm{Sym}^l E$, we see that the weights of $V_{k,l}$ are

$$(a-a')\varepsilon_1 + (b-b')\varepsilon_2 + (c-c')\varepsilon_3, \quad a, b, c, a', b', c' \ge 0,\ a+b+c = k,\ a'+b'+c' = l. \tag{34}$$

From the given definition of the almost complex structure J it is clear that the T^2-representation on $\mathfrak{p}^{1,0}$ splits in three one-dimensional T^2-representations with the weights $\alpha_{12}, \alpha_{31}, \alpha_{23}$.

Since the weights of a tensor product representation are the sums of weights of each factor and since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ on the Lie algebra of the maximal torus of SU_3, we immediately obtain

Corollary 5.11. *The weights of the T^2-representation on $\Lambda^{1,1}\mathfrak{p} \cong \mathfrak{p}^{1,0} \otimes \mathfrak{p}^{0,1}$ are*

$$\pm 3\varepsilon_1,\ \pm 3\varepsilon_2,\ \pm 3\varepsilon_3,\ \textit{and } 0.$$

It remains to compute the Casimir operator of the irreducible SU_3-representations $V_{k,l}$. The highest weight of $V_{k,l}$ is $\gamma = k\varepsilon_1 - l\varepsilon_3$ and $\rho = \varepsilon_1 - \varepsilon_3$, thus

$$\mathrm{Cas}_{V_{k,l}} = \langle \gamma, \gamma + 2\rho \rangle_B = -\tfrac{1}{6}(k(k+2) + l(l+2)). \tag{35}$$

Here we use again the Freudenthal formula and the fact that the Killing form B induces $-\frac{1}{6}$ times the Euclidean scalar product on $\mathfrak{t}^* \subset \mathbb{R}^3$ (easy calculation). Notice that we have $V_{1,1} = \mathfrak{su}_3^{\mathbb{C}}$ and $\mathrm{Cas}_{V_{1,1}} = -1$, which is consistent with $\mathrm{Cas}^G_{\mathrm{ad}} = -1$ as in the previous cases.

It follows that all possible $\bar\Delta$-eigenvalues (with respect to the metric B_0) are of the form $2(k(k+2) + l(l+2))$. Obviously the eigenvalue 12 can only be obtained for $k = l = 1$. Moreover, the restriction of the SU_3-representation $V_{1,1}$ contains the zero weight space. In fact, from (34), the zero weight appears in $V_{k,l}$ if and only if there exist $a, b, c, a', b', c' \geq 0$, $a+b+c = k$, $a'+b'+c' = l$ such that $(a-a')\varepsilon_1 + (b-b')\varepsilon_2 + (c-c')\varepsilon_3 = 0$, which is equivalent to $k = l$. We see that $\dim \mathrm{Hom}_{T^2}(V_{1,1}, \Lambda_0^{(1,1)}\mathfrak{p}) = 2 \cdot 2 = 4$.

Hence the eigenspace of $\bar\Delta$ on primitive $(1,1)$-forms for the eigenvalue 12 is isomorphic to the sum of four copies of $V_{1,1}$, i.e. the eigenvalue 12 has multiplicity $4 \cdot 8 = 32$.

Computing the the smallest eigenvalue and the corresponding eigenspace of the Laplace operator Δ on non-constant functions we find $V_{0,0}$ for the eigenvalue 0 and $V_{1,1}$ for the eigenvalue 12. All other possible representations give a larger eigenvalue. Hence, the Δ-eigenspace on functions $\Omega^0(12)$ is isomorphic to two copies of $V_{1,1}$, i.e. the eigenvalue 12 has multiplicity $8 \cdot 2 = 16$.

Since the dimension of the isometry group of the nearly Kähler manifold SU_3/T^2 is 8, we obtain from (26)

$$\dim(\mathcal{NK}) \leq \dim(\Omega_0^{(1,1)}(12)) - \dim(i(M)) - \dim(\Omega^0(12)) = 8. \tag{36}$$

In the next section we will show by an explicit construction that actually the equality holds, so the flag manifold has an 8-dimensional space of infinitesimal nearly Kähler deformations.

Before describing this construction we note that there are no infinitesimal Einstein deformations other than the nearly Kähler deformations. It follows from (35) that the eigenvalue 2 does not occur in the spectrum of $\bar\Delta$ on $\Lambda_0^{(1,1)}M$. The eigenvalue 6 could be realized on the SU_3-representations $V = V_{1,0}$ or $V = V_{0,1}$. However it is easy to check that $\mathrm{Hom}_{T^2}(V, \Lambda_0^{(1,1)}\mathfrak{p}) = \{0\}$.

Corollary 5.12. *Every infinitesimal Einstein deformation of the homogeneous nearly Kähler metric on $F(1,2) = SU_3/T^2$ is an infinitesimal nearly Kähler deformation.*

6. The infinitesimal nearly Kähler deformations on SU_3/T^2

In this section we describe by explicit computation the space of infinitesimal nearly Kähler deformations of the flag manifold $F(1,2) = SU_3/T^2$. The Lie algebra $\mathfrak{u}_3$ is spanned by $\{h_1, h_2, h_3, e_1, \ldots, e_6\}$, where

$$h_1 = iE_{11}, \qquad h_2 = iE_{22}, \qquad h_3 = iE_{33},$$

$$e_1 = E_{12} - E_{21}, \qquad e_3 = E_{13} - E_{31}, \qquad e_5 = E_{23} - E_{32},$$
$$e_2 = i(E_{12} + E_{21}), \qquad e_4 = i(E_{13} + E_{31}), \qquad e_6 = i(E_{23} + E_{32}).$$

We consider the bi-invariant metric g on SU_3 induced by $-B/12$, where B denotes the Killing form of $\mathfrak{su}_3$. It is easy to check that $|e_i|^2{=}1$ and $|h_i - h_j|^2 = 1$ with respect to g. We extend this metric to U_3 in the obvious way which makes the frame $\{e_i, \sqrt{2}h_j\}$ orthonormal. This defines a metric, also denoted by g, on the manifold $M = F(1,2)$. From now on we identify vectors and 1-forms using this metric and use the notation $e_{ij} = e_i \wedge e_j$, etc.

An easy explicit commutator calculation yields the exterior derivative of the left-invariant 1-forms e_i on U_3:

$$\begin{array}{rcl} de_1 &=& -2e_2 \wedge (h_1 - h_2) + e_{35} + e_{46}, \\ de_2 &=& 2e_1 \wedge (h_1 - h_2) + e_{45} - e_{36}, \\ de_3 &=& 2e_4 \wedge (h_3 - h_1) - e_{15} + e_{26}, \\ de_4 &=& -2e_3 \wedge (h_3 - h_1) - e_{25} - e_{16}, \\ de_5 &=& -2e_6 \wedge (h_2 - h_3) + e_{13} + e_{24}, \\ de_6 &=& 2e_5 \wedge (h_2 - h_3) + e_{14} - e_{23}. \end{array} \tag{37}$$

Let J denote the almost complex structure on $M = F(1,2)$ whose Kähler form is $\omega = e_{12} - e_{34} + e_{56}$ (It is easy to check that ω, which a priori is a left-invariant 2-form on U_3, projects to M because $L_{h_i}\omega = 0$). J induces an orientation on M with volume form $-e_{123456}$. Let $\Psi^+ + i\Psi^-$ denote the associated complex volume form on M defined by the ad_{T^3}-invariant form $(e_2 + iJe_2) \wedge (e_4 + iJe_4) \wedge (e_6 + iJe_6)$. Explicitly,

$$\Psi^+ = e_{136} + e_{246} + e_{235} - e_{145}, \qquad \Psi^- = e_{236} - e_{146} - e_{135} - e_{245}.$$

Using (37) we readily obtain

$$d(e_{12}) = -d(e_{34}) = d(e_{56}) = \Psi^+, \tag{38}$$

so

$$d\omega = 3\Psi^+, \qquad \text{and} \qquad d\Psi^- = -2\omega^2.$$

The pair (g, J) thus defines a nearly Kähler structure on M (a fact which we already knew).

We fix now an element $\xi \in \mathfrak{su}_3 \subset \mathfrak{u}_3$, and denote by X the *right-invariant* vector field on U_3 defined by ξ. Consider the functions

$$x_i = g(X, e_i), \qquad v_i = g(X, h_i). \tag{39}$$

The functions v_i are projectable to M and clearly $v_1 + v_2 + v_3 = 0$. Let us introduce the vector fields on U_3

$$a_1 = x_6 e_5 - x_5 e_6, \qquad a_2 = x_3 e_4 - x_4 e_3, \qquad a_3 = x_2 e_1 - x_1 e_2.$$

One can check that they project to M. Of course, one has

$$Ja_1 = x_5e_5 + x_6e_6, \qquad Ja_2 = x_3e_3 + x_4e_4, \qquad Ja_3 = x_1e_1 + x_2e_2.$$

The commutator relations in SU_3 yield

$$dv_1 = a_2 - a_3, \qquad dv_2 = a_3 - a_1, \qquad dv_3 = a_1 - a_2. \tag{40}$$

Using (37) and some straightforward computations we obtain

$$\begin{array}{ll} d(Ja_1) & = (-a_1 + a_2 + a_3) \lrcorner \Psi^+ + 4(v_2 - v_3)e_{56}, \\ d(Ja_2) & = (a_1 - a_2 + a_3) \lrcorner \Psi^+ + 4(v_1 - v_3)e_{34}, \\ d(Ja_3) & = (a_1 + a_2 - a_3) \lrcorner \Psi^+ + 4(v_1 - v_2)e_{12}. \end{array} \tag{41}$$

We claim that the 2-form

$$\varphi = v_1e_{56} - v_2e_{34} + v_3e_{12} \tag{42}$$

on M is of type (1,1), primitive, co-closed, and satisfies $\Delta\varphi = 12\varphi$. The first two assertions are obvious (recall that $v_1 + v_2 + v_3 = 0$). In order to prove that φ is co-closed, it is enough to prove that $d\varphi \wedge \omega = 0$. Using (38) and (40) we compute:

$$\begin{array}{rcl} d\varphi \wedge \omega & = & [(a_2 - a_3) \wedge e_{56} - (a_3 - a_1) \wedge e_{34} + (a_1 - a_2) \wedge e_{12}] \wedge (e_{12} - e_{34} + e_{56}) \\ & = & (a_1 - a_2) \wedge e_{1256} - (a_3 - a_2) \wedge e_{1234} + (a_1 - a_2) \wedge e_{3456} = 0. \end{array}$$

Finally, using (41), we get

$$\begin{array}{rcl} \Delta\varphi & = & d^*d\varphi = - * d * [(a_2 - a_3) \wedge e_{56} - (a_3 - a_1) \wedge e_{34} + (a_1 - a_2) \wedge e_{12}] \\ & = & - * d[Ja_2 \wedge e_{12} + Ja_3 \wedge e_{34} + Ja_3 \wedge e_{56} - Ja_1 \wedge e_{12} - Ja_1 \wedge e_{34} - Ja_2 \wedge e_{56}] \\ & = & - * [d(Ja_2) \wedge (e_{12} - e_{56}) + d(Ja_3) \wedge (e_{34} + e_{56}) - d(Ja_1) \wedge (e_{12} + e_{34})] \\ & = & - * [(a_1 + a_2 + a_3) \lrcorner \Psi^+ \wedge (e_{12} - e_{56} + e_{34} + e_{56} - e_{12} - e_{34}) \\ & & -2(a_2 \lrcorner \Psi^+) \wedge (e_{12} - e_{56}) - 2(a_3 \lrcorner \Psi^+) \wedge (e_{34} + e_{56}) + 2(a_1 \lrcorner \Psi^+) \wedge (e_{12} + e_{34}) \\ & & +4(v_1 - v_3)e_{34} \wedge (e_{12} - e_{56}) + 4(v_1 - v_2)e_{12} \wedge (e_{34} + e_{56}) \\ & & -4(v_2 - v_3)e_{56} \wedge (e_{12} + e_{34})] \\ & = & - * [4(2v_1 - v_2 - v_3)e_{1234} + 4(v_1 + v_3 - 2v_2)e_{1256} + 4(2v_3 - v_1 - v_2)e_{3456}] \\ & = & - * [12v_1e_{1234} - 12v_2e_{1256} + 12v_3e_{3456}] = 12\varphi. \end{array}$$

Taking into account the inequality (36), we deduce at once the following

Corollary 6.1. *The space of infinitesimal nearly Kähler deformations of the nearly Kähler structure on* $F(1,2)$ *is isomorphic to the Lie algebra of* SU_3. *More precisely, every right-invariant vector field* X *on* SU_3 *defines an element* $\varphi \in \mathcal{NK}$ *via the formulas* (39) *and* (42).

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