TWISTOR FORMS ON RIEMANNIAN PRODUCTS

ANDREI MOROIANU AND UWE SEMMELMANN

Abstract. We study twistor forms on products of compact Riemannian manifolds and show that they are defined by Killing forms on the factors. The result of this note is a necessary step in the classification of Riemannian manifolds with non–generic holonomy carrying twistor forms.

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1. Twistor Forms on Riemannian Manifolds

If (V, g) is a n–dimensional Euclidean vector space, the tensor product $V^* \otimes \Lambda^p V^*$ has the following $O_n(g)$–invariant decomposition:

$$V^* \otimes \Lambda^p V^* \;\cong\; \Lambda^{p-1} V^* \oplus \Lambda^{p+1} V^* \oplus \Lambda^{p,1} V^*$$

where $\Lambda^{p,1}V^*$ – the intersection of the kernels of the wedge and inner products – can be identified with the Cartan product of V^* and $\Lambda^p V^*$. This decomposition immediately translates to Riemannian manifolds (M^n, g):

$$T^*M \otimes \Lambda^p T^*M \;\cong\; \Lambda^{p-1}T^*M \oplus \Lambda^{p+1}T^*M \oplus \Lambda^{p,1}T^*M \tag{1}$$

where $\Lambda^{p,1}T^*M$ denotes the vector bundle corresponding to the vector space $\Lambda^{p,1}$. The covariant derivative $\nabla\psi$ of a p–form ψ is a section of $T^*M \otimes \Lambda^p T^*M$. Its projections onto the summands $\Lambda^{p+1}T^*M$ and $\Lambda^{p-1}T^*M$ are just the differential $d\psi$ and the codifferential $\delta\psi$. Its projection onto the third summand $\Lambda^{p,1}T^*M$ defines a natural first order differential operator T, called the *twistor operator*. The twistor operator $T : \Gamma(\Lambda^p T^*M) \to \Gamma(\Lambda^{p,1}T^*M) \subset \Gamma(T^*M \otimes \Lambda^p T^*M)$ is given for any vector field X by the following formula

$$[\,T\psi\,]\,(X) \;:=\; [\mathrm{pr}_{\Lambda^{p,1}}(\nabla\psi)]\,(X) \;=\; \nabla_X\,\psi \;-\; \tfrac{1}{p+1}\,X \lrcorner\, d\psi \;+\; \tfrac{1}{n-p+1}\,X \wedge \delta\psi\;.$$

Note that here, and in the remaining part of this note, we identify vectors and 1–forms using the metric.

The twistor operator T is a typical example of a so–called Stein–Weiss operator and it was in this context already considered by T. Branson in [1]. It particular it was shown that T^*T is elliptic, which easily follows from computing the principal symbol. Its definition is also similar to the definition of the twistor operator in spin geometry. The tensor product between the spinor bundle and the cotangent bundle decomposes under

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the action of the spinor group into the sum of the spinor bundle and the kernel of the Clifford multiplication. The (spinorial) twistor operator is then defined as the projection of the covariant derivative of a spinor onto the kernel of the Clifford multiplication.

Definition 1.1. *A p–form ψ is called a twistor p–form if and only if ψ is in the kernel of T, i.e. if and only if ψ satisfies*

$$\nabla_X \psi \;=\; \tfrac{1}{p+1} X \lrcorner\, d\psi - \tfrac{1}{n-p+1} X \wedge \delta\psi, \tag{2}$$

for all vector fields X. If the p–form ψ is in addition coclosed, it is called a Killing p–form. This is equivalent to $\nabla\psi \in \Gamma(\Lambda^{p+1}T^*M)$ or to $X \lrcorner \nabla_X\psi = 0$ for any vector field X.

It follows directly from the definition that the Hodge star-operator $*$ maps twistor p–forms into twistor $(n-p)$–forms. In particular, it interchanges closed and coclosed twistor forms. For an introduction to twistor forms, see [2]

2. The Main Result

Let $X = M \times N$ be the Riemannian product of two compact Riemannian manifolds M and N of dimensions m and n respectively. The aim of this note is to prove the following result:

Theorem 2.1. *Every twistor form on X is a sum of forms of the following types: parallel forms, pull–backs of Killing forms on M or N, and wedge products of the volume form of M (or N) with the pull–back of a closed twistor form on N (resp. M).*

Proof. We define the differential operators

$$d^M = \sum_{i=1}^{m} e_i \wedge \nabla_{e_i}, \qquad\qquad d^N = \sum_{j=1}^{n} f_j \wedge \nabla_{f_j},$$

where $\{e_i\}$ and $\{f_j\}$ denote local orthonormal basis of the tangent distributions to M and N. Using Fubini's theorem, we easily see that the adjoint operators to d^M and d^N are

$$\delta^M = -\sum_{i=1}^{m} e_i \lrcorner \nabla_{e_i}, \qquad\qquad \delta^N = -\sum_{j=1}^{n} f_j \lrcorner \nabla_{f_j}.$$

The following relations are straightforward:

$$d = d^M + d^N, \qquad \delta = \delta^M + \delta^N, \qquad (d^M)^2 = (d^N)^2 = (\delta^M)^2 = (\delta^N)^2 = 0,$$

$$0 = d^M d^N + d^N d^M = \delta^M \delta^N + \delta^N \delta^M, \qquad 0 = d^M \delta^N + \delta^N d^M = \delta^M d^N + d^N \delta^M.$$

The vector bundle $\Lambda^p X$ decomposes naturally as

$$\Lambda^p X \cong \oplus_{i=0}^{p} \Lambda^{i,p-i} X,$$

where $\Lambda^{i,p-i}X \cong \Lambda^i M \otimes \Lambda^{p-i}N$. Obviously, d^M and δ^M map $\Lambda^{i,p-i}X$ to $\Lambda^{i+1,p-i}X$ and $\Lambda^{i-1,p-i}X$ respectively, and d^N and δ^N map $\Lambda^{i,p-i}X$ to $\Lambda^{i,p-i+1}X$ and $\Lambda^{i,p-i-1}X$ respectively.

With respect to the above decomposition, every p–form can be written $u = u_0 + \ldots u_p$, where $u_i \in \Lambda^i M \otimes \Lambda^{p-i}N$. For the remaining of this paper, u will denote a twistor p–form on X, where $1 \leq p \leq n+m-1$. The twistor equation reads

$$\nabla_X u = \frac{1}{p+1} X \lrcorner (d^M u + d^N u) - \frac{1}{m+n-p+1} X \wedge (\delta^M u + \delta^N u), \quad \forall X. \tag{3}$$

By projection onto the different irreducible components of $\Lambda^p X$, (3) can be translated into the following two systems of equations:

$$\nabla_X u_k = \frac{1}{p+1} X \lrcorner (d^M u_k + d^N u_{k+1}) - \frac{1}{m+n-p+1} X \wedge (\delta^M u_k + \delta^N u_{k-1}), \quad \forall X \in TM, \tag{4}$$

and

$$\nabla_X u_k = \frac{1}{p+1} X \lrcorner (d^M u_{k-1} + d^N u_k) - \frac{1}{m+n-p+1} X \wedge (\delta^M u_{k+1} + \delta^N u_k), \quad \forall X \in TN. \tag{5}$$

Taking the wedge product with X in (4) and summing over an orthonormal basis of TM yields $d^M u_k = \frac{k+1}{p+1}(d^M u_k + d^N u_{k+1})$, so

$$(p-k) d^M u_k = (k+1) d^N u_{k+1}, \tag{6}$$

and similarly, taking the wedge product with X and summing over an orthonormal basis of TM yields $\delta^M u_k = \frac{m-k+1}{m+n-p+1}(\delta^M u_k + \delta^N u_{k-1})$, thus

$$(n+k-p) \delta^M u_k = (m-k+1) \delta^N u_{k-1}. \tag{7}$$

Suppose first that p is strictly smaller than m and n. For $k < p$, (6) and (7) imply

$$\delta^M d^M u_k = \frac{k+1}{p-k} \delta^M d^N u_{k+1} = -\frac{k+1}{p-k} d^N \delta^M u_{k+1} = -\frac{(k+1)(m-k)}{(p-k)(n+k-p+1)} d^N \delta^N u_k. \tag{8}$$

Integrating over X yields $0 = d^M u_k = \delta^N u_k$, $\forall k < p$. Similarly one gets $0 = d^N u_k = \delta^M u_k$, $\forall k > 0$. Moreover, we have $0 = \delta^N u_p = \delta^M u_0$ (tautologically), so in particular $\delta^M u_k = \delta^N u_k = 0$, $\forall k$. From (4) and (5) we see that $u_1, \ldots u_{p-1}$ are parallel forms on X, and u_0 and u_p are pull-backs of Killing forms on N and M respectively. In other words u can be written in the following form:

$$u = u_M + u_N + \text{parallel form},$$

where u_M and u_N are Killing forms on M and N respectively.

If p is strictly larger than m and n, then the argument above applied to the Hodge dual $*u$ shows that u is equal to a sum

$$u = u_M \wedge vol_N + u_N \wedge vol_M + \text{parallel form},$$

where u_M and u_N are closed twistor forms on M and N and vol_M and vol_N denote the volume forms of M and N respectively.

It remains to treat the case where p is a number between m and n, and we suppose without loss of generality that $m \leq p \leq n$. Obviously $u_{m+1} = \ldots u_p$. Now, for every $k \leq m-1$, we obtain as before, using (8) and integrating over X, that $0 = d^M u_k = \delta^N u_k$, $\forall k \leq m-1$ and similarly, $0 = d^N u_k = \delta^M u_k$, $\;1 \leq k \leq m$. As before, (4) and (5) show that $u_1, \ldots u_{p-1}$ are parallel forms on X, u_0 is the pull–back of a killing form on N, and u_m can be written as $vol_M \wedge u_N$, where u_N is a closed twistor form on N. This proves the theorem.

$\square$

References

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Andrei Moroianu, CMAT, École Polytechnique, UMR 7640 du CNRS, 91128 Palaiseau, France

E-mail address: am@math.polytechnique.fr

Uwe Semmelmann, Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, D-20146 Hamburg, Germany

E-mail address: Uwe.Semmelmann@math.uni-hamburg.de