Eigenvalue Estimates for the Dirac Operator on Quaternionic Kähler Manifolds

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March 27, 1997

Abstract

We consider the Dirac operator on compact quaternionic Kähler manifolds and prove a lower bound for the spectrum. This estimate is sharp since it is the first eigenvalue of the Dirac operator on the quaternionic projective space.

Contents

1 Introduction

On a compact Riemannian spin manifold (M^n, g) with positive scalar curvature κ the eigenvalues of the Dirac operator D satisfy

$$\lambda^2 \geq \frac{n}{n-1}\,\frac{\kappa_0}{4}\,,$$

*Supported by the SFB 256 'Nichtlineare partielle Differentialgleichungen'

†Supported by the Graduiertenkolleg 'Algebraische, analytische und geometrische Methoden und ihre Wechselwirkungen in der modernen Mathematik'

‡Supported by the SFB 256 'Nichtlineare partielle Differentialgleichungen'

where κ_0 is the minimum of the scalar curvature. This estimate has been proven by Th. Friedrich (c. f. [Fri80]). This lower bound is sharp, since it is attained as the first eigenvalue on the sphere. A theorem of O. Hijazi and A. Lichnerowicz ([Hij86], [Lic87]) implies that the case of equality cannot be attained on manifolds with non–trivial parallel k–forms with $k \neq 0, n$. There are two canonical classes of such manifolds which in addition have positive scalar curvature: Kähler manifolds and quaternionic Kähler manifolds. The eigenvalue estimate for manifolds of the first class has been improved by K.–D. Kirchberg in [Kir86] and [Kir90] (see also [Lic90] and [Hij94]). Again, this estimate is sharp: the lower bound is equal to the first eigenvalue on complex projective space in odd complex dimensions and on the product of $\mathbb{P}^{2m+1}(\mathbb{C})$ with the flat 2–torus in even complex dimensions.

The proof of the corresponding result for the second class of manifolds is aim of the present article. A quaternionic Kähler manifold is by definition an oriented 4n–dimensional Riemannian manifold with $n \geq 2$ whose holonomy group is contained in the subgroup $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1) \subset \mathbf{SO}(4n)$. Equivalently they are characterized by the existence of a certain parallel 4–form Ω, the so–called fundamental or Kraines form (c. f. [Bon67], [Kra66]). In even quaternionic dimensions n any quaternionic Kähler manifold possesses a spin structure, whereas for odd n only the quaternionic projective space is spin (c. f. [Sal82]). In this article we prove the following theorem:

Theorem 1.1 *Let (M^{4n}, g) be a compact quaternionic Kähler spin manifold of positive scalar curvature κ. Then any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n+3}{n+2}\,\frac{\kappa}{4}\,.$$

We note that κ is indeed a constant, since any quaternionic Kähler manifold is automatically Einstein. This was first shown by D. V. Alekseevskii in [Ale68-1] and [Ale68-2] (see also [Ish74] and Lemma 3.5).

The spectrum of the Dirac operator D on the quaternionic projective space has been computed by J.–L. Milhorat in [Mil92]. The lower bound $\frac{n+3}{n+2}\,\frac{\kappa}{4}$ turns out to be the first eigenvalue of D^2. Hence, the stated estimate is sharp.

This lower bound was first conjectured by O. Hijazi and J. –L. Milhorat in [HiM95]. They gave first eigenvalue estimates and proved the conjecture for quaternionic dimension $n = 2$ and $n = 3$ (see also [Hij96]). Their approach was to define quaternionic Kähler twistor operators and to use Weitzenböck formulas to prove inequalities for the eigenvalues. We will follow a similar approach by using representation theory of the group $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$ to define natural twistor operators. The formalism we use systematically produces relations between differential operators of second order, showing in particular that the Lichnerowicz–Weitzenböck formula alone is not sufficient to obtain the optimal estimate.

In fact, we obtain a somewhat more general result. The spinor bundle of a quaternionic Kähler manifold M^{4n} splits into a sum of n subbundles and this decomposition is respected by the square of the Dirac operator. We prove that the spectrum of D^2 on any subbundle is bounded from below by the first eigenvalue of D^2 on the corresponding bundle on the quaternionic projective space.

The second author would like to thank D. V. Alekseevskii for many interesting and helpful discussions. All of us would like to thank C. Bär and W. Ballmann for encouragement and support.

2 The Dirac Operator of a Quaternionic Kähler Manifold

2.1 Preliminaries on $\mathbf{Sp}(n)$–Representations

Let (M^{4n}, g) be a quaternionic Kähler manifold. The holonomy group $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1) \subset \mathbf{SO}(4n)$ reduces the $\mathbf{SO}(4n)$–bundle of orthonormal frames to a principal $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$–bundle P, and the Levi–Civita connection on M can be thought of to be given by a 1-form on P. Any representation V of $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ locally gives a vector bundle $\mathbf{V}$ associated to P, which in addition does exist globally provided the representation factors through $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$.

Let H and E be the defining complex representations of $\mathbf{Sp}(1)$ and $\mathbf{Sp}(n)$ with their invariant symplectic forms $\sigma_H \in \Lambda^2 H^*$ and $\sigma_E \in \Lambda^2 E^*$ and their compatible positive quaternionic structures J, e. g.

$$\begin{aligned} J^2 &= -1 \\ \sigma_E(Je_1, Je_2) &= \overline{\sigma_E(e_1, e_2)} \\ \sigma_E(e, Je) &> 0 \quad \text{for } e \neq 0 \ . \end{aligned}$$

The symplectic form σ_E defines an isomorphism $\sharp : E \to E^*$, $e \mapsto e^\sharp := \sigma_E(e, \,.\,)$ with inverse $\flat : E^* \to E$. Using either Gram's determinant or permanent the symplectic form σ_E can be extended to $\Lambda^s E$ or $\mathrm{Sym}^r E$. As an immediate consequence we obtain

$$\sigma_E(e \wedge \eta_1,\, \eta_2) \;=\; \sigma_E(\eta_1,\, e^\sharp \lrcorner\, \eta_2) \quad \eta_1, \eta_2 \in \Lambda^s E \text{ resp. } \mathrm{Sym}^r E\,.$$

The usual formula $\sigma_E(\,.\,, J.\,)$ then defines positive definite hermitian forms, such that e. g. $(e\wedge)^* = (Je)^\sharp \lrcorner$. *Mutatis mutandis* analogous statements are true for H.

Let $\{e_i\}$ and $\{de_i\}$ with $de_i(e_j) = \delta_{ij}$ be a dual pair of bases for E, E^* respectively. In terms of this bases the symplectic form and its — by $\Lambda^2 E \cong \Lambda^2 E^*$ associated — canonical bivector are given by

$$\sigma_E \;=\; \tfrac{1}{2} \sum de_i \wedge e_i^\sharp \in \Lambda^2 E^* \qquad L_E = \tfrac{1}{2} \sum de_i^\flat \wedge e_i \in \Lambda^2 E\,.$$

Wedging with L_E determines a homomorphism $L : \Lambda^{k-2} E \longrightarrow \Lambda^k E$ whereas contracting with σ_E defines its adjoint $\Lambda := L^* : \Lambda^k E \longrightarrow \Lambda^{k-2} E$. The operators L, Λ and $H := [\Lambda, L]$ fulfil the commutator rules of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ and, in addition, $H|_{\Lambda^k E} = (n-k)\mathrm{id}$. Therefore, $\Lambda^k E = \mathrm{im}(L) \oplus \ker(\Lambda)$ splits as $\mathbf{Sp}(n)$–representation and $\Lambda^k_\circ E := \ker(\Lambda)$, the *primitive* space, turns out to be irreducible. Hence, the following decomposition is immediate:

$$\Lambda^q E \;=\; \bigoplus_{k=0}^{[\frac{q}{2}]} \Lambda_\circ^{q-2k} E, \qquad 0 \le q \le n\,.$$

The primitive space is stable under contraction with elements of E^* but it is not preserved by the wedge product. Therefore it is necessary to describe the projection $e \wedge_\circ \omega$ of $e \wedge \omega$ onto $\Lambda^q_\circ E$.

Lemma 2.1 *If $\omega \in \Lambda^*_\circ E$ then $e^\sharp \lrcorner\, \omega \in \Lambda^*_\circ E$. Furthermore*

$$\begin{aligned} \Lambda(e \wedge \omega) &= e^\sharp \lrcorner\, \omega \\ e \wedge_\circ \omega &= e \wedge \omega - \tfrac{1}{n-k+1} L_E \wedge (e^\sharp \lrcorner\, \omega)\,. \end{aligned}$$

To summarize properties of contraction and modified exterior multiplication we have the following lemma:

Lemma 2.2 *Let $\eta, \eta_i \in E^*$ and $e, e_i \in E$. Then following relations are valid on $\Lambda^s_\circ E$:*

$$\begin{aligned} \{\eta_1, \eta_2\} &= 0 & \{\eta \lrcorner\,, e \wedge_\circ\} &= \eta(e) + \tfrac{1}{n-s+1} \eta^\flat \wedge_\circ e^\sharp \lrcorner \\ \{e_1 \wedge_\circ, e_2 \wedge_\circ\} &= 0 & \textstyle\sum de_i \lrcorner\, e_i \wedge_\circ &= (2n-s+2) \tfrac{n-s}{n-s+1}\,\mathrm{id} \\ \textstyle\sum e_i \wedge_\circ de_i \lrcorner &= s\,\mathrm{id}\ . \end{aligned}$$

On H, there are similar equations which relate contraction and symmetric product. It is convenient to modify the contraction on $\mathrm{Sym}^r H$. For $\alpha \in H^*$, we define $\alpha \lrcorner_\circ : \mathrm{Sym}^r H \to \mathrm{Sym}^{r-1} H$ by $\alpha \lrcorner_\circ := \frac{1}{r}\, \alpha \lrcorner$. Let $h\cdot$ denote the symmetric product with $h \in H$.

Lemma 2.3 *Let $h, h_i \in H$ and $\alpha, \alpha_i \in H^*$ Then the following relations are valid on* $\mathrm{Sym}^r H$*:*

$$\begin{aligned} [h_1\cdot, h_2\cdot] &= 0 & [\alpha \lrcorner_\circ, h\cdot] &= -\tfrac{1}{r+1} \alpha^\flat \cdot h^\sharp \lrcorner_\circ \\ [\alpha_1 \lrcorner_\circ, \alpha_2 \lrcorner_\circ] &= 0 & \alpha(h)\mathrm{id} &= h \cdot \alpha \lrcorner_\circ - \alpha^\flat \cdot h^\sharp \lrcorner_\circ \\ \textstyle\sum h_i \cdot dh_i \lrcorner_\circ &= \mathrm{id} & \textstyle\sum dh_i \lrcorner_\circ h_i \cdot &= \tfrac{r+2}{r+1}\,\mathrm{id}\ . \end{aligned}$$

It follows from the Peter–Weyl theorem that any irreducible $\mathbf{Sp}(n)\times\mathbf{Sp}(1)$–module can be realized as a subspace of $H^{\otimes p}\otimes E^{\otimes q}$ for some p and q. The representations of $\mathbf{Sp}(n)\cdot\mathbf{Sp}(1)$ are precisely those with $p+q$ even. Hence, any vector bundle on M associated to P can be expressed in terms of the local bundles $\mathbf{H}$ and $\mathbf{E}$. All structures considered above carry over to the fibres of the associated vector bundles. For example, the complexified tangent bundle is defined by the representation $H\otimes E$, and we fix the identification

$$TM^{\mathbb{C}} \;=\; \mathbf{H}\otimes\mathbf{E}\,.$$

Note that the real structure is simply $\overline{h\otimes e}:=Jh\otimes Je$. Finally, the Riemannian metric on $TM^{\mathbb{C}}$ is given by

$$g_M \;=\; \sigma_H\otimes\sigma_E\,.$$

2.2 Spinor Bundle and Clifford Multiplication

The aim of this section is to give an explicit formula for the Clifford multiplication using the H–E–formalism. The spinor module considered as $\mathbf{Sp}(n)\times\mathbf{Sp}(1)$–representation splits into a sum of n irreducible components. Hence, the spinor bundle of a $4n$–dimensional quaternionic Kähler manifold decomposes into a sum of n subbundles which can be expressed using the locally defined bundles $\mathbf{E}$ and $\mathbf{H}$. Likewise, this decomposition can be defined by considering the Kraines form Ω as endomorphism of the spinor bundle acting via Clifford multiplication.

Proposition 2.1 *The spinor bundle $\mathbf{S}(M)$ of a quaternionic Kähler manifold M decomposes as*

$$\mathbf{S}(M) \;=\; \bigoplus_{r=0}^{n}\mathbf{S}_r(M) \;=\; \bigoplus_{r=0}^{n}\mathrm{Sym}^r\mathbf{H}\otimes\Lambda_\circ^{n-r}\mathbf{E}\,,$$

where each fibre of $\mathbf{S}_r(M)$ is an eigenspace of Ω for the eigenvalue

$$\mu_r = 6n-4r(r+2)\,.$$

The rank of the subbundle $\mathbf{S}_r(M)$ is given by

$$\mathrm{rank}\,(\mathbf{S}_r(M)) = (r+1)\left(\binom{2n}{n-r}-\binom{2n}{n-r-2}\right).$$

The covariant derivative on $\mathbf{S}(M)$ induced by the Levi–Civita connection on (M,g) respects the decomposition given above. For further proceeding it is necessary to express the Clifford multiplication on spinors in terms of the H–E–formalism.

Proposition 2.2 *For any tangent vector $h\otimes e\in\mathbf{H}\otimes\mathbf{E}=TM^{\mathbb{C}}$, the Clifford multiplication $\mu(h\otimes e):\mathbf{S}(M)\to\mathbf{S}(M)$ is given by:*

$$\mu(h\otimes e) \quad=\quad \sqrt{2}\,(h\cdot\otimes e^\sharp\lrcorner \;+\; h^\sharp\lrcorner_\circ\otimes e\wedge_\circ)\,.$$

In particular, the Clifford multiplication maps the subbundle $\mathbf{S}_r(M)$ to the sum $\mathbf{S}_{r-1}(M)\oplus\mathbf{S}_{r+1}(M)$.

Proof. To check whether the above formula indeed defines the Clifford multiplication it suffices to verify the general relation:

$$\mu(X)\circ\mu(Y) \;+\; \mu(Y)\circ\mu(X) \quad=\quad -\,2\,g(X,Y)\,. \tag{2.1}$$

We prove this for vectors $X=h_1\otimes e_1$ and $Y=h_2\otimes e_2$.

$$\{h_1^\sharp\lrcorner_\circ\otimes e_1\wedge_\circ,\, h_2\cdot\otimes e_2^\sharp\lrcorner\} \;=\; [h_1^\sharp\lrcorner_\circ,\, h_2\cdot]\otimes e_1\wedge_\circ e_2^\sharp\lrcorner \;+\; h_2\cdot h_1^\sharp\lrcorner_\circ\otimes\{e_1\wedge_\circ,\, e_2^\sharp\lrcorner\}$$

$$=\; -\tfrac{1}{r+1}\,h_1\cdot h_2^\sharp\lrcorner_\circ\otimes e_1\wedge_\circ e_2^\sharp\lrcorner \;+\; h_2\cdot h_1^\sharp\lrcorner_\circ\otimes(\sigma_E(e_2,e_1)\;+\;\tfrac{1}{n-s+1}\,e_2\wedge_\circ e_1^\sharp\lrcorner)$$

$$=\; -\,h_2\cdot h_1^\sharp\lrcorner_\circ\otimes\sigma_E(e_1,e_2)\;-\;\tfrac{1}{r+1}\,h_1\cdot h_2^\sharp\lrcorner_\circ\otimes e_1\wedge_\circ e_2^\sharp\lrcorner\;+\;\tfrac{1}{n-s+1}\,h_2\cdot h_1^\sharp\lrcorner_\circ\, e_2\wedge_\circ e_1^\sharp\lrcorner$$

If we symmetrize over (1,2) and substitute $s = n - r$ we obtain

$$\begin{aligned}\{\mu(h_1 \otimes e_1),\, \mu(h_2 \otimes e_2)\} &= -2\,(h_2 \cdot h_1^\sharp \lrcorner_\circ \;-\; h_1 \cdot h_2^\sharp \lrcorner_\circ\,) \otimes \sigma_E(e_1,\, e_2)\\ &= -2\,\sigma_H(h_1,\, h_2)\,\sigma_E(e_1,\, e_2)\\ &= -2\,g(h_1 \otimes e_1,\, h_2 \otimes e_2)\,.\end{aligned}$$

The second assertion is clear from the definition. □

Thus, Clifford multiplication splits into two components:

$$\mu^+_- :\quad TM \otimes \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r+1}(M) \qquad \text{and} \qquad \mu^-_+ :\quad TM \otimes \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r-1}(M)\,, \tag{2.2}$$

where $\mu^+_-(e \otimes h \otimes \psi) = \sqrt{2}\,(h \cdot \otimes e \lrcorner\,)\,\psi$ and $\mu^-_+(e \otimes h \otimes \psi) = \sqrt{2}\,(h \lrcorner_\circ \otimes e \wedge_\circ)\,\psi$. We note that this definition makes sense also for $\mathbf{S}_r(M)$ replaced by $\mathrm{Sym}^p\mathbf{H} \otimes \Lambda^q_\circ\mathbf{E}$. In this spirit it is possible to define two operations similar to Clifford multiplication:

$$\begin{aligned}\mu^+_+ : TM \otimes \mathrm{Sym}^p\mathbf{H} \otimes \Lambda^q_\circ\mathbf{E} &\longrightarrow \mathrm{Sym}^{p+1}\mathbf{H} \otimes \Lambda^{q+1}_\circ\mathbf{E}\\ h \otimes e \otimes \psi &\longmapsto \sqrt{2}\,(h \cdot \otimes e \wedge_\circ)\psi\end{aligned}$$

and

$$\begin{aligned}\mu^-_- : TM \otimes \mathrm{Sym}^p\mathbf{H} \otimes \Lambda^q_\circ\mathbf{E} &\longrightarrow \mathrm{Sym}^{p-1}\mathbf{H} \otimes \Lambda^{q-1}_\circ\mathbf{E}\\ h \otimes e \otimes \psi &\longmapsto \sqrt{2}\,(h \lrcorner_\circ \otimes e \lrcorner\,)\psi\,.\end{aligned}$$

As a first application of Lemma 2.2 we will give a new interpretation for the Kraines form acting on the subbundles $\mathbf{S}_r(M)$ of the spinor bundle. For this we have to investigate the Clifford multiplication with 2–forms. On an arbitrary spin manifold it is defined by:

$$\mu(X \wedge Y) = \mu(X)\mu(Y) + g(X,\, Y)\,.$$

The space of 2–forms of a quaternionic Kähler manifold split into the following components:

$$\Lambda^2 TM \;\cong\; \mathrm{Sym}^2\mathbf{H} \oplus \mathrm{Sym}^2\mathbf{E} \oplus (\mathrm{Sym}^2\mathbf{H} \otimes \Lambda^2_\circ\mathbf{E})\,,$$

where $\mathrm{Sym}^2\mathbf{H}$ resp. $\mathrm{Sym}^2\mathbf{E}$ is realized in $\Lambda^2 TM$ as $\mathrm{Sym}^2\mathbf{H} \otimes \sigma_E$ resp. $\sigma_H \otimes \mathrm{Sym}^2\mathbf{E}$.

Proposition 2.3 *Let A be any element in $\mathrm{Sym}^2\mathbf{H} \subset \Lambda^2 TM$. Then Clifford multiplication with A on $\mathbf{S}_r(M) = \mathrm{Sym}^r\mathbf{H} \otimes \Lambda^{n-r}_\circ\mathbf{E}$, $(r \geq 1)$ is given by*

$$\mu(A) = 2\,A \otimes \mathrm{id}\ ,$$

where A acts as an element of $\mathrm{Sym}^2 H \cong \mathfrak{sp}(1)$ on $\mathrm{Sym}^r\mathbf{H}$. In particular, Clifford multiplication with the Kraines form Ω on $\mathbf{S}_r(M)$ corresponds to the action of the Casimir operator $\mathbf{C}$ of $\mathfrak{sp}(1)$, i. e.

$$\mu(\Omega) = 6n\,\mathrm{id} \;+\; 4\,\mathbf{C} \otimes \mathrm{id} \;=\; (6n \,-\, 4r(r+2))\mathrm{id}\ .$$

The next step is to introduce a scalar product on the bundles $\mathrm{Sym}^p\mathbf{H} \otimes \Lambda^q_\circ\mathbf{E}$ which for $q = n - p$ corresponds to the usual Hermitian scalar product on the subbundle $\mathbf{S}_p(M)$ of the spinor bundle. We have already seen that the symplectic forms σ_H and σ_E extend to positive definite Hermitean forms on $\mathrm{Sym}^r H$ and $\Lambda^s_\circ E$. Using this we define on the tensor product $\mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E}$ the following twisted Hermitean form

$$\langle A_1 \otimes \omega_1,\, A_2 \otimes \omega_2\rangle \quad := \quad \tfrac{1}{r!}\,\sigma_H(A_1,\, JA_2)\,\sigma_E(\omega_1,\, J\omega_2)\,.$$

The next lemma is an immediate consequence of this definition.

Lemma 2.4

$$\langle \mu^+_-(X)\psi_1,\, \psi_2\rangle \quad = \quad -\,\langle \psi_1,\, \mu^-_+(\bar{X})\psi_2\rangle \tag{2.3}$$

$$\langle \mu^+_+(X)\psi_1,\, \psi_2\rangle \quad = \quad \langle \psi_1,\, \mu^-_-(\bar{X})\psi_2\rangle\,. \tag{2.4}$$

2.3 Dirac and Twistor Operators

The aim of this section is to describe the decompositon of $TM \otimes \mathbf{S}_r(M)$ and to introduce the corresponding Dirac and twistor operators. Using the Clebsch–Gordan formulas and similar formulas for the fundamental representations of $\mathbf{Sp}(n)$ the tensor product decomposes for $r \geq 1$ as follows:

$$\begin{aligned}
TM \otimes \mathbf{S}_r(M) &\cong (\mathbf{H}\otimes\mathbf{E}) \otimes (\mathrm{Sym}^r\mathbf{H}\otimes\Lambda^{n-r}_\circ\mathbf{E}) \\
&\cong (\mathrm{Sym}^{r+1}\mathbf{H}\otimes\Lambda^{n-r-1}_\circ\mathbf{E}) \,\oplus\, (\mathrm{Sym}^{r-1}\mathbf{H}\otimes\Lambda^{n-r+1}_\circ\mathbf{E}) \\
&\quad \oplus (\mathrm{Sym}^{r+1}\mathbf{H}\otimes\Lambda^{n-r+1}_\circ\mathbf{E}) \,\oplus\, (\mathrm{Sym}^{r-1}\mathbf{H}\otimes\Lambda^{n-r-1}_\circ\mathbf{E}) \\
&\quad \oplus (\mathrm{Sym}^{r+1}\mathbf{H}\otimes K^{n-r}) \,\oplus\, (\mathrm{Sym}^{r-1}\mathbf{H}\otimes K^{n-r}) \\
&\cong \mathbf{S}_{r+1}(M) \,\oplus\, \mathbf{S}_{r-1}(M) \,\oplus\, (\, S^+_r \,\oplus\, S^-_r \,\oplus\, V^+_r \,\oplus\, V^-_r \,)\,.
\end{aligned} \tag{2.5}$$

Here, K^{n-r} is the summand corresponding to the sum of the highest weights in the decomposition of $E\otimes\Lambda^{n-r}_\circ E$. Moreover, we introduced the notation $S^\pm_r = \mathrm{Sym}^{r\pm1}\mathbf{H}\otimes\Lambda^{n-r\pm1}_\circ\mathbf{E}$ and $V^\pm_r = \mathrm{Sym}^{r\pm1}\mathbf{H}\otimes K^{n-r}$. In the case $r=0$ and $r=n$ four of the above summands vanish and we obtain:

$$(\mathbf{E}\otimes\mathbf{H}) \otimes \Lambda^n_\circ\mathbf{E} \cong \mathbf{S}_1(M) \oplus V^+_0 \qquad \text{and} \qquad (\mathbf{E}\otimes\mathbf{H}) \otimes \mathrm{Sym}^n\mathbf{H} \cong \mathbf{S}_{n-1}(M) \oplus S^+_n\,. \tag{2.6}$$

The two components of the Clifford multiplication define natural projections onto the first two summands appearing in the splitting (2.5). The remaining four summands constitute the kernel of the Clifford multiplication. The projections onto S^+_r resp. S^-_r are given by μ^+_+ resp. μ^-_- defined in the preceeding paragraph and, finally, the projectors onto $V^\pm$ will be denoted by $pr_{V^\pm}$. By applying these projectors on the section $\nabla\psi \in \Gamma(TM\otimes\mathbf{S}(M))$ we get the two components of the Dirac operator

$$D^+_- := \mu^+_- \circ \nabla : \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r+1}(M) \qquad D^-_+ := \mu^-_+ \circ \nabla : \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r-1}(M)\,, \tag{2.7}$$

where $D = D^+_- + D^-_+$ is the full Dirac operator, and four twistor operators:

$$\begin{aligned}
D^+_+ &:= \mu^+_+ \circ \nabla : \mathbf{S}_r(M) \longrightarrow S^+_r & D^-_- &:= \mu^-_- \circ \nabla : \mathbf{S}_r(M) \longrightarrow S^-_r \\
T^+ &:= pr_{V^+} \circ \nabla : \mathbf{S}_r(M) \longrightarrow V^+ & T^- &:= pr_{V^-} \circ \nabla : \mathbf{S}_r(M) \longrightarrow V^-\,.
\end{aligned} \tag{2.8}$$

The square of the Dirac operator respects the splitting of the spinor bundle, i. e. $D^2 : \mathbf{S}_r(M) \longrightarrow \mathbf{S}_r(M)$. In particular, we have: $D^+_- D^+_- = 0 = D^-_+ D^-_+$. The adjoint operators are easily computed if one remembers the scalar product introduced in the previous paragraph.

Lemma 2.5

$$(D^+_-)^* = D^-_+\,, \qquad (D^-_+)^* = D^+_-\,, \qquad (D^+_+)^* = -D^-_-\,, \qquad (D^-_-)^* = -D^+_+\,.$$

The proof is an easy consequence of Lemma 2.4.

3 The Curvature Tensor on Quaternionic Kähler Manifolds

3.1 The Linear Space of Curvature Tensors

The exterior algebra ΛV^* of a vector space V defines a natural multiplication $m : \mathrm{Sym}^2\Lambda^2V^* \to \Lambda^4V^*$. Its adjoint is usually called comultiplication and is given by

$$\Delta : \Lambda^4V^* \to \mathrm{Sym}^2\Lambda^2V^*, \quad x\wedge y\wedge z\wedge w \mapsto (x\wedge y)(z\wedge w) + (y\wedge z)(x\wedge w) + (z\wedge x)(y\wedge w)\,.$$

The curvature tensor of a Riemannian manifold M satisfies Bianchi's first identity, i.e. it has to vanish on the image of Δ. Thus it is straightforward to call $\ker m$ the space of curvature tensors on a vector space V. It is easier, however, to adopt the following equivalent

Definition. The space of curvature tensors $\operatorname{Curv} V^*$ on V is the subspace

$$\operatorname{Curv} V^* := \operatorname{span}\{(\alpha\cdot\beta)\times(\gamma\cdot\delta) := (\alpha\wedge\gamma)(\beta\wedge\delta) + (\alpha\wedge\delta)(\beta\wedge\gamma) \text{ with } \alpha,\beta,\gamma,\delta\in V^*\}$$

of $\operatorname{Sym}^2\Lambda^2 V^*$. Note that the generators trivially satisfy the 'dual' Bianchi identity

$$(\alpha\cdot\beta)\times(\gamma\cdot\delta) + (\alpha\cdot\gamma)\times(\delta\cdot\beta) + (\alpha\cdot\delta)\times(\beta\cdot\gamma) = 0$$

and its variants using the apparent symmetries of $(\alpha\cdot\beta)\times(\gamma\cdot\delta)$.

Lemma 3.1 *There exists a natural isomorphism of vector spaces*

$$\begin{array}{ccc} \operatorname{Sym}^2\Lambda^2 V^* & \xrightarrow{\cong} & \operatorname{Curv} V^* \oplus \Lambda^4 V^* \\ (\alpha\wedge\beta)(\gamma\wedge\delta) & \longmapsto & \tfrac{1}{3}\big((\alpha\cdot\gamma)\times(\beta\cdot\delta) - (\alpha\cdot\delta)\times(\beta\cdot\gamma)\big) \;\oplus\; \tfrac{1}{3}\alpha\wedge\beta\wedge\gamma\wedge\delta\,. \end{array}$$

In the same vein

$$\begin{array}{ccc} \operatorname{Sym}^2\operatorname{Sym}^2 V^* & \xrightarrow{\cong} & \operatorname{Curv} V^* \oplus \operatorname{Sym}^4 V^* \\ (\alpha\cdot\beta)(\gamma\cdot\delta) & \longmapsto & \tfrac{1}{3}(\alpha\cdot\beta)\times(\gamma\cdot\delta) \;\oplus\; \tfrac{1}{3}\alpha\cdot\beta\cdot\gamma\cdot\delta\,. \end{array}$$

This isomorphism reveals an alternative realisation of $\operatorname{Curv} V^*$ *as the subspace of sectional curvature tensors in* $\operatorname{Sym}^2\operatorname{Sym}^2 V^*$.

Proof. It is easy to write down an inverse of the first homomorphism using the explicit formula

$$(\alpha\wedge\beta)(\gamma\wedge\delta) = \frac{1}{3}((\alpha\cdot\gamma)\times(\beta\cdot\delta) - (\alpha\cdot\delta)\times(\beta\cdot\gamma)) + \Delta(\frac{1}{3}\alpha\wedge\beta\wedge\gamma\wedge\delta)\,.$$

The inverse of the second homomorphism needs the analogues of Δ and $\times$:

$$\begin{array}{rccc} \Delta: & \operatorname{Sym}^4 V & \longrightarrow & \operatorname{Sym}^2\operatorname{Sym}^2 V \\ & \alpha\cdot\beta\cdot\gamma\cdot\delta & \longmapsto & (\alpha\cdot\beta)(\gamma\cdot\delta) + (\alpha\cdot\gamma)(\delta\cdot\beta) + (\alpha\cdot\delta)(\beta\cdot\gamma) \\ \\ Cr^*: & \operatorname{Sym}^2\Lambda^2 V^* & \longrightarrow & \operatorname{Sym}^2\operatorname{Sym}^2 V \\ & (\alpha\wedge\beta)(\gamma\wedge\delta) & \longmapsto & (\alpha\cdot\gamma)(\beta\cdot\delta) - (\alpha\cdot\delta)(\beta\cdot\gamma)\,. \end{array}$$

With these definitions one finds

$$(\alpha\cdot\beta)(\gamma\cdot\delta) = \Delta(\frac{1}{3}\alpha\cdot\beta\cdot\gamma\cdot\delta) + Cr^*(\frac{1}{3}(\alpha\cdot\beta)\times(\gamma\cdot\delta))\,. \quad \square$$

Remark. Either of the above isomorphisms may be used to deduce

$$\dim \operatorname{Curv} V^* = \frac{N^2}{12}(N^2-1) \qquad \text{with } \mathrm{N} = \dim \mathrm{V}\,.$$

In the next section the space $\operatorname{Curv}(H^*\otimes E^*)$ will be decomposed into its irreducible $\mathbf{SL}\,H\times\mathbf{SL}\,E$-components. For these calculations we need the additional

Lemma 3.2 *There exists a natural isomorphism*

$$\begin{array}{ccc} \operatorname{Sym}^2 V^*\otimes\Lambda^2 V^* & \xrightarrow{\cong} & \Lambda^2\operatorname{Sym}^2 V^* \oplus \Lambda^2\Lambda^2 V^* \\ \alpha\cdot\beta\otimes\gamma\wedge\delta & \longmapsto & \tfrac{1}{2}\big((\alpha\cdot\gamma)\wedge(\beta\cdot\delta) + (\beta\cdot\gamma)\wedge(\alpha\cdot\delta) \;\oplus\; (\alpha\wedge\gamma)\wedge(\beta\wedge\delta) + (\beta\wedge\gamma)\wedge(\alpha\wedge\delta)\big)\,. \end{array}$$

Proof. The partial inverse for the first summand is given by

$$(\alpha\cdot\gamma)\wedge(\beta\cdot\delta) \mapsto \frac{1}{2}\big(\alpha\cdot\beta\otimes\gamma\wedge\delta + \alpha\cdot\delta\otimes\gamma\wedge\beta + \gamma\cdot\beta\otimes\alpha\wedge\delta + \gamma\cdot\delta\otimes\alpha\wedge\beta\big)\,.$$

Marginal changes are needed for the partial inverse of the second summand. $\square$

Lemma 3.3 *Suppose that V is a symplectic vector space with symplectic form σ. The above isomorphism together with the embedding* $\mathrm{Sym}^2V^* \xrightarrow{\otimes\sigma} \mathrm{Sym}^2V^* \otimes \Lambda^2 V^*$ *induces mappings*

$$i_{\mathrm{Sym}} : \mathrm{Sym}^2V^* \to \Lambda^2\mathrm{Sym}^2V^*\,, \qquad i_\Lambda : \mathrm{Sym}^2V^* \to \Lambda^2\Lambda^2V^*\,,$$

whereas i_{Sym} is always injective, i_Λ is injective if and only if $\dim V \neq 2$.

Proof. The symplectic form $\sigma = \sum_i de_i \otimes e_i^\sharp \in V^* \otimes V^*$ defines multiplication operators $\sigma\cdot$ and $\sigma\wedge$ on $\mathrm{Sym}\,V^* \otimes \mathrm{Sym}\,V^*$ and $\Lambda\,V^* \otimes \Lambda\,V^*$ respectively, which fit into commutative diagrams

$$\begin{tikzcd}
\mathrm{Sym}^2V^* \arrow[r, "2i_{\mathrm{Sym}}"] \arrow[d] & \Lambda^2\mathrm{Sym}^2V^* \arrow[d] \\
V^*\otimes V^* \arrow[r, "\sigma\cdot"] & \mathrm{Sym}^2V^*\otimes\mathrm{Sym}^2V^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sym}^2V^* \arrow[r, "2i_\Lambda"] \arrow[d] & \Lambda^2\Lambda^2V^* \arrow[d] \\
V^*\otimes V^* \arrow[r, "\sigma\wedge"] & \Lambda^2V^*\otimes\Lambda^2V^*
\end{tikzcd}$$

because e. g.

$$\begin{aligned}
2i_{\mathrm{Sym}}(\alpha\cdot\beta) &= \frac{1}{2}\sum_i[(\alpha\cdot de_i)\wedge(\beta\cdot e_i^\sharp) + (\beta\cdot de_i)\wedge(\alpha\cdot e_i^\sharp)] \\
&\hookrightarrow \sum_i[de_i\cdot\alpha\otimes e_i^\sharp\cdot\beta + de_i\cdot\beta\otimes e_i^\sharp\cdot\alpha]\,.
\end{aligned}$$

Associated to σ is its canonical bivector L, which in turn defines contraction operators $L\lrcorner = \sum_i de_i^\flat\lrcorner \otimes e_i\lrcorner$ in either case. The pairs of operators $\sigma\cdot$ and $L\lrcorner$ or $\sigma\wedge$ and $L\lrcorner$ each generate a $\mathfrak{sl}_2$-algebra:

$$\begin{array}{rcccll}
H_{\mathrm{Sym}} &:=& [\sigma\cdot, -L\lrcorner] &=& \dim V + l + r & \text{on}\quad \mathrm{Sym}^lV^*\otimes\mathrm{Sym}^rV^* \\
H_\Lambda &:=& [L\lrcorner, \sigma\wedge] &=& \dim V - l - r & \text{on}\quad \Lambda^lV^*\otimes\Lambda^rV^*\,.
\end{array}$$

It remains to appeal to the representation theory of $\mathfrak{sl}_2$ to conclude that $\sigma\cdot$ is always injective, whereas $\sigma\wedge : \Lambda^lV^*\otimes\Lambda^rV^* \to \Lambda^{l+1}V^*\otimes\Lambda^{r+1}V^*$ is injective only for $l + r < \dim V$. □

3.2 The Bianchi Identity for Tensor Products

The aim of this section is to give a description of the space $\mathrm{Curv}(H^*\otimes E^*)$ of curvature tensors on a tensor product $H^*\otimes E^*$ in terms of the first Bianchi identity. To start with, let's make explicit the usual isomorphisms

$$\begin{aligned}
\mathrm{Sym}^2(H^*\otimes E^*) &\stackrel{\cong}{\longrightarrow} \mathrm{Sym}^2H^*\otimes\mathrm{Sym}^2E^* \oplus \Lambda^2H^*\otimes\Lambda^2E^* \\
\Lambda^2(H^*\otimes E^*) &\stackrel{\cong}{\longrightarrow} \mathrm{Sym}^2H^*\otimes\Lambda^2E^* \oplus \Lambda^2H^*\otimes\mathrm{Sym}^2E^*
\end{aligned}$$

setting e. g.

$$(\alpha\otimes\beta)\wedge(\gamma\otimes\delta) \longmapsto \frac{1}{2}[\alpha\cdot\gamma\otimes\beta\wedge\delta \oplus \alpha\wedge\gamma\otimes\beta\cdot\delta]\,.$$

Using all the isomorphisms above, the space $\mathrm{Sym}^2\Lambda^2(H^*\otimes E^*)$ can be decomposed along the lines

$$\begin{aligned}
&\mathrm{Sym}^2\Lambda^2(H^*\otimes E^*) \\
&\cong\ \mathrm{Sym}^2(\mathrm{Sym}^2H^*\otimes\Lambda^2E^*\oplus\Lambda^2H^*\otimes\mathrm{Sym}^2E^*) \\
&\cong\ \mathrm{Sym}^2(\mathrm{Sym}^2H^*\otimes\Lambda^2E^*)\ \oplus\ (\mathrm{Sym}^2H^*\otimes\Lambda^2E^*\otimes\Lambda^2H^*\otimes\mathrm{Sym}^2E^*)\ \oplus\ \mathrm{Sym}(\Lambda^2H^*\otimes\mathrm{Sym}^2E^*) \\
&\cong\ \begin{array}{rcl}
(\mathrm{Curv}\,H^* & \otimes & \mathrm{Curv}\,E^*)_{\mathrm{H}} \\
\mathrm{Curv}\,H^* & \otimes & \Lambda^4E^* \\
\mathrm{Sym}^4H^* & \otimes & \mathrm{Curv}\,E^* \\
\mathrm{Sym}^4H^* & \otimes & \Lambda^4E^* \\
(\Lambda^2\mathrm{Sym}^2H^* & \otimes & \Lambda^2\Lambda^2E^*)_{\mathrm{H}}
\end{array}
\oplus
\begin{array}{rcl}
\Lambda^2\mathrm{Sym}^2H^* & \otimes & \Lambda^2\mathrm{Sym}^2E^* \\
(\Lambda^2\Lambda^2H^* & \otimes & \Lambda^2\mathrm{Sym}^2E^*)_{\mathrm{M}} \\
(\Lambda^2\mathrm{Sym}^2H^* & \otimes & \Lambda^2\Lambda^2E^*)_{\mathrm{M}} \\
\Lambda^2\Lambda^2H^* & \otimes & \Lambda^2\Lambda^2E^*
\end{array}
\oplus
\begin{array}{rcl}
(\mathrm{Curv}\,H^* & \otimes & \mathrm{Curv}\,E^*)_{\mathrm{E}} \\
\mathrm{Curv}\,H^* & \otimes & \mathrm{Sym}^4E^* \\
\Lambda^4H^* & \otimes & \mathrm{Curv}\,E^* \\
\Lambda^4H^* & \otimes & \mathrm{Sym}^4E^* \\
(\Lambda^2\Lambda^2H^* & \otimes & \Lambda^2\mathrm{Sym}^2E^*)_{\mathrm{E}}
\end{array}
\end{aligned}$$

where summands occuring twice are labelled by their respective columns. In exactly the same manner the space $\mathrm{Sym}^2\mathrm{Sym}^2 H^*$ may be decomposed. With $\mathrm{Curv}(H^*\otimes E^*)$ being contained in either space a look at differences and similarities leads to

Lemma 3.4 *In terms of the decomposition of* $\mathrm{Sym}^2\Lambda^2(H^*\otimes E^*)$ *above the first Bianchi identity is equivalent to the five equations*

$$\begin{array}{lrcl}
\mathrm{I} & pr_{(\mathrm{Curv}H^*\otimes\mathrm{Curv}E^*)_H}R & = & pr_{(\mathrm{Curv}H^*\otimes\mathrm{Curv}E^*)_E}R \\
\mathrm{II} & pr_{\mathrm{Sym}^4H^*\otimes\Lambda^4E^*}R & = & 0 \\
\mathrm{II}' & pr_{\Lambda^4H^*\otimes\mathrm{Sym}^4E^*}R & = & 0 \\
\mathrm{III} & pr_{(\Lambda^2\mathrm{Sym}^2H^*\otimes\Lambda^2\Lambda^2E^*)_H}R & = & pr_{(\Lambda^2\mathrm{Sym}^2H^*\otimes\Lambda^2\Lambda^2E^*)_M}R \\
\mathrm{III}' & pr_{(\Lambda^2\Lambda^2H^*\otimes\Lambda^2\mathrm{Sym}^2E^*)_E}R & = & pr_{(\Lambda^2\Lambda^2H^*\otimes\Lambda^2\mathrm{Sym}^2E^*)_M}R
\end{array} \tag{3.9}$$

Proof. In the first place one has to check whether the space of solutions to I–III has the appropriate dimension. Setting $N=\mathrm{dim}E^*$ and $M=\mathrm{dim}H^*$ a first step could be to calculate

$$\mathrm{dim}(\mathrm{Sym}^4E^*\oplus\mathrm{Curv}E^*\oplus\Lambda^4E^*) \;=\; \binom{N+3}{4}+\frac{N^2}{12}(N^2-1)+\binom{N}{4} \;=\; \frac{N^2}{6}(N^2+5)$$

to find the dimension of the space of solutions to be

$$\begin{aligned}
&\frac{N^2}{6}(N^2+5)\mathrm{dim}\,\mathrm{Curv}H^*+\frac{M^2}{6}(M^2+5)\mathrm{dim}\,\mathrm{Curv}E^*-\mathrm{dim}\,\mathrm{Curv}H^*\,\mathrm{dim}\,\mathrm{Curv}E^* \\
&\quad+\;\frac{N^2}{4}(N^2-1)\frac{M^2}{4}(M^2-1) \;=\; \frac{N^2M^2}{12}(N^2M^2-1)\,.
\end{aligned}$$

Thus it is sufficient to show that the generators $(\alpha\otimes\alpha')\cdot(\gamma\otimes\gamma')\times(\beta\otimes\beta')\cdot(\delta\otimes\delta')$ of $\mathrm{Curv}(H^*\otimes E^*)$ in fact satisfy equations I–III, equations II$'$ and III$'$ are then inferred using the symmetry in H and E. To begin with, II is easy, because under the above isomorphisms/projections the element $\big((\alpha\otimes\alpha')\wedge(\beta\otimes\beta')\big)\cdot\big((\gamma\otimes\gamma')\wedge(\delta\otimes\delta')\big)$ is mapped to $\frac{1}{72}\alpha\cdot\beta\cdot\gamma\cdot\delta\otimes\alpha'\wedge\beta'\wedge\gamma'\wedge\delta'\in\mathrm{Sym}^4H^*\otimes\Lambda^4E^*$. Symmetrizing $(\beta\otimes\beta')\leftrightarrow(\delta\otimes\delta')$ to get the image of the above generator yields 0. The argument for equation III. is similar, yet slightly more complicated as two images have to be calculated. Symmetrizing the two results show that the generators are mapped to the same element in the two copies of $\Lambda^2\mathrm{Sym}^2H^*\otimes\Lambda^2\Lambda^2E^*$.

The trickiest part of the proof is I, because the 'dual' Bianchi identity for the generators of $\mathrm{Curv}H^*$ and $\mathrm{Curv}E^*$ is used. Focussing on the component $(\mathrm{Curv}\,H^*\otimes\mathrm{Curv}\,E^*)_H$, the chain of isomorphism/projections maps

$$\begin{aligned}
&\big((\alpha\otimes\alpha')\wedge(\beta\otimes\beta')\big)\cdot\big((\gamma\otimes\gamma')\wedge(\delta\otimes\delta')\big) \\
&\mapsto\;\frac{1}{4}(\alpha\cdot\beta\otimes\alpha'\wedge\beta')(\gamma\cdot\delta\otimes\gamma'\wedge\delta') \\
&\mapsto\;\frac{1}{8}(\alpha\cdot\beta)(\gamma\cdot\delta)\otimes(\alpha'\wedge\beta')(\gamma'\wedge\delta') \\
&\mapsto\;\frac{1}{72}(\alpha\cdot\beta)\times(\gamma\cdot\delta)\otimes\big((\alpha'\cdot\gamma')\times(\beta'\cdot\delta')-(\alpha'\cdot\delta')\times(\beta'\cdot\gamma')\big)\,.
\end{aligned}$$

The same element is projected to $\frac{1}{72}\big((\alpha\cdot\gamma)\times(\beta\cdot\delta)-(\alpha\cdot\delta)\times(\beta\cdot\gamma)\big)\otimes(\alpha'\cdot\beta')\times(\gamma'\cdot\delta')$ in the other copy $(\mathrm{Curv}\,H^*\otimes\mathrm{Curv}\,E^*)_E$. Even after symmetrization the results look different, however the 'dual' Bianchi identity for the generators implies the following identity, which should fix this:

$$\begin{aligned}
&(\alpha\cdot\beta)\times(\gamma\cdot\delta)\otimes(\alpha'\cdot\gamma')\times(\beta'\cdot\delta')+(\alpha\cdot\delta)\times(\gamma\cdot\beta)\otimes(\alpha'\cdot\gamma')\times(\delta'\cdot\beta') \\
&\quad=\;-(\alpha\cdot\gamma)\times(\beta\cdot\delta)\otimes(\alpha'\cdot\gamma')\times(\beta'\cdot\delta') \\
&\quad=\;(\alpha\cdot\gamma)\times(\beta\cdot\delta)\otimes(\alpha'\cdot\beta')\times(\delta'\cdot\gamma')+(\alpha\cdot\gamma)\times(\beta\cdot\delta)\otimes(\alpha'\cdot\delta')\times(\gamma'\cdot\beta')\,. \qquad\square
\end{aligned}$$

3.3 Solutions of the Bianchi Identity

With the first Bianchi identity expressed in terms of the H–E–formalism it is now easy to derive the decomposition of the curvature tensor of a quaternionic Kähler manifold (compare [Ale68-1], [Ale68-2] or [Sal82]) into a linear combination of 'curvature tensors' of the following type:

Definition. Define the following $\mathrm{End}(\mathbf{H}\otimes\mathbf{E})$-valued 2-forms on $\mathbf{H}\otimes\mathbf{E}$:

$$\begin{aligned} R^H_{h_1\otimes e_1,h_2\otimes e_2} &= \sigma_E(e_1,e_2)(h_1h_2\otimes \mathrm{id}_E) \\ R^E_{h_1\otimes e_1,h_2\otimes e_2} &= \sigma_H(h_1,h_2)(\mathrm{id}_H\otimes e_1e_2) \\ R^{hyper}_{h_1\otimes e_1,h_2\otimes e_2} &= \sigma_H(h_1,h_2)(\mathrm{id}_H\otimes \mathfrak{R}_{e_1,e_2}), \end{aligned} \tag{3.10}$$

where $\mathfrak{R}\in\mathrm{Sym}^4E^*$ and $\mathfrak{R}_{e_1,e_2}: e\mapsto \mathfrak{R}(e_1,e_2,e,.)^\flat$ is an endomorphism of E.

Lemma 3.5 *A quaternionic Kähler manifold M is Einstein, and its curvature tensor is given by*

$$R = -\frac{\kappa}{8n(n+2)}(R^H+R^E)+R^{hyper}\,. \tag{3.11}$$

where κ is the scalar curvature of M and the symmetric 4-form $\mathfrak{R}$ is necessarily the symmetrisation:

$$\mathfrak{R}(e_1,e_2,e_3,e_4) = \frac{1}{24\sigma_H(h_1,h_2)\sigma_H(h_3,h_4)}\sum_{\tau\in S_4}\langle R_{h_1\otimes e_{\tau 1},h_2\otimes e_{\tau 2}}h_3\otimes e_{\tau 3},h_4\otimes e_{\tau 4}\rangle\,.$$

which is independent of the choice of the h_i as long as $\sigma_H(h_1,h_2)\sigma_H(h_3,h_4)\neq 0$.

Proof. By definition the curvature tensor is a 2-form on $TM^{\mathbb{C}}\cong\mathbf{H}\otimes\mathbf{E}$ with values in $\mathfrak{sp}H\oplus\mathfrak{sp}E$, i. e. it is an element of $R\in\mathrm{Sym}^2(\mathrm{Sym}^2H^*\otimes\sigma_E\oplus\sigma_H\otimes\mathrm{Sym}^2E^*)$. Of course (σ_E) considered as an element of $\Lambda(\Lambda^2\mathbf{E})$ satisfies $(\sigma_E)\wedge(\sigma_E)=0$, though $\sigma_E\wedge\sigma_E\neq 0$ for $\dim E\neq 2$. Thus the E-symmetric part of R can be written as

$$\frac{1}{2}(\sigma_H)^2\otimes r^E\oplus\frac{1}{2}(\sigma_H)^2\otimes\mathfrak{R}$$

for some $r^E\in\mathrm{Curv}E^*$ and $\mathfrak{R}\in\mathrm{Sym}^4E^*$. Equation II$'$ is then trivially satisfied, because $\sigma_H\wedge\sigma_H=0$. The H-symmetric part, however, reduces to

$$r^H\otimes\frac{1}{2}(\sigma_E)^2$$

for some $r^H\in\mathrm{Curv}H^*$, as a nonzero contribution from Sym^4H^* is incompatible with equation II with $\sigma_E\wedge\sigma_E$ being nonzero.

Equation III then implies that the mixed part of R, an element of $\mathrm{Sym}^2H^*\otimes\sigma_E\otimes\sigma_H\otimes\mathrm{Sym}^2E^*$, is mapped to 0 under the projection to $\Lambda^2\mathrm{Sym}^2H^*\otimes\Lambda^2\Lambda^2E^*$. On the other hand, Lemma (3.3) ensures this projection to be injective on this particular subspace of $\mathrm{Sym}^2H^*\otimes\Lambda^2E^*\otimes\Lambda^2H^*\otimes\mathrm{Sym}^2E^*$, so that the mixed curvature part has to vanish.

Finally equation I shows that r^E is a scalar multiple of $pr_{\mathrm{Curv}\,E^*}(\frac{1}{2}(\sigma_E)^2)$ and that r^H is the same multiple of $pr_{\mathrm{Curv}\,H^*}(\frac{1}{2}(\sigma_H)^2)$, because this is the only way to satisfy

$$r^E\otimes pr_{\mathrm{Curv}H^*}\Big(\frac{1}{2}(\sigma_H)^2\Big) = pr_{\mathrm{Curv}E^*}\Big(\frac{1}{2}(\sigma_E)^2\Big)\otimes r^H\,.$$

Thus the first Bianchi identity implies that at any point of M the curvature tensor of the quaternionic Kähler manifold is a linear combination of R^H+R^E and R^{hyper} with:

$$\begin{aligned} R^H &= pr_{\mathrm{Curv}H^*}\Big(\frac{1}{2}(\sigma_H)^2\Big)\otimes\frac{1}{2}(\sigma_E)^2 \\ R^E &= \frac{1}{2}(\sigma_H)^2\otimes pr_{\mathrm{Curv}E^*}\Big(\frac{1}{2}(\sigma_E)^2\Big) \\ R^{hyper} &= \frac{1}{2}(\sigma_H)^2\otimes\mathfrak{R}\,. \end{aligned} \tag{3.12}$$

To determine R completely, it is convenient to calculate the Ricci curvature of M, as its definition

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R_{Z,X}Y), \quad X,Y,Z \in TM$$

is easy to handle. Note that for real vectors X, Y the endomorphism $R_{.,X}Y$ is already defined over $\mathbb{R}$, so that its trace may be calculated over $\mathbb{R}$ or $\mathbb{C}$. The contributions from the different components of R are

- $\mathrm{Ric}^{hyper}(h_1 \otimes e_1, h_2 \otimes e_2)$ is the trace of the factorizable endomorphism

  $$h \otimes e \mapsto \sigma_H(h,h_1)h_2 \otimes \mathfrak{R}(e,e_1,e_2,.)^\flat = -(h_1^\sharp \otimes h_2)h \otimes \mathfrak{R}_{e_1,e_2}e\,.$$

  Its trace is thus the product of the partial traces, however, $\mathfrak{R}_{e_1,e_2} \in \mathrm{Sym}^2 E^* \cong \mathfrak{sp}E$ is trace-free.

- $\mathrm{Ric}^E(h_1 \otimes e_1, h_2 \otimes e_2)$ is the trace of the endomorphism

  $$h \otimes e \mapsto \sigma_H(h,h_1)h_2 \otimes \big(\sigma_E(e,e_2)e_1 + \sigma_E(e_1,e_2)e\big) = -(h_1^\sharp \otimes h_2)h \otimes \big(-e_2^\sharp \otimes e_1 + \sigma_E(e_1,e_2)\mathrm{id}_E\big)e\,.$$

  which factorizes, too. Its trace is thus

  $$\mathrm{Ric}^E(h_1 \otimes e_1, h_2 \otimes e_2) = -(2n+1)\sigma_H(h_1,h_2)\sigma_E(e_1,e_2)\,.$$

- The same argument goes through for Ric^H; the different dimensions of H and E account for the slightly changed result:

  $$\mathrm{Ric}^H(h_1 \otimes e_1, h_2 \otimes e_2) = -3\sigma_H(h_1,h_2)\sigma_E(e_1,e_2)\,.$$

The Ricci curvature being a multiple of the metric the quaternionic Kähler manifold M is Einstein and the scalar curvature κ is thus constant on M. The coefficient of $R^H + R^E$ in R is then fixed by

$$\mathrm{Ric}(X,Y) = \frac{\kappa}{4n}g(X,Y) = -\frac{\kappa}{8n(n+2)}(\mathrm{Ric}^H + \mathrm{Ric}^E)(X,Y)\,. \quad \square \tag{3.13}$$

At the end of this section a strange and remarkable feature of the hyperkähler part R^{hyper} of the curvature tensor should be pointed out. This feature is convenient for deriving the Weitzenböck formulas below, nevertheless it implies that essentially new ideas are needed to discuss the limit case in the eigenvalue estimate: R^{hyper} eludes all Weitzenböck formulas!

To be more precise, the symmetric 4-form $\mathfrak{R}$ induces a natural endomorphism on every vector bundle associated to the principal $\mathbf{Sp}(1)\mathbf{Sp}(n)$–bundle in the following way:

$$\begin{array}{ccccc} \mathrm{Sym}^4 E^* & \xrightarrow{\Delta} & \mathrm{Sym}^2 E^* \otimes \mathrm{Sym}^2 E^* & \xrightarrow{m} & \mathcal{U}(\mathfrak{sp}E) \\ \mathfrak{R} & & \longmapsto & & Q_{\mathfrak{R}}\,. \end{array}$$

where $\mathcal{U}(\mathfrak{sp}E)$ is the universal envelopping algebra of $\mathfrak{sp}E$. This mapping is injective by Poincaré–Birkhoff–Witt, nevertheless the endomorphism induced on the spinor bundle is trivial:

Lemma 3.6 *Any symmetric 4-form in* $\mathrm{Sym}^4 E^*$ *induces the trivial endomorphism on* ΛE.

Proof. As the 4th powers span $\mathrm{Sym}^4 E^*$, it is sufficient to check the lemma only for an element of the form $\frac{1}{24}\alpha^4$. The comultiplication maps it to $\frac{1}{2}\alpha^2 \otimes \frac{1}{2}\alpha^2$. Considered as an endomorphism $\frac{1}{2}\alpha^2$ denotes $A = \alpha \otimes \alpha^\flat$, extended as derivation to ΛE. As such it satisfies $A^2 = 0$:

$$\begin{aligned} A^2(e_1 \wedge \ldots \wedge e_s) &= \sum_{i=1}^{s} e_1 \wedge \ldots \wedge A^2 e_i \wedge \ldots \wedge e_s \\ &\qquad + 2\sum_{i<j} e_1 \wedge \ldots \wedge Ae_i \wedge \ldots \wedge Ae_j \wedge \ldots \wedge e_s\,. \end{aligned}$$

In fact, the first sum is zero, because already $A^2 = 0$ on E, and the second vanishes identically, because the endomorphism A of E has rank 1. $\square$

Corollary 3.1 *The comultiplication of $\mathfrak{R}$ may be written as*

$$\Delta\mathfrak{R} = \frac{1}{2}\sum_{i,j} de_i \cdot de_j \otimes \mathfrak{R}(e_i, e_j, ., .)$$

such that

$$\sum_{i,j} \mathrm{id} \otimes (de_j^\flat \wedge_\circ de_i \lrcorner + de_i^\flat \wedge_\circ de_j \lrcorner)\mathfrak{R}_{e_i,e_j} = 0 \tag{3.14}$$

holds as an operator identity on the spinor bundle.

4 The Universal Weitzenböck Formula

In this section we develop the main tool for the proof of the lower bound. It turns out that all necessary information can be encoded in a single matrix. To motivate the further proceeding, it is useful to remember the situation in the Riemannian case. For deriving the eigenvalue estimate in that case, two things are needed: the Lichnerowicz–Weitzenböck formula and the twistor operator.

4.1 The Riemannian Estimate Revisited

Let (M^n, g) be Riemennian spin manifold with tangent resp. spinor bundle TM resp. $\mathbf{S}(M)$ associated to the representations V and Σ of $\mathbf{Spin}(V)$. The associativity of the tensor product $(V\otimes V)\otimes\Sigma \cong V\otimes(V\otimes\Sigma)$ can be thought of as two different ways of decomposing $V\otimes V\otimes\Sigma$ into irreducible $\mathbf{Spin}(V)$–representations; the isomorphism induced is expressed by an invertible matrix $\mathcal{W}$ for each isotypical component.

Accordingly, the special section $\nabla^2\psi \in \Gamma(TM\otimes TM\otimes\mathbf{S}(M))$ splits into sections of the associated bundles inducing two different sets of 2nd order differential operators which are related by the same matrix. We will only be interested in differential operators from the spinor bundle to itself.

In the Riemannian case $V\otimes V\otimes\Sigma$ contains two copies of Σ, such that the associativity as above is expressed by a 2×2–matrix. Let's look at $(V\otimes V)\otimes\Sigma$ first. Due to the obvious decomposition

$$V\otimes V \cong \Lambda^2 V \oplus \mathrm{Sym}^2_\circ V \oplus \mathbb{C},$$

the projection onto the two summands in $(V\otimes V)\otimes\Sigma$ is given by contracting of the two V factors with the metric g and the operation $\Lambda^2 V \cong \mathfrak{spin}(V)$ on Σ. The space $\mathrm{Sym}^2_\circ V\otimes\Sigma$ contains no copy of Σ. To summarize:

$$\begin{array}{rccl} pr_{\mathbb{C}}: & V\otimes V\otimes\Sigma & \longrightarrow & \Sigma \\ & e_1\otimes e_2\otimes\psi & \longmapsto & g(e_1,e_2)\psi\,, \end{array}$$

$$\begin{array}{rccl} pr_{\Lambda^2}: & V\otimes V\otimes\Sigma & \longrightarrow & \Sigma \\ & e_1\otimes e_2\otimes\psi & \longmapsto & e_1e_2\psi + g(e_1,e_2)\psi\,. \end{array}$$

Strictly speaking, there is a factor $\frac{1}{2}$ missing in pr_{Λ^2}, but this is of no concern at this point. On the other hand, the projection to the first Σ in $V\otimes(V\otimes\Sigma)$ is merely twice the Clifford multiplication. The complementary projection involves the kernel $K\subset V\otimes\Sigma$ of the Clifford multiplication and is contraction of the two V–factors in $V\otimes K$:

$$\begin{array}{rccl} pr_{\Sigma}: & V\otimes V\otimes\Sigma & \longrightarrow & \Sigma \\ & e_1\otimes e_2\otimes\psi & \longmapsto & e_1e_2\psi\,, \end{array}$$

$$\begin{array}{rccccl} pr_K: & V\otimes V\otimes\Sigma & \longrightarrow & V\otimes K\subset V\otimes V\otimes\Sigma & \longrightarrow & \Sigma \\ & e_1\otimes e_2\otimes\psi & \longmapsto & e_1\otimes(e_2\otimes\psi + \frac{1}{n}\sum_i e_i\otimes e_i\cdot e_2\cdot\psi) & \longmapsto & g(e_1,e_2)\psi + \frac{1}{n}e_1e_2\psi\,. \end{array}$$

With these calculations it is now easy to provide a first example of a matrix $\mathcal{W}$ as above, called the Weitzenböck matrix of a Riemannian manifold. It is the matrix appearing in the explicit formula for the isomorphism induced by associativity:

$$\begin{pmatrix} pr_{\mathbb{C}} \\ \\ pr_{\Lambda^2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{n} & 1 \\ \\ \frac{n-1}{n} & 1 \end{pmatrix} \begin{pmatrix} pr_\Sigma \\ \\ pr_K \end{pmatrix}.$$

The last thing to do is to identify the projectors with differential operators like $\nabla^*\nabla$ and D^2 on the manifold. For a section $\psi \in \Gamma(\mathbf{S}(M))$ the section $\nabla^2\psi$ can be expanded in a sum over an orthonormal basis leading to the following images of the projections onto Σ:

$$\begin{aligned} pr_{\mathbb{C}}(\nabla^2\psi) &= \textstyle\sum_{i,j} g(e_i, e_j)\nabla^2_{e_i,e_j}\psi = -\nabla^*\nabla\psi \\ pr_{\Lambda^2}(\nabla^2\psi) &= \textstyle\sum_{i,j}(e_ie_j + g(e_i,e_j))\nabla^2_{e_i,e_j}\psi = \frac{\kappa}{4}\psi \end{aligned}$$

and

$$\begin{aligned} pr_\Sigma(\nabla^2\psi) &= \textstyle\sum_{i,j} e_ie_j\nabla^2_{e_i,e_j}\psi = D^2\psi \\ pr_K(\nabla^2\psi) &= \textstyle\sum_{i,j}\left(g(e_i,e_j) + \frac{1}{n}e_ie_j\nabla^2_{e_i,e_j}\right)\psi = -T^*T\psi\,. \end{aligned}$$

The only projection which does not lead to a differential operator is the well–known curvature term appearing in the Lichnerowicz–Weitzenböck formula. The final form of the Riemannian version of the Weitzenböck matrix formula reads:

$$\begin{pmatrix} -\nabla^*\nabla\psi \\ \\ \frac{\kappa}{4}\psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{n} & 1 \\ \\ \frac{n-1}{n} & 1 \end{pmatrix} \begin{pmatrix} D^2\psi \\ \\ -T^*T\psi \end{pmatrix}. \tag{4.15}$$

This matrix equation is a nice tool to produce some well–known formulas by multiplying it with row vectors from the left, e. g. multiplying with $(-1\ \ 1)$ yields the Lichnerowicz–Weitzenböck formula. Taking the L^2-product with ψ, we get its integrated version:

$$\begin{pmatrix} -\|\nabla\psi\|^2 \\ \\ \frac{1}{4}\langle\psi, \kappa\psi\rangle \end{pmatrix} = \begin{pmatrix} -\frac{1}{n} & 1 \\ \\ \frac{n-1}{n} & 1 \end{pmatrix} \begin{pmatrix} \|D\psi\|^2 \\ \\ -\|T\psi\|^2 \end{pmatrix}. \tag{4.16}$$

Estimating the twistor operator term to zero, the second row immediately gives the well-known estimate of Friedrich [Fri80] for the first eigenvalue of the Dirac operator. This rather simple example of the Riemannian case shows the principle by which the eigenvalue estimate in the quaternionic case will be obtained.

4.2 The Weitzenböck Matrix

The same strategy is pursued in the quaternionic Kähler case, but under holonomy $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$ the splitting of $TM \otimes TM \otimes \mathbf{S}(M)$ is much more delicate. Like in the Riemannian case, the splitting of $\nabla^2\psi$, considered as section in $TM \otimes TM \otimes \mathbf{S}(M)$ yields two different sets of differential operators related by a matrix $\mathcal{W}$. To calculate it, we have to look at the splitting of

$$V \otimes V \otimes \Sigma = (H \otimes E) \otimes (H \otimes E) \otimes \bigoplus_{r=0}^{n} \left(\mathrm{Sym}^r H \otimes \Lambda^{n-r}_\circ E\right)$$

into irreducibles, find the Σ-summands and determine how they are embedded in $V \otimes V \otimes \Sigma$.

There is a simple observation which makes the following calculations feasible. All representations occuring are products of representations of $\mathbf{Sp}(1)$ or $\mathbf{Sp}(n)$. Hence it suffices to look at the H–part resp. the E–part of the splitting, that means, one can look at the spaces $H \otimes H \otimes \mathrm{Sym}^r H$ and $E \otimes E \otimes \Lambda^{n-r}_\circ E$ separately to search for their $\mathrm{Sym}^r H$– resp. $\Lambda^{n-r}_\circ E$–summands.

First, let's consider the H–part. The two sides of the isomorphism

$$(H \otimes H) \otimes \mathrm{Sym}^r H \cong H \otimes (H \otimes \mathrm{Sym}^r H)$$

need to be decomposed, where on the left side $H \otimes H$ is acting as endomorphisms on $\mathrm{Sym}^r H$ and on the right side H is acting by its part of the Clifford multiplication. On the right side the situation is easy:

$$
\begin{array}{rcccccl}
pr_{-+}: & H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & H \otimes \mathrm{Sym}^{r+1} H & \longrightarrow & \mathrm{Sym}^r H \\
& h_1 \otimes h_2 \otimes s & \longmapsto & h_1 \otimes h_2 \cdot s & \longmapsto & h_1^\sharp \lrcorner_\circ (h_2 \cdot s)\,, \\[2ex]
pr_{+-}: & H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & H \otimes \mathrm{Sym}^{r-1} H & \longrightarrow & \mathrm{Sym}^r H \\
& h_1 \otimes h_2 \otimes s & \longmapsto & h_1 \otimes h_2^\sharp \lrcorner_\circ s & \longmapsto & h_1 \cdot (h_2^\sharp \lrcorner_\circ s)\,.
\end{array}
$$

Using Lemma 2.3 the expression for the first projection can be rewritten:

$$pr_{-+}(h_1 \otimes h_2 \otimes s) = \sigma_H(h_1, h_2)s + \frac{r}{r+1} h_1 \cdot h_2^\sharp \lrcorner_\circ s\,.$$

On the left hand side, we use the isomorphism

$$H \otimes H \cong \mathrm{Sym}^2 H \oplus \Lambda^2 H \cong \mathrm{Sym}^2 H \oplus \mathbb{C},$$

where $\mathbb{C}$ denotes the trivial representation, spanned by σ_H. Hence,

$$
\begin{array}{rcl}
pr_{\mathbb{C}} : H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & \mathrm{Sym}^r H \\
h_1 \otimes h_2 \otimes s & \longmapsto & \sigma_H(h_1, h_2)s\,.
\end{array}
$$

Elements of $\mathrm{Sym}^2 H$ operate in the obvious manner as endomorphims on H, namely as $(h_1 h_2)(h) = \sigma_H(h_1, h)h_2 + \sigma_H(h_2, h)h_1$, extended as derivation to the symmetric algebra:

$$
\begin{array}{rcl}
pr_{\mathrm{Sym}^2 H} : H \otimes H \otimes \mathrm{Sym}^r H & \longrightarrow & \mathrm{Sym}^r H \\
h_1 \otimes h_2 \otimes s & \longmapsto & (h_1 h_2)(s)\,.
\end{array}
$$

This can be written explicitely:

$$(h_1 h_2)(s) = (h_2 \cdot h_1^\sharp \lrcorner \;\; + h_1 \cdot h_2^\sharp \lrcorner \;\;)s = r\sigma_H(h_1, h_2)s + 2h_1 \cdot h_2^\sharp \lrcorner\, s = r\sigma_H(h_1, h_2)s + 2rh_1 \cdot h_2^\sharp \lrcorner_\circ s\,.$$

To get an overview of this situation, a diagram will be helpful. The notation $[.]_{\mathrm{Sym}^r H}$ refers to the irreducible $\mathrm{Sym}^r H$-summand of the space in question.

$$
\begin{tikzcd}
\begin{matrix} [\mathbb{C} \otimes \mathrm{Sym}^r H]_{\mathrm{Sym}^r H} \\ \sigma_H(h_1, h_2)s \end{matrix} & & \begin{matrix} \mathrm{Sym}^r H \\ \sigma_H(h_1, h_2)s + \frac{r}{r+1} h_1 \cdot h_2^\sharp \lrcorner_\circ s \end{matrix} \\
& \begin{matrix} H \otimes H \otimes \mathrm{Sym}^r H \\ h_1 \otimes h_2 \otimes s \end{matrix} \arrow[ul, "pr_{\mathbb{C}}"] \arrow[ur, "pr_{-+}"'] \arrow[dl, "pr_{\mathrm{Sym}^2 H}"'] \arrow[dr, "pr_{+-}"] & \\
\begin{matrix} [\mathrm{Sym}^2 H \otimes \mathrm{Sym}^r H]_{\mathrm{Sym}^r H} \\ r\sigma_H(h_1, h_2)s + 2rh_1 \cdot h_2^\sharp \lrcorner_\circ s \end{matrix} & & \begin{matrix} \mathrm{Sym}^r H \\ h_1 \cdot h_2^\sharp \lrcorner_\circ s \end{matrix}
\end{tikzcd}
$$

From this diagram the H–part of the Weitzenböck matrix is evident:

$$\mathcal{W}_H = \begin{pmatrix} 1 & -\frac{r}{r+1} \\ r & \frac{r(r+2)}{r+1} \end{pmatrix}. \tag{4.17}$$

Turning to the E–part of the Weitzenböck matrix, one has to look at the two sides of the obvious isomorphism again:

$$(E \otimes E) \otimes \Lambda_\circ^{n-r} E \cong E \otimes (E \otimes \Lambda_\circ^{n-r} E)\,. \tag{4.18}$$

In difference to the H–part there exists a kernel $Kn-r$ of multiplication and contraction on the right side:

$$E \otimes \Lambda_\circ^{n-r} E \cong \Lambda_\circ^{n-r+1} E \oplus \Lambda_\circ^{n-r-1} E \oplus K^{n-r} .$$

Therefore, in the decomposition of $E \otimes (E \otimes \Lambda_\circ^{n-r} E)$ the representation $\Lambda_\circ^{n-r} E$ occurs three times, namely twice by the E–part of the Clifford multiplication and in addition as an irreducible summand of $E \otimes K^{n-r}$. The following lemma gives an explicit expression for the projection onto K^{n-r}:

Lemma 4.1

$$\begin{array}{rrcl} \widetilde{pr}_K : & E \otimes \Lambda_\circ^{n-r} E & \longrightarrow & K^{n-r} \\ & e \otimes \omega & \longmapsto & e \otimes \omega - \frac{1}{n-r+1} \sum_i e_i \otimes de_i \lrcorner\, e \wedge_\circ \omega \\ & & & - \frac{r+2}{(n+r+3)(r+1)} \sum_i de_i^\flat \otimes e_i \wedge_\circ e^\sharp \lrcorner\, \omega . \end{array}$$

Proof. Simple use of the number operators introduced in Lemma 2.2 immediately shows that the projection of $e \otimes \omega$ onto the kernel of the $\wedge_\circ$-product is $e \otimes \omega - \frac{1}{n-r+1} \sum_i e_i \otimes de_i \lrcorner\, e \wedge_\circ \omega$. By the same argument $e \otimes \omega$ plus the third summand is the projection onto the kernel of contraction:

$$\begin{aligned} \sum_i de_i \lrcorner\, e_i \wedge_\circ e^\sharp \lrcorner\, \omega &= \sum_i \left(\sigma_E(de_i^\flat, e_i) - e_i \wedge_\circ de_i \lrcorner\, + \tfrac{1}{r+2} de_i^\flat \wedge_\circ e_i^\sharp \lrcorner\, \right) e^\sharp \lrcorner\, \omega \\ &= (2n - (n-r-1) - \tfrac{n-r-1}{r+2}) e^\sharp \lrcorner\, \omega \\ &= \tfrac{(n+r+3)(r+1)}{r+2} e^\sharp \lrcorner\, \omega . \end{aligned}$$

It remains to show that the second resp. the third summand already lies in the kernel of contraction resp. multiplication. This is clear for the second summand because $\sigma_E \lrcorner\, e^\sharp \wedge_\circ \omega = 0$ by definition. Upon $\wedge_\circ$–multiplication, the third summand reduces to the projection of $L_E \wedge e^\sharp \lrcorner\, \omega$ onto $\Lambda_\circ E$ which is zero by definition, too. $\square$

Tensoring once again, the projection onto the irreducible $\Lambda_\circ^{n-r} E$–summand contained in $E \otimes K^{n-r} \subset E \otimes E \otimes \Lambda_\circ^{n-r} E$ can be thought of as contracting the two E–factors. All ingrediences for computing the right hand side are now at hand:

$$\begin{array}{rrclcl} pr_{-+} : & E \otimes E \otimes \Lambda_\circ^{n-r} E & \longrightarrow & E \otimes \Lambda_\circ^{n-r+1} E & \longrightarrow & \Lambda_\circ^{n-r} E \\ & e_1 \otimes e_2 \otimes \omega & \longmapsto & e_1 \otimes e_2 \wedge_\circ \omega & \longmapsto & e_1^\sharp \lrcorner\, e_2 \wedge_\circ \omega , \\[1ex] pr_{+-} : & E \otimes E \otimes \Lambda_\circ^{n-r} E & \longrightarrow & E \otimes \Lambda_\circ^{n-r-1} E & \longrightarrow & \Lambda_\circ^{n-r} E \\ & e_1 \otimes e_2 \otimes \omega & \longmapsto & e_1 \otimes e_2^\sharp \lrcorner\, \omega & \longmapsto & e_1 \wedge_\circ e_2^\sharp \lrcorner\, \omega , \\[1ex] pr_K : & E \otimes E \otimes \Lambda_\circ^{n-r} E & \longrightarrow & E \otimes K^{n-r} & \longrightarrow & \Lambda_\circ^{n-r} E \\ & e_1 \otimes e_2 \otimes \omega & \longmapsto & e_1 \otimes \widetilde{pr}_K(e_2 \otimes \omega) & \longmapsto & \sigma_E(e_1, e_2)\omega - \frac{1}{n-r+1} e_1^\sharp \lrcorner\, e_2 \wedge_\circ \omega \\ & & & & & + \frac{r+2}{(n+r+3)(r+1)} e_1 \wedge_\circ e_2^\sharp \lrcorner\, \omega . \end{array}$$

On the left side of (4.18), $E \otimes E$ is seen as space of endomorphisms, acting as derivations on $\Lambda_\circ^{n-r} E$:

$$E \otimes E \cong \mathrm{Sym}^2 E \oplus \Lambda_\circ^2 E \oplus \mathbb{C},$$

where $\Lambda_\circ^2 E$ resp. $\mathbb{C}$ denotes the trace–free part resp. the trace part of $\Lambda^2 E$. To calculate the trace–free part of an element $e_1 \wedge e_2 \in \Lambda^2 E$, one observes that the action on E is given by

$$(e_1 \wedge e_2)(e) = \sigma_E(e_1, e) e_2 - \sigma_E(e_2, e) e_1 ,$$

its trace being $2\sigma_E(e_1, e_2)$. Its trace–free part is thus $e_1 \wedge e_2 - \frac{1}{n} \sigma_E(e_1, e_2) \mathrm{id}$. Extended as derivation to $\Lambda_\circ^{n-r} E$, the identity id acts as multiplication with $n-r$, and *a forteriori*, the trace–free part of $e_1 \wedge e_2$ becomes the operator $e_2 \wedge_\circ e_1^\sharp \lrcorner\, - e_1 \wedge_\circ e_2^\sharp \lrcorner\, - \frac{n-r}{n} \sigma_E(e_1, e_2)$.

With the preceeding calculations, the form of the projections is obvious:

$$
\begin{array}{rrcl}
pr_{\mathbb{C}}: & E\otimes E\otimes\Lambda^{n-r}_\circ E & \longrightarrow & \Lambda^{n-r}_\circ E\\
& e_1\otimes e_2\otimes\omega & \longmapsto & \sigma_E(e_1,e_2)\omega\,,\\[2ex]
pr_{\mathrm{Sym}^2E}: & E\otimes E\otimes\Lambda^{n-r}_\circ E & \longrightarrow & \Lambda^{n-r}_\circ E\\
& e_1\otimes e_2\otimes\omega & \longmapsto & e_2\wedge_\circ e_1^\sharp\lrcorner\,\omega+e_1\wedge_\circ e_2^\sharp\lrcorner\,\omega\\[1ex]
& & & =\big(\frac{r+2}{r+1}e_1\wedge_\circ e_2^\sharp\lrcorner\ -e_1^\sharp\lrcorner\, e_2\wedge_\circ+\sigma_E(e_1,e_2)\big)\omega\,,\\[2ex]
pr_{\Lambda^2_\circ E}: & E\otimes E\otimes\Lambda^{n-r}_\circ E & \longrightarrow & \Lambda^{n-r}_\circ E\\
& e_1\otimes e_2\otimes\omega & \longmapsto & e_2\wedge_\circ e_1^\sharp\lrcorner\,\omega-e_1\wedge_\circ e_2^\sharp\lrcorner\,\omega-\frac{n-r}{n}\sigma_E(e_1,e_2)\omega\\[1ex]
& & & =\big(-\frac{r}{r+1}e_1\wedge_\circ e_2^\sharp\lrcorner\ -e_1^\sharp\lrcorner\, e_2\wedge_\circ+\frac{r}{n}\sigma_E(e_1,e_2)\big)\omega.
\end{array}
$$

To summarize all calculations above let's consider a diagram:

$$
\begin{tikzcd}
\begin{array}{l}[\mathbb{C}\otimes\Lambda^{n-r}_\circ E]_{\Lambda^{n-r}_\circ E}\\ \sigma_E(e_1,e_2)\omega\end{array} & & \begin{array}{l}\Lambda^{n-r}_\circ E\\ e_1^\sharp\lrcorner\, e_2\wedge_\circ\end{array}\\
\begin{array}{l}[\mathrm{Sym}^2E\otimes\Lambda^{n-r}_\circ E]_{\Lambda^{n-r}_\circ E}\\ (e_2\wedge_\circ e_1^\sharp\lrcorner\ +e_1\wedge_\circ e_2^\sharp\lrcorner\ )\omega\end{array} & \begin{array}{l}E\otimes E\otimes\Lambda^{n-r}_\circ E\\ e_1\otimes e_2\otimes\omega\end{array} \arrow[ul, "pr_{\mathbb{C}}"'] \arrow[l, "pr_{\mathrm{Sym}^2E}"'] \arrow[dl, "pr_{\Lambda^2_\circ}"] \arrow[ur, "pr_{-+}"] \arrow[r, "pr_{+-}"] \arrow[dr, "pr_K"] & \begin{array}{l}\Lambda^{n-r}_\circ E\\ e_1\wedge_\circ e_2^\sharp\lrcorner\,\omega\end{array}\\
\begin{array}{l}[\Lambda^2_\circ E\otimes\Lambda^{n-r}_\circ E]_{\Lambda^{n-r}_\circ E}\\ e_2\wedge_\circ e_1^\sharp\lrcorner\,\omega-e_1\wedge_\circ e_2^\sharp\lrcorner\,\omega\\ -\frac{n-r}{n}\sigma_E(e_1,e_2)\omega\end{array} & & \begin{array}{l}[E\otimes K^{n-r}]_{\Lambda^{n-r}_\circ E}\\ \sigma_E(e_1,e_2)\omega-\frac{1}{n-r+1}e_1^\sharp\lrcorner\, e_2\wedge_\circ\omega\\ +\frac{r+2}{(n+r+3)(r+1)}e_1\wedge_\circ e_2^\sharp\lrcorner\,\omega\end{array}
\end{tikzcd}
$$

Finally, we are able to give the E–part of the Weitzenböck matrix $\mathcal{W}_E$:

$$
\mathcal{W}_E=\begin{pmatrix}
\frac{1}{n-r+1} & -\frac{r+2}{(n+r+3)(r+1)} & 1\\[1ex]
-\frac{n-r}{n-r+1} & \frac{(n+r+2)(r+2)}{(n+r+3)(r+1)} & 1\\[1ex]
-\frac{(n-r)(n+1)}{n(n-r+1)} & -\frac{r(n+r+2)(n+1)}{n(n+r+3)(r+1)} & \frac{r}{n}
\end{pmatrix}. \tag{4.19}
$$

The full Weitzenböck matrix is a 6×6–matrix which is calculated as Kronecker product $\mathcal{W}=\mathcal{W}_E\otimes\mathcal{W}_H$ of the two partial Weitzenböck matrices, i. e. the nine 2×2-blocks of $\mathcal{W}$ are obtained by multiplying $\mathcal{W}_H$

with the corresponding entry in $\mathcal{W}_E$.

$$\mathcal{W} = \begin{pmatrix} \frac{1}{n-r+1} & -\frac{r}{(n-r+1)(r+1)} & -\frac{r+2}{(n+r+3)(r+1)} & \frac{r(r+2)}{(n+r+3)(r+1)^2} & 1 & -\frac{r}{r+1} \\ \frac{r}{n-r+1} & \frac{r(r+2)}{(n-r+1)(r+1)} & -\frac{r(r+2)}{(n+r+3)(r+1)} & -\frac{r(r+2)^2}{(n+r+3)(r+1)^2} & r & \frac{r(r+2)}{r+1} \\ -\frac{n-r}{n-r+1} & \frac{r(n-r)}{(n-r+1)(r+1)} & \frac{(n+r+2)(r+2)}{(n+r+3)(r+1)} & -\frac{r(n+r+2)(r+2)}{(n+r+3)(r+1)^2} & 1 & -\frac{r}{r+1} \\ -\frac{(n-r)r}{n-r+1} & -\frac{r(r+2)(n-r)}{(n-r+1)(r+1)} & \frac{r(n+r+2)(r+2)}{(n+r+3)(r+1)} & \frac{r(n+r+2)(r+2)^2}{(n+r+3)(r+1)^2} & r & \frac{r(r+2)}{r+1} \\ -\frac{(n-r)(n+1)}{n(n-r+1)} & \frac{r(n-r)(n+1)}{n(n-r+1)(r+1)} & -\frac{r(n+r+2)(n+1)}{n(n+r+3)(r+1)} & \frac{r^2(n+r+2)(n+1)}{n(n+r+3)(r+1)^2} & \frac{r}{n} & -\frac{r^2}{n(r+1)} \\ -\frac{r(n-r)(n+1)}{n(n-r+1)} & -\frac{r(r+2)(n-r)(n+1)}{n(n-r+1)(r+1)} & -\frac{r^2(n+r+2)(n+1)}{n(n+r+3)(r+1)} & -\frac{r^2(r+2)(n+r+2)(n+1)}{n(n+r+3)(r+1)^2} & \frac{r^2}{n} & \frac{r^2(r+2)}{n(r+1)} \end{pmatrix} \tag{4.20}$$

4.3 Associated Differential Operators

In this section the two different sets of projectors onto copies of $\mathbf{S}(M)$ related by the associativity isomorphism

$$(TM \otimes TM) \otimes \mathbf{S}(M) \cong TM \otimes (TM \otimes \mathbf{S}(M)) \tag{4.21}$$

are applied to the special section $\nabla^2\psi \in \Gamma(TM \otimes TM \otimes \mathbf{S}(M))$. This defines two sets of differential operators on sections of the spinor bundle related by the matrix $\mathcal{W}$. In the sequel we determine all these differential operators.

On the right hand side of (4.21) four of the projections of $\nabla^2\psi$ are easily identified with expressions like $\frac{1}{2}D_+^- D_-^+\psi$. Due to the definition of the Clifford multiplication, the explicit expressions have an additional factor $\frac{1}{2}$.

$$\begin{aligned} pr_{-+} \otimes pr_{-+} : \quad \nabla^2\psi \quad &\longmapsto \quad \textstyle\sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} (dh_a \lrcorner_\circ \otimes de_i \lrcorner\,)(dh_b^\flat \cdot \otimes de_j^\flat \wedge_\circ) \nabla^2_{h_a \otimes e_i, h_b \otimes e_j} \psi \\ &\qquad = \tfrac{1}{2} D_-^- D_+^+ \psi = -\tfrac{1}{2} (D_+^+)^* D_+^+ \psi \,, \\ pr_{+-} \otimes pr_{-+} : \quad \nabla^2\psi \quad &\longmapsto \quad \textstyle\sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} (dh_a^\flat \cdot \otimes de_i \lrcorner\,)(dh_b \lrcorner_\circ \otimes de_j^\flat \wedge_\circ) \nabla^2_{h_a \otimes e_i, h_b \otimes e_j} \psi \\ &\qquad = \tfrac{1}{2} D_-^+ D_+^- \psi \,, \\ pr_{-+} \otimes pr_{+-} : \quad \nabla^2\psi \quad &\longmapsto \quad \textstyle\sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} (dh_a \lrcorner_\circ \otimes de_i^\flat \wedge_\circ)(dh_b^\flat \cdot \otimes de_j \lrcorner\,) \nabla^2_{h_a \otimes e_i, h_b \otimes e_j} \psi \\ &\qquad = \tfrac{1}{2} D_+^- D_-^+ \psi \,, \\ pr_{+-} \otimes pr_{+-} : \quad \nabla^2\psi \quad &\longmapsto \quad \textstyle\sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} (dh_a^\flat \cdot \otimes de_i^\flat \wedge_\circ)(dh_b \lrcorner_\circ \otimes de_j \lrcorner\,) \nabla^2_{h_a \otimes e_i, h_b \otimes e_j} \psi \\ &\qquad = \tfrac{1}{2} D_+^+ D_-^- \psi = -\tfrac{1}{2} (D_-^-)^* D_-^- \psi \,. \end{aligned}$$

This is only a particular case of a more general phenomenon:

Lemma 4.2 *Let V and Σ be the representations spaces for TM and $\mathbf{S}(M)$ and let $p : V \otimes \Sigma \to W$ be an $\mathbf{Sp}(n)\mathbf{Sp}(1)$–equivariant mapping. Consider the equivariant mapping $\Phi : V \otimes (V \otimes \Sigma) \to \Sigma$ given by* $\mathrm{id} \otimes p^* p$ *followed by contraction of the two V–factors. Then the differential operator $\Phi \circ \nabla^2$ is equal to $-P^*P$, where $P = p \circ \nabla$.*

This lemma can be applied to the remaining two projectors. To begin with $\Phi = pr_{-+} \otimes pr_K$ we consider the multiplication $m : H \otimes \mathrm{Sym}^r H \to \mathrm{Sym}^{r+1} H$. Its adjoint m^* is given by $\sum h_i \otimes dh_i \lrcorner_\circ$. The adjoint of the projection $\widetilde{pr}_K : E \otimes \Lambda_\circ^{n-r} E \to K^{n-r}$ is simply the inclusion. By definition, $pr_K : E \otimes E \otimes \Lambda_\circ^{n-r} E \to \Lambda_\circ^{n-r} E$ is $\mathrm{id}_E \otimes \widetilde{pr}_K$ followed by contraction of the two E-factors. Similarly, it is easy to show that $pr_{-+} : H \otimes H \otimes$

$\mathrm{Sym}^r H \to \mathrm{Sym}^r H$ is $\mathrm{id}_H \otimes m^* m$ followed by contraction. The assumption of the lemma is thus satisfied and we conclude that $(pr_{-+} \otimes pr_K) \circ \nabla^2 = -(T^+)^* T^+$.

The same argument goes through for the projector $pr_{+-} \otimes pr_K$. The adjoint of the contraction operator $c : H \otimes \mathrm{Sym}^r H \to \mathrm{Sym}^{r-1} H$, however, is given by $c^* = -\sum h_i \otimes dh_i^\flat \cdot$. Hence, $(pr_{+-} \otimes pr_K) \circ \nabla^2 = (T^-)^* T^-$.

The computed projections of $\nabla^2 \psi$ corresponding to the splitting on the right hand side of (4.21) can be collected as entries of a vector:

$$\Big(-\tfrac{1}{2}(D_+^+)^* D_+^+ \psi \quad \tfrac{1}{2} D_-^+ D_+^- \psi \quad \tfrac{1}{2} D_+^- D_-^+ \psi \quad -\tfrac{1}{2}(D_-^-)^* D_-^- \psi \quad -(T^+)^* T^+ \psi \quad (T^-)^* T^- \psi \Big) . \tag{4.22}$$

To determine the differential operators occuring on the left hand side of the isomorphism, some technical lemmata are needed.

Lemma 4.3

$$\sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} \big((dh_b^\flat \cdot dh_a \lrcorner_\circ + dh_a^\flat \cdot dh_b \lrcorner_\circ) \otimes \sigma_E(de_i^\flat, de_j^\flat) \big) \nabla^2_{h_a \otimes e_i, h_b \otimes e_j} \psi = \tfrac{r(r+2)}{n+2} \tfrac{\kappa}{4} \psi .$$

Proof. The left hand side factorizes over the projection of $\nabla^2 \psi \in \Gamma(TM \otimes TM \otimes \mathbf{S}(M))$ onto $\Gamma(\mathrm{Sym}^2 H \otimes \mathbb{C} \otimes \mathbf{S}(M) \subset \Lambda^2 TM \otimes \mathbf{S}(M))$. Hence it is a curvature term, and only the H–symmetric part $\frac{-\kappa}{8n(n+2)} R_H$ contributes according to Lemma 3.5.

$$\begin{aligned}
&\sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} \big((dh_b^\flat \cdot dh_a \lrcorner_\circ + dh_a^\flat \cdot dh_b \lrcorner_\circ) \otimes \sigma_E(de_i^\flat, de_j^\flat) \big) \nabla^2_{h_a \otimes e_i, h_b \otimes e_j} \\
= \;& \tfrac{1}{2} \sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} \big((dh_b^\flat \cdot dh_a \lrcorner_\circ + dh_a^\flat \cdot dh_b \lrcorner_\circ) \otimes \sigma_E(de_i^\flat, de_j^\flat) \big) \\
& \cdot \tfrac{-\kappa}{8n(n+2)} \big((h_b \cdot h_a^\sharp \lrcorner_\circ + h_a \cdot h_b^\sharp \lrcorner_\circ) \otimes \sigma_E(e_i, e_j) \big) \\
= \;& \tfrac{-\kappa}{4(n+2)} \sum_{a,b=1}^{2} dh_a^\flat \cdot dh_b \lrcorner_\circ (h_a \cdot h_b^\sharp \lrcorner_\circ + h_b \cdot h_a^\sharp \lrcorner_\circ) \\
= \;& \tfrac{-\kappa}{4(n+2)} \sum_{a,b=1}^{2} \big(\sigma_H(dh_b^\flat, h_a) dh_a^\flat \cdot h_b^\sharp \lrcorner_\circ + dh_a^\flat \cdot h_a \cdot dh_b \lrcorner_\circ h_b^\sharp \lrcorner_\circ \\
& + dh_a^\flat \cdot h_b \cdot dh_b \lrcorner_\circ h_a^\sharp \lrcorner_\circ + \sigma_H(dh_b^\flat, h_b) dh_a^\flat \cdot h_a^\sharp \lrcorner_\circ \big) \\
= \;& \tfrac{-\kappa}{4(n+2)} \big(-r - (r-1)r - 2r \big) \\
= \;& \tfrac{r(r+2)}{n+2} \tfrac{\kappa}{4} . \quad \square
\end{aligned}$$

Lemma 4.4

$$\sum_{i,j=0}^{2n} \sum_{a,b=1}^{2} \big(\sigma_H(dh_a^\flat, dh_b^\flat) \otimes (de_j^\flat \wedge_\circ de_i \lrcorner + de_i^\flat \wedge_\circ de_j \lrcorner) \big) \nabla^2_{h_a \otimes e_i, h_b \otimes e_j} = \tfrac{(n+r+2)(n-r)}{n(n+2)} \tfrac{\kappa}{4} .$$

Proof. The situation here is exactly the same as in the preceeding lemma only with the roles of H and E interchanged. The result only depends on the E–symmetric part $\frac{-\kappa}{8n(n+2)} R_E$ because R^{hyper} does not contribute as has been shown in Corollary 3.14.

$$\sum_{i,j=0}^{2n}\sum_{a,b=1}^{2}\left(\sigma_H(dh_a^\flat,dh_b^\flat)\otimes(de_j^\flat\wedge_\circ de_i\lrcorner\ +de_i^\flat\wedge_\circ de_j\lrcorner\ )\right)\nabla^2_{h_a\otimes e_i,h_b\otimes e_j}$$

$$\begin{aligned}
&= \tfrac{1}{2}\textstyle\sum_{i,j=0}^{2n}\sum_{a,b=1}^{2}\left(\sigma_H(dh_a^\flat,dh_b^\flat)\otimes(de_j^\flat\wedge_\circ de_i\lrcorner\ +de_i^\flat\wedge_\circ de_j\lrcorner\ )\right)\\
&\quad\cdot\tfrac{-\kappa}{8n(n+2)}0\left(\sigma_H(h_a,h_b)\otimes(e_j\wedge_\circ e_i^\sharp\lrcorner\ +e_i\wedge_\circ e_j^\sharp\lrcorner\ )\right)\\
&= \tfrac{-\kappa}{4n(n+2)}\textstyle\sum_{i,j=0}^{2n} de_i^\flat\wedge_\circ de_j\lrcorner\,(e_i\wedge_\circ e_j^\sharp\lrcorner\ +e_j\wedge_\circ e_i^\sharp\lrcorner\ )\\
&= \tfrac{-\kappa}{4n(n+2)}\textstyle\sum_{i,j=0}^{2n}\Big(de_i^\flat\wedge_\circ e_i\wedge_\circ de_j\lrcorner\, e_j^\sharp\lrcorner\ +\sigma_E(de_j^\flat,e_i)de_i^\flat\wedge_\circ e_j^\sharp\lrcorner\ +\tfrac{1}{r+1}de_i^\flat\wedge_\circ de_j\wedge_\circ e_i^\sharp\lrcorner\, e_j^\sharp\lrcorner\\
&\quad -de_i^\flat\wedge_\circ e_j\wedge_\circ de_j\lrcorner\, e_i^\sharp\lrcorner\ +\sigma_E(de_j^\flat,e_j)de_i^\flat\wedge_\circ e_i^\sharp\lrcorner\ +\tfrac{1}{r+1}de_i^\flat\wedge_\circ de_j\wedge_\circ e_j^\sharp\lrcorner\, e_i^\sharp\lrcorner\Big)\\
&= \tfrac{-\kappa}{4n(n+2)}\textstyle\sum_{i,j=0}^{2n}\big(-(n-r)+(n-r-1)(n-r)-2n(n-r)\big)\\
&= \tfrac{(n+r+2)(n-r)}{n(n+2)}\tfrac{\kappa}{4}\,. \quad\square
\end{aligned}$$

Lemma 4.5

$$pr_{\mathrm{Sym}^2H}\otimes pr_{\Lambda^2_\circ E}(\nabla^2\psi)=0\,.$$

Proof. The left hand side is the projection of $\nabla^2\psi$ onto $\Gamma(\mathrm{Sym}^2H\otimes\Lambda^2_\circ E\otimes\mathbf{S}(M))\subset\Gamma(\Lambda^2TM\otimes\mathbf{S}(M))$. Hence, it is again a part of the curvature tensor of the manifold. But Lemma 3.5 shows that this contribution does not exist.

The rest is easy. With help of the preceeding lemmata, the projections of $\nabla^2\psi$ onto the irreducible $\mathbf{S}(M)$–summands of the left hand side of $(TM\otimes TM)\otimes\mathbf{S}(M)\cong TM\otimes(TM\otimes\mathbf{S}(M))$ can be given:

$$\begin{aligned}
pr_{\mathbb{C}}\otimes pr_{\mathbb{C}}:\ \nabla^2\psi &\longmapsto \textstyle\sum_{i,j=0}^{2n}\sum_{a,b=1}^{2}\sigma_H(dh_a^\flat,dh_b^\flat)\sigma_E(de_i^\flat,de_j^\flat)\nabla^2_{h_a\otimes e_i,h_b\otimes e_j}\psi\\
&\quad = -\nabla^*\nabla\psi\,,\\
pr_{\mathrm{Sym}^2H}\otimes pr_{\mathbb{C}}:\ \nabla^2\psi &\longmapsto \tfrac{r(r+2)}{n+2}\tfrac{\kappa}{4}\psi\,,\\
pr_{\mathbb{C}}\otimes pr_{\mathrm{Sym}^2E}:\ \nabla^2\psi &\longmapsto \tfrac{(n+r+2)(n-r)}{n(n+2)}\tfrac{\kappa}{4}\psi\,,\\
pr_{\mathrm{Sym}^2H}\otimes pr_{\mathrm{Sym}^2E}:\ \nabla^2\psi &\longmapsto \textstyle\sum_{i,j=0}^{2n}\sum_{a,b=1}^{2}(dh_b^\flat\cdot dh_a\lrcorner_\circ+dh_a^\flat\cdot dh_b\lrcorner_\circ)\\
&\quad \otimes(de_j^\flat\wedge_\circ de_i\lrcorner\ +de_i^\flat\wedge_\circ de_j\lrcorner\ )\nabla^2_{h_a\otimes e_i,h_b\otimes e_j}\psi\\
&\quad =:\mathcal{C}\psi\,,\\
pr_{\mathbb{C}}\otimes pr_{\Lambda^2_\circ E}:\ \nabla^2\psi &\longmapsto \textstyle\sum_{i,j=0}^{2n}\sum_{a,b=1}^{2}\sigma_H(dh_a^\flat,dh_b^\flat)\\
&\quad \otimes\big(de_j^\flat\wedge_\circ de_i\lrcorner\,\omega-de_i^\flat\wedge_\circ de_j\lrcorner\,\omega-\tfrac{n-r}{n}\sigma_E(de_i^\flat,de_j^\flat)\big)\psi\\
&\quad =:\mathcal{L}\psi\,,\\
pr_{\mathrm{Sym}^2H}\otimes pr_{\Lambda^2_\circ E}:\ \nabla^2\psi &\longmapsto 0\,.
\end{aligned}$$

Here, $\mathcal{C}$ and $\mathcal{L}$ are mysterious 2nd order differential operators which are simply defined by the expressions above. As for the right hand the computed projections can be collected in a vector:

$$\Big(-\nabla^*\nabla\psi \quad \tfrac{r(r+2)}{n+2}\tfrac{\kappa}{4}\psi \quad \tfrac{(n+r+2)(n-r)}{n(n+2)}\tfrac{\kappa}{4}\psi \quad \mathcal{C}\psi \quad \mathcal{L}\psi \quad 0\Big)\,. \tag{4.23}$$

Summarizing all results, the full Weitzenböck matrix equation reads as follows:

$$\begin{pmatrix} -\nabla^*\nabla\psi \\ \tfrac{r(r+2)}{n+2}\tfrac{\kappa}{4}\psi \\ \tfrac{(n+r+2)(n-r)}{n(n+2)}\tfrac{\kappa}{4}\psi \\ \mathcal{C}\psi \\ \mathcal{L}\psi \\ 0 \end{pmatrix} = \mathcal{W}\cdot \begin{pmatrix} -\tfrac12 (D_+^+)^* D_+^+\psi \\ \tfrac12 D_-^+ D_+^-\psi \\ \tfrac12 D_+^- D_-^+\psi \\ -\tfrac12 (D_-^-)^* D_-^-\psi \\ -(T^+)^*T^+\psi \\ (T^-)^*T^-\psi \end{pmatrix}. \tag{4.24}$$

5 Proof of the Theorem

The proof of the lower bound for the Dirac spectrum is an easy application of our Weitzenböck matrix $\mathcal{W}$. Nevertheless, before deriving this result we want to give a few other consequences showing how much information is contained in $\mathcal{W}$.

From equation (4.24) it is clear that the matrix $\mathcal{W}$ yields six lineary independent Weitzenböck formulas involving Dirac and twistor operators. Since we have no further information about the operators $\mathcal{C}$ and $\mathcal{L}$, we can use only four of them and look for suitable linear combinations to eliminate certain operators. This is easily done by multiplying equation (4.24) from the left by a row vector $(a_1\ \dots\ a_6)$, where the a_j are arbitrary coefficients. For example, we can eliminate the two twistor operators $T^\pm$ by multiplying the second equation with $\frac{r}{n}$ and subtracting the last. This is obtained by multiplying with $(0\ \ \frac{r}{n}\ \ 0\ \ 0\ \ 0\ \ -1)$, and thus the corresponding Weitzenböck formula can be written as

$$\tfrac{r}{2}\, D_-^-\, D_+^+ \;+\; \tfrac{r(r+2)}{2(r+1)}\, D_-^+\, D_+^- \;+\; \tfrac{r^2}{2(r+1)}\, D_+^-\, D_-^+ \;+\; \tfrac{r^2(r+2)}{2(r+1)^2}\, D_+^+\, D_-^- \;=\; \tfrac{r^2(r+2)}{n(n+2)}\,\tfrac{\kappa}{4}\,. \tag{5.25}$$

This equation also follows from a theorem of Y. Nagatomo and T. Nitta. In [Nag96] they consider Dirac and twistor operators on arbitrary bundles $S^pH\otimes\Lambda^qE$ and prove a general Weitzenböck formula.

Since we have four independent equations for six unknowns ($D_-^-D_+^+\psi$, $D_-^+D_+^-\psi$ etc.) it is possible to eliminate all four twistor operators at once. The corresponding row vector for $r\neq 0$ is $(-1\ \ \frac{1}{n}\ \ 1\ \ 0\ \ 0\ \ -\frac{1}{r})$, and we obtain the well–known Lichnerowicz formula:

$$\nabla^*\nabla + \tfrac{\kappa}{4} \;=\; D_-^+\, D_+^- + D_+^-\, D_-^+ \;=\; D^2\,.$$

Before coming to the most important application of this formalism, the proof of the eigenvalue estimate, let's have a look on the particular form of eigenspinors of D. Since D^2 respects the splitting into the subbundles $\mathbf{S}_r(M)$, some assumptions on the form of an eigenspinor of D can be made. Let $\psi_r\in\Gamma(\mathbf{S}_r(M))$ be an eigenspinor for D^2 with eigenvalue λ^2. By D, it is mapped to $D\psi_r =: \psi_{r-1}+\psi_{r+1}\in\Gamma(\mathbf{S}_{r-1}(M)\oplus\mathbf{S}_{r+1}(M))$. Because of $D^2\psi_r\in\Gamma(\mathbf{S}_r(M))$ we get $D_+^-\psi_{r-1} = 0 = D_-^+\psi_{r+1}$. But starting with ψ_{r-1}, which itself is an eigenspinor of D^2 with the eigenvalue λ^2, we see that $\lambda\psi_{r-1}\pm D\psi_{r-1}\in\Gamma(\mathbf{S}_{r-1}(M)\oplus\mathbf{S}_r(M))$ is an eigenspinor of D with eigenvalue $\pm\lambda$. The analogous argument goes through with ψ_{r+1}. Hence we always can assume that an eigenspinor for D is of the form $\psi=\psi_r+\psi_{r+1}\in\Gamma(\mathbf{S}_r(M)\oplus\mathbf{S}_{r+1}(M))$.

Theorem 5.1 *Let (M^{4n},g) be a compact quaternionic Kähler spin manifold of positive scalar curvature κ and let $\psi=\psi_r+\psi_{r+1}\in\Gamma(\mathbf{S}_r(M)\oplus\mathbf{S}_{r+1}(M))$ be an eigenspinor for D with eigenvalue λ. Then*

$$\lambda^2 \;\geq\; \frac{n+r+3}{n+2}\,\frac{\kappa}{4}\,.$$

Proof. We apply formula (4.24) to the spinor ψ_r and multiply it from the left with the row vector $(0 \quad \frac{n+r+2}{n} \quad r+2 \quad 0 \quad 0 \quad -\frac{r+2}{r})$. This eliminates the operators T^- and D_-^- and we find after taking the L^2-product with ψ_r.

$$-\tfrac{r+1}{n-r+1}\,\|D_+^+\psi_r\|^2 \;+\; (r+2)\,\|D_+^-\psi_r\|^2 \;+\; \tfrac{(r+2)(n+r+2)}{n+r+3}\,\|D_-^+\psi_r\|^2 \;-\; 2(r+1)\,\|T^+\psi_r\|^2$$

$$= \;\tfrac{(r+2)(n+r+2)}{n+2}\,\tfrac{\kappa}{4}\,\|\psi_r\|^2\,.$$

Using $D_+^-\psi_r = 0$ and estimating $\|D_+^+\psi\|^2$ and $\|T^+\psi\|^2$ by zero leads to the inequality

$$\|D_-^+\psi_r\|^2 \;\geq\; \tfrac{n+r+3}{n+2}\,\tfrac{\kappa}{4}\,\|\psi_r\|^2\,.$$

The same procedure is carried out with ψ_{r+1}. The only difference is that in this case $D_-^+\psi_{r+1} = 0$, hence the same inequality holds:

$$\|D_-^+\psi_{r+1}\|^2 \;\geq\; \tfrac{n+r+3}{n+2}\,\tfrac{\kappa}{4}\,\|\psi_{r+1}\|^2\,. \quad \square$$

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