dg-ga/9704002
MPI 97-29

Killing spinors are Killing vector fields in Riemannian Supergeometry

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Abstract

A supermanifold M is canonically associated to any pseudo Riemannian spin manifold (M_0, g_0). Extending the metric g_0 to a field g of bilinear forms $g(p)$ on T_pM, $p \in M_0$, the pseudo Riemannian supergeometry of (M, g) is formulated as G-structure on M, where G is a supergroup with even part $G_0 \cong Spin(k, l)$; (k, l) the signature of (M_0, g_0). Killing vector fields on (M, g) are, by definition, infinitesimal automorphisms of this G-structure. For every spinor field s there exists a corresponding odd vector field X_s on M. Our main result is that X_s is a Killing vector field on (M, g) if and only if s is a twistor spinor. In particular, any Killing spinor s defines a Killing vector field X_s.

1 Introduction to supergeometry

First we introduce the supergeometric language which is needed to formulate the main result of the paper. Standard references on supergeometry are [M], [L] and [K].

1.1 **Supermanifold.** We consider pairs $(M_0, \mathcal{A})$, where M_0 is a C^∞-manifold and $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras; $\dim M_0 = m$.

Example 1: We denote by $\mathcal{C}^\infty_{M_0}$ the sheaf of (smooth) functions of M_0. It associates to an open set $U \subset M_0$ the algebra $\mathcal{C}^\infty_{M_0}(U) = C^\infty(U)$ of smooth functions on U. Let E be a (smooth) vector bundle over M_0 and $\mathcal{E}$ the corresponding locally free sheaf of $\mathcal{C}^\infty_{M_0}$-modules: $\mathcal{E}$ associates to an open set $U \subset M_0$ the $C^\infty(U)$-module $\mathcal{E}(U) = \Gamma(U, E)$

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of sections of E over U. Conversely, any locally free sheaf $\mathcal{E}$ of $\mathcal{C}^\infty_{M_0}$-modules defines a vector bundle $E \to M_0$. The exterior sheaf $\wedge\mathcal{E} = \wedge^{ev}\mathcal{E} + \wedge^{odd}\mathcal{E}$ is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras on M_0.

Definition 1 *The pair $M = (M_0, \mathcal{A})$ is called a (differentiable) supermanifold of dimension $m|n$ over M_0 if for all $p \in M_0$ there exists an open neighborhood $U \ni p$ and a rank n free sheaf $\mathcal{E}_U$ of $\mathcal{C}^\infty_U$-modules over U such that $\mathcal{A}|_U \cong \wedge\mathcal{E}_U$ (as sheaves of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras). The (local) sections of $\mathcal{A}$ are called (local) functions on M.*

From Def. 1 it follows that there exists a canonical epimorphism $\epsilon : \mathcal{A} \to \mathcal{C}^\infty_{M_0}$, which is called the **evaluation map**. Its kernel is the ideal $\mathcal{J}$ generated by $\mathcal{A}_1$: $\ker\epsilon = \mathcal{J} = \langle\mathcal{A}_1\rangle = \mathcal{A}_1 + \mathcal{A}_1^2$. By the construction of Example 1 to any vector bundle $E \to M_0$ we have associated a supermanifold $M(E) = (M_0, \mathcal{A} = \wedge\mathcal{E})$. In this case the exact sequence

$$0 \to \mathcal{J} = \langle\mathcal{E}\rangle \to \mathcal{A} = \wedge\mathcal{E} \xrightarrow{\epsilon} \mathcal{C}^\infty_{M_0} \to 0$$

of sheafs of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras has a canonical splitting $\mathcal{C}^\infty_{M_0} \hookrightarrow \wedge\mathcal{E} = \mathcal{C}^\infty_{M_0} + \langle\mathcal{E}\rangle$.

Let $(x^1, \ldots, x^m)$ be local coordinates for M_0 defined on an open set $U \subset M_0$ such that $\mathcal{A}|_U \cong \wedge\mathcal{E}_U$, where $\mathcal{E}_U$ is a rank n free sheaf of $\mathcal{C}^\infty_U$-modules, cf. Def. 1. Let $\theta_1, \ldots, \theta_n$ be sections of $\mathcal{E}_U$ trivializing the vector bundle E_U associated to the sheaf $\mathcal{E}_U$. Note that $x^1, \ldots, x^m, \theta_1, \ldots, \theta_n$ can be considered as local functions on the supermanifold M. Moreover, any local function $f \in \mathcal{A}(U)$ is of the form

$$f = \sum_{\alpha\in\mathbb{Z}_2^n} f_\alpha(x^1, \ldots, x^m)\theta^\alpha\,, \quad f_\alpha(x^1, \ldots, x^m) \in C^\infty(U) = \mathcal{C}^\infty_{M_0}(U)\,, \tag{1}$$

where $\theta^\alpha := \theta_1^{\alpha_1} \wedge \ldots \wedge \theta_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Definition 2 *The tupel $(x^i, \theta_j) = (x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)$ is called a* **local coordinate system** *for M over U.*

The evaluation map applied to a (local) function $f = f(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)$ with expansion (1) is given by:

$$\epsilon(f) = f(x^1, \ldots, x^m, 0, \ldots, 0) = f_{(0,\ldots,0)}(x^1, \ldots, x^m)\,.$$

Let $M = (M_0, \mathcal{A})$ and $N = (N_0, \mathcal{B})$ be supermanifolds.

Definition 3 *A* **morphism** *$\Phi : M \to N$ is a pair $\Phi = (\varphi, \phi)$, where $\varphi : M_0 \to N_0$ is a differentiable map and $\phi : \mathcal{B} \to \varphi_*\mathcal{A}$ is a morphism of sheaves of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras. Φ is called an* **isomorphism** *if φ is a diffeomorphism and ϕ is an isomorphism. An isomorphism $\Phi : M \to M$ is called* **automorphism** *of M.*

In local coordinate systems (x^i, θ_j) for M and $(\tilde{x}^k, \tilde{\theta}_l)$ for N a morphism Φ is expressed by p even functions $\tilde{x}^k(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)$, $k = 1, \ldots, p$, and odd q functions $\tilde{\theta}_l(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n)$, $l = 1, \ldots, q$; where $(p, q) = \dim N$.

1.2 **Tangent vector/vector field.** Let $M = (M_0, \mathcal{A})$ be a supermanifold. For any point $p \in M_0$ the evaluation map $\epsilon : \mathcal{A} \to \mathcal{C}^\infty_{M_0}$ induces an epimorphism $\epsilon_p : \mathcal{A}_p \to \mathbb{R}$, $\epsilon_p(f) := \epsilon(f)(p)$, where $\mathcal{A}_p$ denotes the stalk of $\mathcal{A}$ at p. For $\alpha \in \mathbb{Z}_2 = \{0, 1\}$ we define

$$(T_pM)_\alpha := \{v : \mathcal{A}_p \to \mathbb{R} \quad \mathbb{R}\text{-linear} | v(fg) = v(f)\epsilon_p(g) + (-1)^{\alpha \tilde{f}} \epsilon_p(f) v(g)\} ,$$

where the equation is required for all $f, g \in \mathcal{A}_p$ of pure degree and $\tilde{f} \in \{0, 1\}$ denotes the degree of f.

Definition 4 *The* **tangent space** *of M at $p \in M_0$ is the $\mathbb{Z}_2$-graded vector space $T_pM = (T_pM)_0 + (T_pM)_1$. The elements of T_pM are called* **tangent vectors**. *Any morphism $\Phi = (\varphi, \phi) : M = (M_0, \mathcal{A}) \to N = (N_0, \mathcal{B})$ induces linear maps $d\Phi(p) : T_pM \to T_{\varphi(p)}N$, defined by $(d\Phi(p)v)(f) := v(\phi_p(f))$, $p \in M_0$, $v \in T_pM$, $f \in \mathcal{B}_{\varphi(p)}$, where $\phi_p : \mathcal{B}_{\varphi(p)} \to \mathcal{A}_p$ is the morphism of stalks associated to $\phi : \mathcal{B} \to \varphi_* \mathcal{A}$. The map $d\Phi(p)$ is called the* **differential at** *p of Φ.*

The sheaf $Der\mathcal{A}$ of derivations of $\mathcal{A}$ over $\mathbb{R}$ is a sheaf of $\mathbb{Z}_2$-graded $\mathcal{A}$-modules: $Der\mathcal{A} = (Der\mathcal{A})_0 + (Der\mathcal{A})_1$, where

$$(Der\mathcal{A})_\alpha = \{X : \mathcal{A} \to \mathcal{A} \quad \mathbb{R}\text{-linear} | X(fg) = X(f)g + (-1)^{\alpha \tilde{f}} f X(g)\} ,$$

where the equation is required for all $f, g \in \mathcal{A}$ of pure degree.

Definition 5 *The sheaf $\mathcal{T}_M = Der\mathcal{A}$ is called the* **tangent sheaf** *of $M = (M_0, \mathcal{A})$. The sections of $\mathcal{T}_M$ are called* **vector fields**.

Any local coordinate system (x^i, θ_j) over U gives rise to even vector fields $\frac{\partial}{\partial x^i}$ and odd vector fields $\frac{\partial}{\partial \theta_j}$ over U. The action of the vector fields $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta_j}$ on a function f with expansion (1) is given by:

$$\frac{\partial f}{\partial x^i} = \sum_\alpha \frac{\partial f_\alpha(x^1, \ldots, x^m)}{\partial x^i} \theta^\alpha ,$$

$$\frac{\partial f}{\partial \theta_j} = \sum_\alpha \alpha_j (-1)^{\alpha_1 + \cdots + \alpha_{j-1}} f_\alpha(x^1, \ldots, x^m) \theta_1^{\alpha_1} \wedge \ldots \wedge \theta_j^{\alpha_j - 1} \wedge \ldots \wedge \theta_n^{\alpha_n} .$$

Any vector field X on M over U can be written as

$$X = \sum_{i=1}^m X^i(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n) \frac{\partial}{\partial x^i} + \sum_{j=1}^n Y^j(x^1, \ldots, x^m, \theta_1, \ldots, \theta_n) \frac{\partial}{\partial \theta_j} ,$$

where $X^i, Y^j \in \mathcal{A}(U)$.

If $\Phi = (\varphi, \phi) : M = (M_0, \mathcal{A}) \to N = (N_0, \mathcal{B})$ is an isomorphism then φ^{-1} and $\phi^{-1} : \varphi_* \mathcal{A} \to \mathcal{B}$ exist and give rise to an isomorphism $\mathcal{A} \to \varphi_*^{-1} \mathcal{B}$. The induced isomorphism between the corresponding sheaves of derivations is denoted by

$$d\Phi : \mathcal{T}_M \to \varphi_*^{-1} \mathcal{T}_N$$

and is called the **differential** of Φ. For any open $U \subset M_0$ the differential $d\Phi$ is expressed by an $\mathcal{A}(U)$-linear map $d\Phi_U : \mathcal{T}_M(U) \to \mathcal{T}_N(\varphi(U))$, where the action of $\mathcal{A}(U)$ on $\mathcal{T}_N(\varphi(U))$ is defined using the isomorphism $\mathcal{A}(U) \xrightarrow{\sim} \mathcal{B}(\varphi(U))$ induced by ϕ^{-1}.

Let X be a vector field defined on some open set $U \subset M_0$ and $p \in U$. Then we can define the **value** $X(p) \in T_pM$ of X at p:

$$X(p)(f) := \epsilon_p(X(f)) , \quad f \in \mathcal{A}_p .$$

However, unless $\dim M = m|n = m|0$, a vector field is not determined by its values.

Finally, we relate the tangent spaces and tangent sheaves of M and M_0. Any even tangent vector $v \in (T_pM)_0$ annihilates the ideal $\mathcal{J} = \ker \epsilon$ in the exact sequence

$$0 \to \mathcal{J} \to \mathcal{A} \xrightarrow{\epsilon} \mathcal{C}^\infty_{M_0} \to 0 \tag{2}$$

and hence defines a tangent vector to M_0. More explicitly, we define a map $\epsilon : T_pM \to T_pM_0$ by the equation

$$\epsilon(v)(\epsilon(f)) = v_0(f) ,$$

where $v = v_0 + v_1 \in (T_pM)_0 + (T_pM)_1$, $f \in \mathcal{A}_p$ and $f \mapsto \epsilon(f)$ is the evaluation map of stalks $\epsilon : \mathcal{A}_p \to (\mathcal{C}^\infty_{M_0})_p$.

Proposition 1 *There is a canonical exact sequence of $\mathbb{Z}_2$-graded vector spaces:*

$$0 \to (T_pM)_1 \to T_pM \xrightarrow{\epsilon} T_pM_0 \to 0 .$$

In particular, ϵ induces a canonical isomorphism $(T_pM)_0 \xrightarrow{\sim} T_pM_0$.

Similarly, on the level of tangent sheaves we define $\epsilon : \mathcal{T}_M \to \mathcal{T}_{M_0}$ by the equation

$$\epsilon(X)(\epsilon(f)) = \epsilon(X_0(f)) ,$$

where $X = X_0 + X_1 \in (\mathcal{T}_M(U))_0 + (\mathcal{T}_M(U))_1$, $f \in \mathcal{A}(U)$ and $U \subset M_0$ open.

Proposition 2 *There is a canonical exact sequence of sheaves of $\mathcal{A}$-modules*

$$0 \to \ker \epsilon \to \mathcal{T}_M \xrightarrow{\epsilon} \mathcal{T}_{M_0} \to 0 , \tag{3}$$

where $\ker \epsilon = (\mathcal{T}_M)_1 + \mathcal{J}\mathcal{T}_M$. In particular, there is the following exact sequence of $\mathcal{A}$-modules:

$$0 \to (\mathcal{J}\mathcal{T}_M)_0 \to (\mathcal{T}_M)_0 \to \mathcal{T}_{M_0} \to 0 .$$

1.3 **Frame/frame field/local coordinates.**

Definition 6 *Let $V = V_0 + V_1$ be a $\mathbb{Z}_2$-graded vector space of* **rank** $m|n$, *i.e.* $\dim V_0 = m$ *and* $\dim V_1 = n$. *A* **basis** *of V is a tupel $(b_1, \ldots, b_{m+n})$ such that $(b_1, \ldots, b_m)$ is a basis of V_0 and $(b_{m+1}, \ldots, b_{m+n})$ is a basis of V_1. Let $M = (M_0, \mathcal{A})$ be a supermanifold and $p \in M_0$. A* **frame** *at p is a basis of T_pM. A tupel $(X_1, \ldots X_{m+n})$ of vector fields defined on an open subset $U \subset M_0$ is called a* **frame field** *if $(X_1(p), \ldots X_{m+n}(p))$ is a frame at all points $p \in U$. We denote by $\mathcal{F}(U)$ the set of all frame fields over U. The sheaf of sets $U \mapsto \mathcal{F}(U)$ is called the* **sheaf of frame fields**.

Any local coordinate system (x^i, θ_j) over U gives rise to the frame field $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta_j})$ over U.

1.4 **Supergroup.** Let $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$ be an associative $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra with unit. We will always assume that $\mathbf{A}$ is **supercommutative**, i.e. $ab = (-1)^{\tilde{a}\tilde{b}}ba$ for all $a, b \in \mathbf{A}_0 \cup \mathbf{A}_1$. Under this assumption any left-$\mathbf{A}$-module carries a canonical right-$\mathbf{A}$-module structure and vice versa; so we will simply speak of $\mathbf{A}$-modules. For any supermanifold $M = (M_0, \mathcal{A})$ the algebra of functions $\mathcal{A}(M_0)$ is supercommutative, associative and has a unit.

For any set Σ and non-negative integers r, s we denote by $Mat(r, s, \Sigma)$ the set of $r \times s$-matrices with entries in Σ and put $Mat(r, \Sigma) := Mat(r, r, \Sigma)$. Any partition $(r = m + n, s = k + l)$ defines a $\mathbb{Z}_2$-grading on the $\mathbf{A}$-module $V = Mat(r, s, \mathbf{A})$:

$$\begin{aligned} V_0 &= \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in Mat(m, k, \mathbf{A}_0),\ D \in Mat(n, l, , \mathbf{A}_0), \\ &\qquad B \in Mat(m, l, \mathbf{A}_1),\ C \in Mat(n, k, \mathbf{A}_1)\}\,, \\ V_1 &= \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in Mat(m, k, \mathbf{A}_1),\ D \in Mat(n, l, , \mathbf{A}_1), \\ &\qquad B \in Mat(m, l, \mathbf{A}_0),\ C \in Mat(n, k, \mathbf{A}_0)\}\,. \end{aligned}$$

The $\mathbb{Z}_2$-graded $\mathbf{A}$-module $V = V_0 + V_1$ is denoted by $Mat(m|n, k|l, \mathbf{A})$. Matrix multiplication turns $Mat(m|n, \mathbf{A}) := Mat(m|n, m|n, \mathbf{A})$ into an associative $\mathbb{Z}_2$-graded algebra with unit.

Definition 7 *A* **super Lie bracket** *on a $\mathbb{Z}_2$-graded vector space $V = V_0 + V_1$ is a bilinear map $[\cdot, \cdot] : V \times V \to V$ such that for all $x, y, z \in V_0 \cup V_1$ we have:*

i) $\widetilde{[x, y]} = \tilde{x} + \tilde{y}$,

ii) $[x, y] = -(-1)^{\tilde{x}\tilde{y}}[y, x]$ *and*

iii) $[x, [y, z]] = [[x, y], z] + (-1)^{\tilde{x}\tilde{y}}[y, [x, z]]$.

The pair $(V, [\cdot,\cdot])$ *is called a* **super Lie algebra**.

The supercommutator

$$[X,Y] = XY - (-1)^{\tilde{X}\tilde{Y}} YX\,, \quad X,Y \in Mat(m|n,\mathbf{A})_0 \cup Mat(m|n,\mathbf{A})_1$$

defines a super Lie bracket on the $\mathbb{Z}_2$-graded vector space $Mat(m|n,\mathbf{A})$. The super Lie algebra $(Mat(m|n,\mathbf{A}), [\cdot,\cdot])$ is denoted by $\mathfrak{gl}_{m|n}(\mathbf{A})$. We put

$$GL_{m|n}(\mathbf{A}) := \{g \in Mat(m|n,\mathbf{A})_0 | g \quad \text{is invertible}\}\,.$$

Similarly, if V is a $\mathbb{Z}_2$-graded $\mathbf{A}$-module $End_{\mathbf{A}}(V)$ carries a canonical super Lie algebra structure, which is denoted by $\mathfrak{gl}_{\mathbf{A}}(V)$. By definition $GL_{\mathbf{A}}(V)$ is the group of invertible elements of $End_{\mathbf{A}}(V)$. Finally, we will use the convention $\mathfrak{gl}_{m|n} := \mathfrak{gl}_{m|n}(\mathbb{R})$, $\mathfrak{gl}(V) := \mathfrak{gl}_{\mathbb{R}}(V)$, $GL(V) := GL_{\mathbb{R}}(V)$.

Definition 8 *A* **supergroup** G *is a contravariant functor* $M \mapsto G(M)$ *from the category of supermanifolds into the category of groups. Let* H*,* G *be supergroups. We say that* H *is a super subgroup of* G *and write* $H \subset G$ *if* $H(M) \subset G(M)$ *is a subgroup and* $H(\Phi) = G(\Phi)|H(N)$ *for all supermanifolds* M*,* N *and morphisms* $\Phi : M \to N$*.*

Example 2: The **general linear supergroup** $GL_{m|n}$ is the supergroup $M \to GL_{m|n}(M)$ obtained as composition of the following two functors:

i) the contravariant functor $M = (M_0, \mathcal{A}) \to \mathcal{A}(M_0)$ from the category of supermanifolds into that of asssociative, supercommutative algebras with unit,

ii) the covariant functor $\mathbf{A} \to GL_{m|n}(\mathbf{A})$ from the category of associative, supercommutative algebras with unit into that that of groups.

Definition 9 *A* **linear super Lie algebra** $\mathfrak{g}$ *is a super Lie subalgebra* $\mathfrak{g} \subset \mathfrak{gl}_{m|n}$ *(for some* $m|n$*). A* **linear supergroup** *is a super subgroup* $G \subset GL_{m|n}$ *(for some* $m|n$*).*

Example 3: Let $\mathfrak{g} \subset \mathfrak{gl}_{m|n}$ be a linear super Lie algebra. For any associative, supercommutative algebra with unit $\mathbf{A}$ we can consider the super Lie algebra $\mathfrak{g} \otimes \mathbf{A} \subset \mathfrak{gl}_{m|n}(\mathbf{A})$. Its even part $\mathfrak{g}(\mathbf{A}) := (\mathfrak{g} \otimes \mathbf{A})_0$ is a Lie algebra. If $\mathbf{A} = \mathcal{A}(M_0)$ is the algebra of functions of a supermanifold $M = (M_0, \mathcal{A})$ then it is easy to see that the exponential series

$$\sum_{n=0}^{\infty} \frac{1}{n!} X^n\,, \quad X \in Mat(m|n,\mathbf{A})\,,$$

converges (locally uniformly) to an element $\exp X \in GL_{m|n}(\mathbf{A})$. Now let $G(\mathbf{A})$ be the subgroup of $GL_{m|n}(\mathbf{A})$ generated by $\exp \mathfrak{g}(\mathbf{A})$. then the functor $M = (M_0, \mathcal{A}) \mapsto G(M) := G(\mathcal{A}(M_0))$ is a linear supergroup, which we denote by $\exp \mathfrak{g}$.

1.5 **G-structure.** Let $M = (M_0, \mathcal{A})$ be a super manifold of $\dim M = m|n$. For any open subset $U \subset M_0$ we consider the supermanifold $M|_U := (U, \mathcal{A}|_U)$. The general linear supergroup $GL_{m|n}$ induces a sheaf $\mathcal{GL}_M$ of groups over M_0: $\mathcal{GL}_M(U) := GL_{m|n}(M|_U) = GL_{m|n}(\mathcal{A}(U))$, $U \subset M_0$ open. The group $\mathcal{GL}_M(U)$ acts naturally (from the right) on the set $\mathcal{F}(U)$ of frame fields over U. This action turns $\mathcal{F}$ into a sheaf of $\mathcal{GL}_M$-sets. Now let $G \subset GL_{m|n}$ be a linear supergroup and $\mathcal{G}$ the corresponding sheaf of groups, i.e. $\mathcal{G}(U) = G(M|_U)$ for all open $U \subset M_0$. Since $\mathcal{G}$ is a sheaf of subgroups $\mathcal{G} \subset \mathcal{GL}_M$ the sheaf $\mathcal{F}$ of frame fields of M is, in particular, a sheaf of $\mathcal{G}$-sets.

Definition 10 *Let $M = (M_0, \mathcal{A})$, $\dim M = m|n$, be a supermanifold and $G \subset GL_{m|n}$ a linear supergroup. A* **G-structure** *on M is a sheaf $\mathcal{F}_G$ of $\mathcal{G}$-subsets $\mathcal{F}_G \subset \mathcal{F}$ such that for all $p \in M_0$ there exists an open neighborhood $U \ni p$ for which $\mathcal{G}(U)$ acts simply transitively on $\mathcal{F}_G(U)$.*

Example 4: For any supermanifold M, $\dim M = m|n$, the sheaf of frame fields $\mathcal{F}$ is a $GL_{m|n}$-structure.

1.6 **Automorphism of G-structure.** We denote by $Aut(M)$ the group of all automorphisms of the supermanifold M, see Def. 3. The differential $d\Phi : \mathcal{T}_M \to \varphi_*^{-1}\mathcal{T}_M$ of any $\Phi = (\varphi, \phi) \in Aut(M)$ induces an isomorphism $\mathcal{F} \to \varphi_*^{-1}\mathcal{F}$, again denoted by $d\Phi$. Now let $\mathcal{F}_G \subset \mathcal{F}$ be a G-structure on M, for some linear supergroup $G \subset GL_{m|n}$. For simplicity we can assume that $G = \exp \mathfrak{g}$ as in Example 3.

Definition 11 *$\Phi = (\varphi, \phi) \in Aut(M)$ is called an* **automorphism** *of the G-structure $\mathcal{F}_G$ if $d\Phi\mathcal{F}_G \subset \varphi_*^{-1}\mathcal{F}_G$.*

We recall that any $p \in M_0$ has an open neighborhood U such that $\mathcal{G}(U)$ acts simply transitively on $\mathcal{F}_G(U)$. Such open sets $U \subset M_0$ will be called **small**. If $U \subset M_0$ is small then $\mathcal{F}_G(U) = E\mathcal{G}(U)$ for any frame field $E \in \mathcal{F}_G(U)$. Here the right-action of the group $\mathcal{G}(U)$ on $\mathcal{F}_G(U)$ is simply denoted by juxtaposition.

Proposition 3 *$\Phi \in Aut(M)$ is an automorphism of the G-structure $\mathcal{F}_G$ iff*

$$d\Phi_{U'} E|_{U'} \in E|_{\varphi(U')}\mathcal{G}(\varphi(U'))$$

for all small $U \subset M_0$, $E \in \mathcal{F}_G(U)$ and open $U' \subset U$ such that $\varphi(U') \subset U$.

For any open set $U \subset M_0$ the vector space $\mathcal{T}_M(U)^{m+n}$ of $(m+n)$-tupels of vector fields is naturally a right-module of the associative, $\mathbb{Z}_2$-graded algebra $Mat(m|n, \mathcal{A}(U))$. In particular, it is a right-module of the super Lie algebra $\mathfrak{g} \otimes \mathcal{A}(U) \subset \mathfrak{g}_{m|n}(\mathcal{A}(U))$. On the other hand, $\mathcal{T}_M(U)$ (and hence $\mathcal{T}_M(U)^{m+n}$) is naturally a left-module for the super Lie algebra $\mathcal{T}_M(U)$ of local vector fields. The action on $\mathcal{T}_M(U)$ is given by the adjoint representation, i.e. by the supercommutator $ad_X Y = X \circ Y - (-1)^{\tilde{X}\tilde{Y}} Y \circ X$, $X, Y \in \mathcal{T}_M(U)$ of pure degree. The corresponding action on $\mathcal{T}_M(U)^{m+n}$ is denoted by L_X ("Lie derivative"):

$$L_X E := ([X, X_1], \ldots, [X, X_{m+n}]), \quad E = (X_1, \ldots, X_{m+n}) \in \mathcal{T}_M(U)^{m+n}.$$

Proposition 3 motivates the following definition.

Definition 12 *A vector field X on M is an* **infinitesimal automorphism** *of the G-structure $\mathcal{F}_G$ if*

$$L_{X|_U} E|_U \in E|_U(\mathfrak{g} \otimes \mathcal{A}(U))$$

for all small $U \subset M_0$, $E \in \mathcal{F}_G(U)$.

2 Supergeometry associated to the spinor bundle

2.1 **The supermanifold M(S).** Let (M_0, g_0) be a (smooth) pseudo Riemannian spinmanifold with spinor bundle $S \to M_0$. The corresponding locally free sheaf of $\mathcal{C}^\infty_{M_0}$-modules will be denoted by $\mathcal{S}$; $\mathcal{S}(U) = \Gamma(U, S)$, $U \subset M_0$ open. To the vector bundle $S \to M_0$ we associate the supermanifold $M : M(S) = (M_0, \mathcal{A} = \wedge \mathcal{S})$.

Consider the $\mathbb{Z}_2$-graded vector bundle $TM_0 + S^* \to M_0$ with even part TM_0 and odd part S^*.

Proposition 4 *For any $p \in M_0$ there is a canonical isomorphism of $\mathbb{Z}_2$-graded vector spaces $\iota_p : T_pM_0 + S^*_p \xrightarrow{\sim} T_pM$.*

Proof: We define $\iota_p^{-1}|(T_pM)_0 := \epsilon|(T_pM)_0$, see Prop. 1. Now it is sufficient to construct a canonical isomorphism $S^* \xrightarrow{\sim} (T_pM)_1$. For any section $s \in \Gamma(U, S^*)$ interior multiplication $\iota(s)$ by s defines an odd derivation of the $\mathbb{Z}_2$-graded algebra $\mathcal{A}(U) = \Gamma(U, \wedge S)$, i.e. a vector field $X_s := \iota(s) \in \mathcal{T}_M(U)_1$. The value $X_s(p) \in (T_pM)_1$ depends only on $s(p) \in S^*_p$ and we can define $\iota_p(s(p)) := X_s(p)$. $\square$

Using the embedding $\mathcal{C}^\infty_{M_0} \hookrightarrow \wedge \mathcal{S}$, we can consider $\mathcal{T}_M$ as a sheaf of $\mathcal{C}^\infty_{M_0}$-modules. Interior multiplication $s \mapsto \iota(s) = X_s$ defines a monomorphism $S^* \hookrightarrow (\mathcal{T}_M)_1$ of sheaves of $\mathcal{C}^\infty_{M_0}$-modules. We want to extend this map to $\iota : \mathcal{T}_{M_0} + \mathcal{S}^* \to \mathcal{T}_M$. For a local vector field $X \in \mathcal{T}_{M_0}(U)$ on M_0 we put

$$\iota(X) := \nabla_X \in \mathcal{T}_M(U)_0,$$

where ∇ is the canonical connection on $\wedge S$, i.e. the one induced by the Levi-Civita-connection on (M_0, g_0).

Proposition 5 *The map* $\iota : \mathcal{T}_{M_0} + \mathcal{S}^* \to \mathcal{T}_M$ *is a monomorphism of sheaves of* $\mathbb{Z}_2$*-graded* $\mathcal{C}^\infty_{M_0}$*-modules. Moreover,* $\iota|\mathcal{T}_{M_0}$ *defines a splitting of the sequence (2), i.e.* $\epsilon \circ \iota|\mathcal{T}_{M_0} = id$.

Note that given any vector bundle E and connection D on E we can canonically define $\iota_{E,D} : \mathcal{T}_{M_0} + \mathcal{E}^* \hookrightarrow \mathcal{T}_M$, where $M = M(E)$ and $\mathcal{E}$ is the sheaf of local sections of E. In Prop. 5 we have $\iota = \iota_{S,\nabla}$.

2.2 **The coadjoint representation of the Poincaré super Lie algebras.** Let $(V_0, \langle\cdot,\cdot\rangle)$ be a pseudo Euclidean vector space of signature (k, l), $k + l = m$, and V_1 the spinor module of the group $Spin(V_0)$, $n := \dim V_1 = 2^{[\frac{m}{2}]}$. Put $V := V_0 + V_1$. The vector space $\mathfrak{p}(V) := \mathfrak{spin}(V_0) + V$ carries the structure of $\mathfrak{spin}(V_0)$-module. We want to extend this structure to a super Lie bracket $[\cdot,\cdot]$ on $\mathfrak{p}(V)$ which satisfies $[V_0, V] = 0$ and $[V_1, V_1] \subset V_0$. Such an extension is precisely given by a $Spin(V_0)$-equivariant map $\pi : \vee^2 V_1 \to V_0$; here $\vee^2$ denotes the symmetric square.

Definition 13 *The structure of super Lie algebra defined on* $\mathfrak{p}(V)$ *by the map* π *is called a* **Poincaré super Lie algebra**.

We denote by $\rho : V_0 \to End(V_1)$ the (standard) Clifford multiplication.

Definition 14 *A bilinear form* β *on the spinor module is called* **admissible** *if*

1) *β is symmetric or skew symmetric. We define the symmetry σ of β to be $\sigma(\beta) = +1$ in the first case and $\sigma(\beta) = -1$ in the second.*

2) *Clifford multiplication $\rho(v)$, $v \in V_0$, is either symmetric or skew symmetric. Accordingly, we define the type τ of β to be $\tau(\beta) = \pm 1$.*

An admissible form β *is called* **suitable** *if* $\sigma(\beta)\tau(\beta) = +1$.

Given a suitable bilinear form β on V_1 we define $\pi = \pi_{\rho,\beta} : \vee^2 V_1 \to V_0$ by

$$\langle \pi(s_1 \vee s_2), v\rangle = \beta(\rho(v)s_1, s_2), \quad s_1, s_2 \in V_1, \quad v \in V_0 . \tag{4}$$

The map π is $Spin(V_0)$-equivariant. Hence it defines on the vector space $\mathfrak{p}(V)$ the structure of Poincaré super Lie algebra. The following theorem was proved in [A-C].

Theorem 1 *Any* $Spin(V_0)$*-equivariant map* $\vee^2 V_1 \to V_0$ *is a linear combination of maps* π_{ρ,β_i}, β_i *suitable.*

All admissible bilinear forms on the spinor module were explicitly determined in [A-C]. The spinor module carries a **non-degenerate** suitable bilinear form β for all values of $m = k+l$ and $s = k-l$ except for $(m,s) = (5,7)$, $(6,0)$, $(6,6)$ and $(7,7) \pmod{(8,8)}$. Now we assume that a non-degenerate suitable bilinear form β on V_1 is given. The map $\pi = \pi_{\rho,\beta}$ defines on $\mathfrak{p}(V)$ the structure of Poincaré super Lie algebra such that $[V_1, V_1] = V_0$.

Given a super Lie algebra $\mathfrak{g}$ the **coadjoint representation** $ad^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$, $x \mapsto ad^*_x$, is defined by the equation

$$ad^*_x(y^*) = -(-1)^{\tilde{x}\tilde{y^*}} y^* \circ ad_x \,,$$

for $x \in \mathfrak{g}$ and $y^* \in \mathfrak{g}^*$ of pure degree.

Proposition 6 *The coadjoint representation of $\mathfrak{p}(V)$ preserves the subspace $V^\perp = \{x^* \in \mathfrak{p}(V)^* | x^*(V) = 0\} \subset \mathfrak{p}(V)^*$ and hence induces a representation $\alpha : \mathfrak{p}(V) \to \mathfrak{gl}(V^*)$ on $V^* \cong \mathfrak{p}(V)^*/V^\perp$. It has kernel $\ker\alpha = V_0$ and therefore induces a faithful representation of the super Lie algebra $\mathfrak{p}(V)/V_0$ on V^*.*

Once we choose a basis $b = (b_1, \ldots, b_{m+n})$ of V^*, we can identify $\alpha(\mathfrak{p}(V)) \subset \mathfrak{gl}(V^*)$ with a subalgebra $\alpha(\mathfrak{p}(V))^b \subset \mathfrak{gl}_{m|n}$, where $A \mapsto A^b$ denotes the isomorphism $\mathfrak{gl}(V^*) \to \mathfrak{gl}_{m|n}$ defined by b. If moreover $(b_1, \ldots, b_m)$ is an orthonormal basis of $V_1^\perp \cong V_0^*$ then the even part $\alpha(\mathfrak{p}(V))_0^b \cong \mathfrak{spin}(k,l)$ is a canonically embedded spinor Lie algebra, i.e.

$$\alpha(\mathfrak{p}(V))_0^b = \mathfrak{spin}_\sigma := \{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} | A \in \mathfrak{so}(k,l) \subset \mathfrak{gl}_m \} \,,$$

where $\sigma : \mathfrak{so}(k,l) \to \mathfrak{gl}_n$ is equivalent to the spinor representation.

The linear group $Spin_\sigma \subset GL_{m|n}(\mathbb{R})$ generated by the Lie algebra $\mathfrak{spin}_\sigma \subset (\mathfrak{gl}_{m|n})_0 \cong \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ acts on the set of bases of V^* from the right.

Proposition 7 *Assume that $\alpha(\mathfrak{p}(V))_0^b = \mathfrak{spin}_\sigma$ and $b' = bg$ for some $g \in Spin_\sigma$. Then $\alpha(\mathfrak{p}(V))^b = \alpha(\mathfrak{p}(V))^{b'}$.*

Proof: This follows from the fact that $\alpha(\mathfrak{p}(V))_0^b = \mathfrak{spin}_\sigma$ and $\alpha(\mathfrak{p}(V))_1^b = \alpha(V_1)^b$ are invariant under $\mathfrak{spin}_\sigma = \alpha(\mathfrak{spin}(V_0))^b$. $\square$

Now let $(e_1, \ldots, e_m)$ be an orthonormal basis of V_0 and $(\theta^1, \ldots, \theta^n)$ a basis of V_1. The dual bases of V_0^* and V_1^* will be denoted by (e^i) and (θ_j).

Proposition 8 *With respect to the basis $b = (e^1, \ldots, e^m, \theta_1, \ldots, \theta_n)$ of $V^* \cong V_0^* + V_1^*$ the super Lie algebra $\alpha(\mathfrak{p}(V)) \subset \mathfrak{gl}(V^*)$ is identified with*

$$\alpha(\mathfrak{p}(V))^b = \{ \begin{pmatrix} A & 0 \\ C & \sigma(A) \end{pmatrix} | A \in \mathfrak{so}(k,l),\ C^{ji} = e^i(\pi(s \vee \theta^j)),\ s \in V_1 \} \,,$$

where $C = (C^{ji})$, $j = 1, \ldots, n$, $i = 1, \ldots, m$, and $\sigma : \mathfrak{so}(k,l) \to \mathfrak{gl}_n$ is equivalent to the spinor representation.

2.3 **The (pseudo) Riemannian supergeometry associated to the spinor bundle.** Now we carry over the construction of 2.2 to the $\mathbb{Z}_2$-graded vector bundle $V := TM_0 + S$ over M_0. We assume that M_0 is simply connected. The vector bundle V carries the canonical connection induced by the Levi-Civita connection of the pseudo Riemannian manifold (M_0, g_0). The holonomy algebra of V at $p \in M_0$ is a subalgebra of $\mathfrak{spin}(T_pM_0) \subset \mathfrak{gl}(V_p)_0$. This implies, in particular, that the bundle of $Spin(TM_0)$-invariant bilinear forms on S is flat. Let g_1 be a parallel non-degenerate suitable bilinear form on S, see Def. 14 and the remarks following Thm. 1.

The $Spin(TM_0)$-invariant bilinear form $g = g_0 + g_1$ on V should be thought of as a pseudo Riemannian metric for the supermanifold $M = M(S)$. Note that, due to Prop. 4, $g(p)$ induces a non-degenerate bilinear form on T_pM. However, recall that g_1 is symmetric or skew-symmetric. The map $\pi = \pi_{\rho,g_1} : \vee^2 S \to TM_0$ defines on $\mathfrak{p}(V) = \mathfrak{spin}(TM_0) + S \subset \mathfrak{gl}(V)$ the structure of bundle of Poincaré super Lie algebras. $\mathfrak{p}(V)$ is a parallel bundle. Now let $\alpha : \mathfrak{p}(V) \to \mathfrak{gl}(V^*)$ be the field of representations induced by the coadjoint representation, cf. Prop. 6. The image $\alpha(\mathfrak{p}(V)) \subset \mathfrak{gl}(V^*)$ is a parallel bundle of super Lie algebras.

Proposition 9 *The frame bundle of $V^* \to M$ has a subbundle P_{Spin_σ} with structure group $Spin_\sigma \subset GL_{m|n}(\mathbb{R})$, $Spin_\sigma \cong Spin(k,l)$, such that for all $b = (e^i, \theta_j) \in (P_{Spin_\sigma})_p$:*

1) (e^i) is an orthonormal basis of $T_p^*M_0$ and

2) $\alpha(\mathfrak{p}(V_p))$ is identified via b with the subalgebra $\mathfrak{g} = \alpha(\mathfrak{p}(V_p))^b \subset \mathfrak{gl}_{m|n}(\mathbb{R})$, where

$$\mathfrak{g}_0 = \mathfrak{spin}_\sigma = \{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} | A \in \mathfrak{so}(k,l)\} \quad \textit{and}$$

$$\mathfrak{g}_1 = \{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} | C = (C^{ji}),\ C^{ji} = e^i(\pi(s \vee \theta^j)),\ s \in S_p\}$$

are independent of b and p. Here (θ^j) is the basis of S_p dual to (θ_j).

Proof: This follows from the holonomy reduction and Propositions 7 and 8. □

We denote by $\mathcal{V}$ the sheaf of local sections of V. Identifying TM_0 and T^*M_0 via g_0, the map ι of Prop. 5 corresponds to a monomorphism $\iota : \mathcal{V} = \mathcal{T}^*_{M_0} + \mathcal{S}^* \hookrightarrow \mathcal{T}_M$. This induces a map

$$\iota : \Gamma(U, P_{Spin_\sigma}) \to \mathcal{F}(U)\,,$$

where $\mathcal{F}(U)$ is the set of frame fields of M over the open set $U \subset M_0$. The image of ι generates a $Spin_\sigma$-structure on M, where $Spin_\sigma$ is now considered as (purely even) linear supergroup $Spin_\sigma \subset GL_{m|n}$. More precisely, recall that $Spin_\sigma(\mathcal{A}(U))$ is the group generated by $\exp \mathfrak{spin}_\sigma(\mathcal{A}(U)) \subset GL_{m|n}(\mathcal{A}(U))$. It acts on $\mathcal{F}(U)$ from the right. Put

$$\mathcal{F}_{Spin_\sigma}(U) := \iota(\Gamma(U, P_{Spin_\sigma}))Spin_\sigma(\mathcal{A}(U)).$$

Proposition 10 *$\mathcal{F}_{Spin_\sigma}$ is a $Spin_\sigma$-structure on M.*

Denote by G the linear supergroup defined by the linear super Lie algebra $\mathfrak{g}$, see Example 3. Since $\mathfrak{spin}_\sigma \subset \mathfrak{g} \subset \mathfrak{gl}_{m|n}(\mathbb{R})$, we have the following inclusions of linear supergroups:

$$Spin_\sigma \subset G \subset GL_{m|n}. \tag{5}$$

Put $\mathcal{F}_G(U) := \mathcal{F}_{Spin_\sigma}(U)G(\mathcal{A}(U))$ for all open $U \subset M_0$.

Proposition 11 *$\mathcal{F}_G$ is a G-structure on M.*

Definition 15 *A* **Killing vector field** *on (M, g) is an infinitesimal automorphism of the G-structure $\mathcal{F}_G$, see Def. 12.*

2.4 **Twistor spinors as Killing vector fields.**

Definition 16 *A section s of the spinor bundle $S \to M_0$ is called a* **twistor spinor** *if there exists a section $\tilde{s}$ of S such that*

$$\nabla_X s = \rho(X)\tilde{s} \tag{6}$$

for all vector fields X on M_0. Here $\rho(X) : S \to S$ is Clifford multiplication. A twistor spinor s is called a Killing spinor if $\tilde{s} = \lambda s$ for some constant $\lambda \in \mathbb{R}$

Remark: From (6) it follows that $\tilde{s} = -\frac{1}{m}Ds$, where D is the Dirac operator.

The non-degenerate bilinear form g_1 on S induces the isomorphism

$$S \xrightarrow{\sim} S^*, \quad s \mapsto s^* := g_1(s, \cdot).$$

Recall that $\iota|\mathcal{S}^* : \mathcal{S}^* \hookrightarrow \mathcal{T}_M$ is simply given by interior multiplication, s. 2.1. To any spinor field S we associate the odd vector field $X_s := \iota(s^*)$ on M. Now we can state the main result of this paper.

Theorem 2 *Let (M_0, g_0) be a pseudo Riemannian spin manifold with spinor bundle (S, g_1); g_1 a parallel non-degenerate suitable bilinear form on S, see Def. 14 and 2.3. Consider the supermanifold $M = M(S)$ with the bilinear form $g = g_0 + g_1$ and let s be a section of S. The vector field X_s is a Killing vector field on (M, g) iff s is a twistor spinor, see Def. 15 and 16.*

Corollary 1 *A Killing vector field X_s for an extension g of g_0 is a Killing vector field for any other extension; the extensions beeing as in 2.3.*

Lemma 1 *For all sections s^*, t^* of S^* and X of TM_0 we have:*

i) $[\iota(s^*), \iota(t^*)] = 0$,

ii) $[\iota(s^*), \iota(X)] = [\iota(s^*), \nabla_X] = -\iota((\nabla_X)^*)$.

Proof: i) By definition of the supercommutator $[\cdot,\cdot]$ on $\mathcal{T}_M$, we have $[\iota(s^*), \iota(t^*)] = \iota(s^*) \circ \iota(t^*) + \iota(t^*) \circ \iota(s^*) = 0$.
ii) Recall that $s^* = g_1(s, \cdot)$. If t is a section of S we have $[\iota(s^*), \iota(X)](t) = s^*(\nabla_X t) - \nabla_X s^*(t) = g_1(s, \nabla_X t) - \nabla_X g_1(s,t) = -g_1(\nabla_X s, t) = -(\nabla_X s)^*(t)$. $\Box$

Proposition 12 *Let s be a twistor spinor. For all vector fields X and spinor fields t on M_0 we have:*

i) $[\iota(s^*), \iota(X)] = -\iota((\rho(X)\tilde{s})^*) = -\tau(g_1)\iota(\rho(X)^*\tilde{s}^*)$, *where $\tau(g_1) \in \{\pm 1\}$ is the type of g_1, see Def. 14.*

ii) $[\iota(s^*), \iota(X)](t) = -g_1(\rho(X)\tilde{s}, t) = -g_0(\pi(\tilde{s} \vee t), X)$.

Proof: The first equation of i) follows from Lemma 1 ii), since $\nabla_X s = \rho(X)\tilde{s}$. Now the second equation of i) and the first equation of ii) follow from the definition of the type τ: $(\rho(X)\tilde{s})^*(t) = g_1(\rho(X)\tilde{s}, t) = \tau(g_1) g_1(\tilde{s}, \rho(X)t)$. The last equation of ii) is simply the definition of $\pi = \pi_{\rho, g_1}$, cf. (4). $\Box$

Proof (of Theorem 2): Let $(e^i, \theta_j) \in \Gamma(U, P_{Spin_\sigma})$, $U \subset M_0$ open, and (e_i, θ^j) the dual local frame for $V = TM_0 + S$. Put

$$E := (\iota(e^i), \iota(\theta_j)) \in \Gamma(U, \mathcal{F}_{Spin_\sigma}) \subset \Gamma(U, \mathcal{F}_G) \,.$$

Since (e_i) is orthonormal, i.e. $g_0(e_i, e_j) = \varepsilon_i \delta_{ij}$, $\varepsilon_i \in \{\pm 1\}$, we have $e^i = \varepsilon_i g_0(e_i, \cdot)$. Hence, by definition of ι on $\mathcal{T}^*_{M_0}$, we have $\iota(e^i) = \varepsilon_i \iota(e_i)$. Therefore by Lemma 1 for any $s \in \Gamma(U, S)$ we have

$$L_{X_s} E = ([X_s, \iota(e^i)], [X_s, \iota(\theta_j)]) = (-\varepsilon_i \iota((\nabla_{e_i} s)^*), 0) \,, \tag{7}$$

$$(\nabla_{e_i} s)^*(\theta^j) = g_1(\nabla_{e_i} s, \theta^j) \,. \tag{8}$$

From this computation it follows that $L_{X_s} E \in E(\mathfrak{g} \otimes \mathcal{A}(U))$ iff there exists a $t \in \Gamma(U, S)$ such that

$$L_{X_s} E = E C_t \,, \quad \text{where} \tag{9}$$

$$C_t = \begin{pmatrix} 0 & 0 \\ (C_t^{ji}) & 0 \end{pmatrix} \in \mathfrak{g} \otimes \mathcal{A}(U)\,, \quad C_t^{ji} = e^i(\pi(t \vee \theta^j))\,, \tag{10}$$

see Prop. 8. By (7), (8) and (10) equation (9) is equivalent to

$$g_1(\nabla_{e_i} s, \theta^j) = -\varepsilon_i e^i(\pi(t \vee \theta^j))\,, \quad i = 1, \ldots, m,\ j = 1, \ldots, n. \tag{11}$$

The right-hand-side is

$$-\varepsilon_i e^i(\pi(t \vee \theta^j)) = -g_0(\pi(t \vee \theta^j), e_i) = -g_1(\rho(e_i)t, \theta^j)\,, \tag{12}$$

hence (11) is equivalent to the twistor equation (6) with $\tilde{s} = -t$. $\Box$

Acknowledgments. D.V.A. is grateful to Max-Planck-Institut für Mathematik and V.C. thanks the Mathematical Sciences Research Institute, S.-S. Chern and R. Osserman, for hospitality and support. V.C. would also like to thank W. Kramer, E. Poletaeva and V. Serganova for discussions related to the subject of this paper. Three of us (D.V.A., C..D., U.S.) should like to thank the Mathematisches Forschungsinstitut Oberwolfach, where this work was begun, for hospitality in the framework of the Research-in-Pairs programme supported by the Volkswagen Stiftung.

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