KILLING FORMS ON SYMMETRIC SPACES

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Abstract. Killing forms on Riemannian manifolds are differential forms whose covariant derivative is totally skew–symmetric. We show that a compact simply connected symmetric space carries a non–parallel Killing p–form ($p \geq 2$) if and only if it isometric to a Riemannian product $S^k \times N$, where S^k is a round sphere and $k > p$.

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1. Introduction

There are two equivalent definitions of Killing vector fields on Riemannian manifolds. A vector field X is Killing if its local flow consists of isometries. Equivalently, X is Killing if the covariant derivative $\nabla X^\flat$ of the dual 1–form $X^\flat$ is skew–symmetric.

This second definition can be generalized to forms of higher degree as follows. A p–form u is called *Killing* if its covariant derivative is totally skew–symmetric, *i.e.* if it exists some $p+1$–form τ such that

$$\nabla u = \tau.$$

It is easy to check that in that case τ is necessarily equal to $\frac{1}{p+1}du$. In contrast to Killing 1–forms, which are just dual to infinitesimal isometries, there is no geometrical interpretation of Killing p–forms for $p \geq 2$.

The aim of this paper is to show the following

Theorem 1.1. *If a symmetric space M of compact type carries a non–parallel Killing p–form and $p \geq 2$, then the universal cover $\tilde{M}$ is either a round sphere, or has a factor isometric to a round sphere in its de Rham decomposition.*

Together with the fact that a Killing form on a product splits as a product of Killing forms on the factors [4], we get therefore a complete description of all Killing forms on locally symmetric spaces of compact type.

This result can be thought of as a generalization of the following weaker assertion:

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Proposition 1.2. *A symmetric space admitting real Killing spinors is locally conformally flat.*

To see that Theorem 1.1 implies Proposition 1.2, we recall the fact that a manifold M which carry Killing spinors is locally irreducible (cf. [3]) and using the squaring construction one can construct non–parallel Killing p–forms for some $p \geq 2$ starting from a Killing spinor. Thus the universal cover of M has to be the sphere, so M is conformally flat.

Of course, one can give a more direct proof of Proposition 1.2. If Ψ is a Killing spinor, an immediate calculation shows that M is Einstein and $W(X,Y) \cdot \Psi = 0$, where W is the Weyl tensor, for all $X, Y \in TM$ (cf. [3]). Differentiating this relation several times and using the fact that the Weyl tensor is parallel, we obtain that for every vectors X and Y, the Clifford product of the 2–form $W(X,Y)$ with every spinor vanishes, so finally $W = 0$.

The strategy for the proof of Theorem 1.1 is somewhat similar, although much more involved. We first show that if M is a Riemannian product, then at least one of the factors carries a Killing p–form, too. We then interpret Killing p–forms as parallel sections of $\Lambda^p M \oplus \Lambda^{p+1} M$ with respect to some modified natural connection $\bar{\nabla}$ acting on this bundle. The curvature $\bar{R}$ of this connection can be computed explicitly in terms of the Riemannian curvature of M, and the sections of $\Lambda^p M \oplus \Lambda^{p+1} M$ which lie in the kernel of $\bar{R}$ define some ∇–parallel sub–bundle $E_0 \oplus F_0$ of $\Lambda^p M \oplus \Lambda^{p+1} M$. The point is that this sub–bundle is not necessarily $\bar{\nabla}$–invariant. By an inductive procedure, one can construct a sequence of ∇–parallel sub–bundles $E_{i+1} \oplus F_{i+1} \subset E_i \oplus F_i$ such that $\bar{\nabla}(E_{i+1}) \subset F_i$ and $\bar{\nabla}(F_{i+1}) \subset E_i$. This sequence is of course stationary, and defines some ∇– and $\bar{\nabla}$–parallel sub–bundle $E \oplus F = E_k \oplus F_k$ of $\Lambda^p M \oplus \Lambda^{p+1} M$ for some k large enough.

A tricky argument (which is the core of the paper and is described in detail in Section 3) allows one to show that the projection F of $E \oplus F$ onto $\Lambda^{p+1} M$ is either zero, or the whole space. The first case just says that every Killing form has to be parallel, while in the second situation it is easy to show that the Weyl curvature of M has to vanish.

2. Preliminaries

Throughout this paper we use Einstein's summation conventions on double subscripts. Vectors and 1–forms are identified via the metric. In the sequel, $\{e_i\}$ will denote a local orthonormal basis of the tangent bundle, parallel at some point.

Definition 2.1. *A p–form u is called a Killing p–form if and only if*

$$\nabla_X u = \frac{1}{p+1} X \lrcorner\, du, \tag{1}$$

for all vector fields X.

Let u be a Killing p–form. Obviously $X \lrcorner \nabla_X u = 0$ for all vectors X so in particular $\delta u = 0$. Let us take the covariant derivative in (1) with respect to some vector field Y, wedge with X and sum over an orthonormal basis $X = e_i$ (X, Y and e_i are supposed to be parallel at a point):

$$\begin{aligned} e_i \wedge R_{e_i,Y} u &= e_i \wedge \nabla_{e_i} \nabla_Y u - e_i \wedge \nabla_Y \nabla_{e_i} u = e_i \wedge \nabla_{e_i} (\frac{1}{p+1} Y \lrcorner du) - \nabla_Y du \\ &= -\frac{1}{p+1} Y \lrcorner e_i \wedge \nabla_{e_i} du + \frac{1}{p+1} \nabla_Y du - \nabla_Y du \\ &= -\frac{1}{p+1} Y \lrcorner d(du) - \frac{p}{p+1} \nabla_Y du = -\frac{p}{p+1} \nabla_Y du. \end{aligned}$$

Thus, denoting by R^+ the operator

$$R^+ : TM \to \mathrm{End}(\Lambda^p M, \Lambda^{p+1} M), \qquad R^+(X)u := e_i \wedge R_{X,e_i} u,$$

the above equation reads

$$\nabla_X du = \frac{p+1}{p} R^+(X) u. \tag{2}$$

Consider the connection $\tilde{\nabla}$ on $\Lambda^p M \oplus \Lambda^{p+1} M$ given by

$$\tilde{\nabla}_X (u, v) := (\nabla_X u - \frac{1}{p+1} X \lrcorner v, \nabla_X v - \frac{p}{p+1} R^+(X) u).$$

We have just shown that (u, du) is a $\tilde{\nabla}$–parallel section of $\Lambda^p M \oplus \Lambda^{p+1} M$ for every Killing p–form u. We will not discuss here the consequences of this important fact, but rather refer to [6] for details.

Taking the covariant derivative with respect to some vector field Y in (1), skew–symmetrising in X and Y and using (2) yields

$$R_{X,Y} u = -\frac{1}{p} (X \lrcorner R^+(Y) u - Y \lrcorner R^+(X) u) \quad \forall X, Y \in TM, \tag{3}$$

which we rewrite as

$$(\mathcal{I} \wedge R^+) u = -p\, R\, u, \ \forall u \in \Lambda^p M \text{ Killing form.} \tag{4}$$

Here $\mathcal{I}$ stands for the interior product, $\mathcal{I}(X)u := X \lrcorner u$, and R^+ for the operator defined before, both of which are viewed as a 1–form with values in $End(\Lambda^* M)$. Therefore their exterior product is a 2–form with values in $End(\Lambda^* M)$, and so is the curvature operator R. Note that $\mathcal{I}$ decreases and R^+ increases the degree of the form by 1.

If M is locally symmetric, then R is parallel, and so is R^+ ($\mathcal{I}$ is always parallel). Thus, taking the covariant derivative with respect to Y in (2), skew–symmetrising in X and Y and using (1) yields

$$R_{X,Y} du = -\frac{1}{p} (R^+(X) Y \lrcorner du - R^+(Y) X \lrcorner du) \quad \forall X, Y \in TM, \tag{5}$$

or, equivalently,

$$(R^+ \wedge \mathcal{I})du = -p\, R\, du,\ \forall u \in \Lambda^p M \text{ Killing form.} \tag{6}$$

We end up this section by deriving some useful algebraic relations satisfied by R^+.

Lemma 2.2. *On any manifold we have*

$$R^+ \wedge \mathcal{I} = \mathcal{I} \wedge R^+ + R,\ \textit{ or, equivalently, } 2[R^+ \wedge \mathcal{I}] = R. \tag{7}$$

Here we define, for two 1–forms A, B with values in some algebra bundle Ω, their *commutator* $[A, B] : TM \otimes TM \to \Omega$ by $[A, B](X \otimes Y) := A(X)B(Y) - B(X)A(Y)$, and $[A \wedge B] : \Lambda^2(TM) \to \Omega$, resp. $[A \odot B] : S^2(TM) \to \Omega$ denote its skew–symmetric, resp. symmetric, part.

Proof.

$$\begin{aligned} R^+(X)\mathcal{I}(Y)u &= e_i \wedge R_{X,e_i}(Y \lrcorner u) = e_i \wedge R_{X,e_i}Y \lrcorner u + e_i \wedge Y \lrcorner R_{X,e_i}u \\ &= e_i \wedge R_{X,e_i}Y \lrcorner u - \mathcal{I}(Y) \lrcorner R^+(X)(u) + R_{X,Y}u, \end{aligned}$$

which, after skew–symmetrization and using the Bianchi identity on the first term of the right hand side, yields the desired result. ☐

Corollary 2.3. *Let E be a parallel (i.e., ∇–stable) sub–bundle of $\Lambda^p M$. The induced operator $\widetilde{[R^+, \mathcal{I}]} : TM \otimes TM \to Hom(E, \Lambda^p M/E)$ is a symmetric 2–tensor.*

Lemma 2.4. *Let E be as before. Then the induced 3–tensor*

$$\widetilde{\mathcal{I}^2 R^+} : TM^{\otimes 3} \to \Lambda^{p-1}M/(\mathcal{I}(E) + \mathcal{I}R^+\mathcal{I}(E))$$

vanishes identically.

Proof. First note that

$$\mathcal{I}^2 R^+ = \mathcal{I}R^+\mathcal{I} + \mathcal{I}[\mathcal{I}, R^+] = \mathcal{I}R^+\mathcal{I} - \frac{1}{2}\mathcal{I}R + \mathcal{I}[\mathcal{I} \odot R^+],$$

therefore

$$\widetilde{\mathcal{I}^2 R^+} \equiv \mathcal{I}[\mathcal{I} \odot R^+] \pmod{\mathcal{I}(E) + \mathcal{I}R^+\mathcal{I}(E)}.$$

The tensor $\widetilde{\mathcal{I}^2 R^+}$ is clearly skew–symmetric in the first two arguments, because two interior products anti–commute. On the other hand, the previous equation shows that $\widetilde{\mathcal{I}^2 R^+}$ is equal to the co–restriction to $Hom(E, \Lambda^p M/(\mathcal{I}(E) + \mathcal{I}R^+\mathcal{I}(E))$ of the tensor $\mathcal{I}[\mathcal{I} \odot R^+]$, which is, by its very definition, symmetric in the last two arguments.

But a 3–tensor which is symmetric in the last two arguments and skew–symmetric in the first two arguments is necessarily zero. ☐

3. Form bundles on symmetric spaces

The results in this section could have been stated in terms of abstract representation theory, but we prefer the more geometric presentation below.

Let M be a symmetric space with curvature tensor R. We define the following vector bundles on M:

$$E_0 := \{u \in \Lambda^p M \mid Ru = -\frac{1}{p}\mathcal{I} \wedge R^+ u\},$$

$$F_0 := \{v \in \Lambda^{p+1} M \mid Rv = -\frac{1}{p} R^+ \wedge \mathcal{I} v\}.$$

We then define inductively the vector bundles

$$E_k := \{u \in E_{k-1} \mid R^+(X)u \in F_{k-1} \quad \forall X \in TM\},$$

$$F_k := \{v \in F_{k-1} \mid X \lrcorner v \in E_{k-1} \quad \forall X \in TM\}.$$

Since R is parallel, we see that E_k and F_k are parallel vector bundles for every k. We denote by

$$E := \bigcap_k E_k, \qquad F := \bigcap_k F_k. \tag{8}$$

By definition, for every sections u and v of E and F respectively we have

$$Ru = -\frac{1}{p}\mathcal{I} \wedge R^+ u \tag{9}$$

$$Rv = -\frac{1}{p} R^+ \wedge \mathcal{I} v \tag{10}$$

$$R^+(X)u \in F \qquad \text{and} \qquad X \lrcorner v \in E \quad \forall X \in TM \tag{11}$$

Notice that from (11) we get

$$\mathcal{I} R^+(E) \subset E. \tag{12}$$

Lemma 3.1. *Let k be an integer $k \geq 1$. For every tangent vectors $X_1, \cdots, X_k$, $Y_1, \cdots, Y_k$ and for every section u of E we have*

$$X_1 \lrcorner \ldots \lrcorner X_k \lrcorner R^+(Y_1) \ldots R^+(Y_k) u \in E. \tag{13}$$

Proof. The statement claims that $\mathcal{I}^k (R^+)^k$, as a $2k$–tensor with values in $Hom(E, \Lambda^p M)$, takes actually values in $End(E)$.

We use induction on k. For $k = 1$ the result follows from (12).

Step 1. First we show (also by induction), using the Lemma 2.4, that

$$\mathcal{I}^l R^+(V) \subset \mathcal{I} R^+ \mathcal{I}^{l-1}(V) + I^{l-1}(V), \tag{14}$$

for any $l \geq 1$ and any ∇–parallel form sub–bundle V. Indeed, for $l = 2$ this is exactly Lemma 2.4, and for $l > 2$, we get from the induction step that $\mathcal{I}^l R^+(V) \subset \mathcal{I}^2 R^+ \mathcal{I}^{l-2}(V) + \mathcal{I}(\mathcal{I}^{l-2}(V))$, from which the claim follows using again Lemma 2.4.

Step 2. Let us denote now by $E' := (R^+)^{k-1}(E) \subset \Lambda^{p+k-1}$ the image of the last factors in the product $\mathcal{I}^k (R^+)^k$. Then E' is a parallel bundle. We have to show that

$\mathcal{I}^k R^+(E') \subset E$. From (14) we get $\mathcal{I}^k R^+(E') \subset \mathcal{I} R^+ \mathcal{I}^{k-1}(E') + \mathcal{I}^{k-1}(E')$, and from the induction step $\mathcal{I}^{k-1}(E') \subset E$, so we get in the end $\mathcal{I}^k (R^+)^k(E) \subset E + \mathcal{I} R^+(E) \subset E$ by (12).

□

Corollary 3.2. *If there exist $n-p$ tangent vectors $Y_1, \ldots, Y_{n-p}$ and a p–form $u \in E$ such that $R^+(Y_1) \ldots R^+(Y_{n-p})u$ is non–zero, then $E = \Lambda^p M$.*

Proof. This follows simply because the different contractions of the volume form with $n-p$ vectors span $\Lambda^p M$.

□

We now examine under which circumstances the hypothesis in the corollary above can fail, that is, what can one say about E if $R^+(Y_1) \ldots R^+(Y_{n-p})u = 0$ for all $Y_1, \ldots, Y_{n-p} \in TM$ and $u \in E$.

Lemma 3.3. *Let $V \subset \Lambda^q M$ be some irreducible summand in the decomposition of q–forms under the holonomy representation.*

(i) *If $R^+(X)u = 0$ for all tangent vectors X and $u \in V$ then the holonomy representation on V is trivial.*

(ii) *Suppose that the holonomy representation is irreducible on TM and that M is not Kähler. If there exists some sub–bundle W of $\Lambda^{q+1}M$ on which the holonomy representation is trivial and such that $R^+(X)u \in W$ for every tangent vector X and $u \in V$, then either $q = n-1$, or $R^+(X)u = 0\ \forall X$.*

Proof. (i) Taking the interior product with X and making the sum over an orthonormal basis yields
$$0 = e_i \lrcorner R^+(e_i)u = -q(R)u.$$
But $q(R)$, being the Casimir operator of the holonomy group on V, is a non–negative constant. Moreover, this constant is zero if and only if V is the trivial representation.

(ii) Let H be the holonomy group of M. Since M is irreducible, it has to be Einstein, and we denote its Einstein constant by r. The operator R^+ defines an equivariant map $R^+ : TM \otimes V \to W$. We have two possibilities: either V is isomorphic (as H–representation) to TM, or not. In the last case, the real Schur Lemma shows that the map R^+ vanishes. We now suppose that V is isomorphic to TM. In particular, $q(R)$ acts on V by multiplication with the scalar r (remember that $q(R) = \mathrm{Ric}$ on $TM \simeq \Lambda^1 M$). Let $\{v_\alpha\}$, $\alpha = 1, \ldots, \dim(W)$ be an orthonormal basis of W. Since the curvature acts trivially on W we can write for every $u \in V$
$$\begin{aligned} \langle R^+(X)u, v_\alpha \rangle &= \langle e_i \wedge R_{X,e_i}u, v_\alpha \rangle = -\langle u, R_{X,e_i}(e_i \lrcorner v_\alpha) \rangle \\ &= -\langle u, (R_{X,e_i}e_i) \lrcorner v_\alpha \rangle = -r \langle u, X \lrcorner v_\alpha \rangle, \end{aligned}$$
whence
$$R^+(X)u = \langle R^+(X)u, v_\alpha \rangle v_\alpha = -r \langle u, X \lrcorner v_\alpha \rangle v_\alpha.$$

Taking the interior product in this formula yields

$$u = \frac{1}{r} q(R)u = -\frac{1}{r} e_i \lrcorner R^+(e_i)u = e_i \lrcorner \langle u, e_i \lrcorner v_\alpha \rangle v_\alpha = \langle u, e_i \lrcorner v_\alpha \rangle e_i \lrcorner v_\alpha. \qquad (15)$$

Since M is not Kähler, it turns out that every H–equivariant bilinear form on TM is a real multiple of the metric. In particular, we get

$$\langle X \lrcorner v_\alpha, Y \lrcorner v_\beta \rangle = c_{\alpha\beta} \langle X, Y \rangle.$$

Taking $X = Y = e_i$ and summing yields

$$nc_{\alpha\beta} = c_{\alpha\beta}\langle e_i, e_i \rangle = \langle e_i \lrcorner v_\alpha, e_i \lrcorner v_\beta \rangle = \langle e_i \wedge e_i \lrcorner v_\alpha, v_\beta \rangle = (q+1)\delta_{\alpha\beta}, \qquad (16)$$

so finally

$$\langle X \lrcorner v_\alpha, Y \lrcorner v_\beta \rangle = \frac{q+1}{n} \delta_{\alpha\beta} \langle X, Y \rangle. \qquad (17)$$

We take the scalar product with $e_j \lrcorner v_\beta$ in (15) and use (17) to obtain

$$\langle u, e_j \lrcorner v_\beta \rangle = \langle u, e_i \lrcorner v_\alpha \rangle \langle e_i \lrcorner v_\alpha, e_j \lrcorner v_\beta \rangle = \frac{q+1}{n} \langle u, e_j \lrcorner v_\beta \rangle.$$

Since u is non–zero by assumption, this shows that $q = n-1$.

□

Corollary 3.4. *Let M be a compact irreducible non–Kählerian symmetric space. If the bundle E defined in (8) is a proper sub–bundle of $\Lambda^p M$ then $R^+(X)u = 0$ for all $X \in TM$ and $u \in E$.*

Proof. Let k be the smallest positive integer such that $R^+(Y_1)\dots R^+(Y_k)u = 0$ for all $Y_1, \dots Y_k \in TM$ and $u \in E$. Clearly $1 \le k \le n - p + 1$. If E is strictly included in $\Lambda^p M$, Corollary 3.2 shows that $k \le n - p$. Let W be the maximal sub–bundle of $\Lambda^{p+k-1}M$ on which the holonomy group acts trivially. From Lemma 3.3 (i) we see that $R^+(Y_1)\dots R^+(Y_{k-1})u \in W$ for all $Y_1, \dots Y_{k-1} \in TM$ and $u \in E$.

Suppose that $k \ge 2$. Lemma 3.3 (ii) shows that either $R^+(Y_1)\dots R^+(Y_{k-1})u = 0$ for all $Y_1, \dots Y_{k-1} \in TM$ and $u \in E$, or $p + k - 2 = n - 1$. The first case contradicts the minimality of k, and the second case contradicts the inequality $k \le n - p$. This shows that $k = 1$, thus proving our assertion. □

4. Killing forms on symmetric spaces

Let $u \in \Lambda^p M$ be a Killing form on a compact simply connected symmetric space M.

Proposition 4.1. *The pair (u, du) is a section of the bundle $E \oplus F$ defined in the previous section.*

Proof. From (3) and (5) we see that (u, du) is a section of $E_0 \oplus F_0$. Moreover (1) and (2) show that if (u, du) is a section of $E_k \oplus F_k$ for some $k \ge 0$, then it is also a section of $E_{k+1} \oplus F_{k+1}$. A simple induction argument ends up the proof. □

Suppose now that $p \geq 2$. Then M is not Kählerian:

Lemma 4.2. *A Killing p–form on a compact Kähler manifold is parallel if $p \geq 2$.*

Proof. We make use of the classical Kählerian operators

$$d^c := \sum Je_i \wedge \nabla_{e_i}, \qquad \delta^c := -\sum Je_i \lrcorner \nabla_{e_i},$$

$$J = \sum Je_i \wedge e_i \lrcorner\,, \qquad \Lambda := \frac{1}{2}\sum e_i \lrcorner Je_i \lrcorner$$

acting on forms, which satisfy the well-known relations

$$[d, J] = -d^c, \qquad [d, \Lambda] = \delta^c, \qquad d^c\delta + \delta d^c = 0 = \delta^c\delta + \delta\delta^c$$

on Kähler manifolds.

Let u be a Killing p–form. We take the wedge product with JX in (1) and sum over an orthonormal basis $X = e_i$ to obtain:

$$d^c u := Je_i \wedge \nabla_{e_i} u = \frac{1}{p+1} J du = \frac{1}{p+1}(dJu + d^c u),$$

whence

$$pd^c u = dJu.$$

Since $\delta u = 0$ and d^c anti–commutes with δ, the two members of this relation are L^2–orthogonal, so they both vanish.

Taking now the interior product with JX in (1) and summing over an orthonormal basis $X = e_i$ yields

$$\delta^c u = -Je_i \lrcorner \nabla_{e_i} u = \frac{2}{p+1}\Lambda du = \frac{2}{p+1}(d\Lambda u + \delta^c u),$$

so

$$(p-1)\delta^c u = d\Lambda u.$$

By L^2–orthogonality again, (using also the fact that $p \geq 2$) we get $\delta^c u = 0$. Thus $\Delta u = d^c\delta^c u + \delta^c d^c u = 0$, showing that $du = 0$, and by (1), $\nabla u = 0$.

□

This result shows that we can assume that M is not Kählerian. If M is reducible, then u is a sum of pull–backs of Killing forms on the factors (see [4]). We can therefore suppose, without lost of generality, that M is irreducible. From Corollary (3.4) we deduce that either $R^+(X)u = 0$ for every X, or $E = \Lambda^p M$. The first case implies that du is parallel, so in particular $\Delta u = 0$. The Weitzenböck formula (cf. [6])

$$\Delta u = (p+1)\nabla^*\nabla u$$

then shows that u is parallel.

Consider now the second possibility: $E = \Lambda^p M$.

Lemma 4.3. *If (3) holds for any p–form u, $2 \leq p \leq n-2$, then the Weyl tensor of M vanishes.*

Proof. The equation (3) is $O(n)$–invariant, so if it holds for a given non-zero curvature tensor, it must hold for all curvature tensors belonging to the corresponding $O(n)$–invariant space.

The equation holds trivially for the scalar part of the curvature tensor, and, on the other hand, all symmetric spaces are Einstein, so we only need to consider Ricci–flat curvature tensors. Therefore, if (3) holds for the curvature tensor R, it equally holds for W, where W is the Weyl component of R. If this is non–zero, (3) must hold for all tensors of Weyl type, because the space of Weyl tensors is $O(n)$–irreducible for $n \geq 4$. (For $n = 4$ there are two $SO(4)$–irreducible components, but these are distinguished by the orientation only, so they are not $O(4)$–invariant).

We will give an example of a Weyl tensor in dimension 4, and of a particular 2–form u_0, for which (3) fails.

In higher dimensions we complete this Weyl tensor in the trivial way, and for higher degree forms we simply take products of u_0 with some $p - 2$ form depending only on the last $n - 4$ variables. Note that this operation will produce examples of Weyl tensors and p–forms that do not satisfy (3), as long as $2 \leq p \leq n - 2$.

Consider $\alpha := g(I\cdot,\cdot), \beta := g(J\cdot,\cdot), \gamma := g(K\cdot,\cdot)$ a basis of self–dual 2–forms Λ^+ in $\mathbb{R}^4$, obtained by composing the Euclidean metric with three orthogonal complex structures on $\mathbb{R}^4$ that induce the quaternionic structure on $\mathbb{R}^4 = \mathbb{H}$. In suitable coordinates we have

$$\begin{array}{rcl} \alpha &=& e_1 \wedge e_2 + e_3 \wedge e_4, \\ \beta &=& e_1 \wedge e_3 - e_2 \wedge e_4, \\ \gamma &=& e_1 \wedge e_4 + e_2 \wedge e_3. \end{array}$$

Define the curvature tensor R by $R(\alpha) = \alpha \simeq I$, $R(\beta) = -\beta \simeq -J, R(\gamma) = 0$, and extend it by 0 on anti–self–dual 2–forms. It is a Ricci–flat, self–dual curvature tensor.

Let us compute $R^+(X)u$, for any 2–form u:

$$R^+(X)u = e_i \wedge R_{X,e_i}u = e_i \wedge \left(\frac{1}{2}\langle X \wedge e_i, \alpha\rangle I - \frac{1}{2}\langle X \wedge e_i, \beta\rangle J \right) u,$$

where the factors 1/2 come from the fact that $\|\alpha\|^2 = \|\beta\|^2 = 2$.

Setting $u = u_0 := \beta$ and using $I\beta = 2\gamma$ and $J\beta = 0$, we get

$$R^+(X)\beta = e_i \wedge \gamma\langle X \wedge e_i, \alpha\rangle = e_i \wedge \gamma\langle X, I(e_i)\rangle = -I(X) \wedge \gamma = J(X) \lrcorner\, \omega,$$

where $\omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ is the volume form.

We get

$$e_1 \lrcorner\, R^+(e_2)\beta - e_2 \lrcorner\, R^+(e_1)\beta = \gamma,$$

but $R_{e_1,e_2}\beta = 0$, which contradicts (3). □

So, if $E = \Lambda^p(M)$, and the locally symmetric space M is not a space form, then p must be either 1 — in which case we get the Killing vector fields — or $p = n - 1$.

In this latter case, the Hodge dual — say ξ — of u is a closed 1–form satisfying the equation

$$\nabla_X \xi = -\frac{1}{n}\delta\xi X. \tag{18}$$

In particular $d\xi = 0$, and since M is Einstein, the Bochner formula yields

$$n\xi = \mathrm{Ric}(\xi) = \Delta\xi - \nabla^*\nabla\xi = \frac{n}{n-1}d\delta\xi,$$

so $\Delta(\delta\xi) = (n-1)\delta\xi$. If $\delta\xi = 0$, (18) shows that ξ – and thus u – is parallel. Otherwise, the Obata theorem (cf. [5]) shows that $\delta\xi$ is a characteristic function of the round sphere, so the Weyl curvature of M vanishes. This proves Theorem 1.1.

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