Vanishing Theorems for Quaternionic Kähler Manifolds

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Abstract

In this article we discuss a peculiar interplay between the representation theory of the holonomy group of a Riemannian manifold, the Weitzenböck formula for the Hodge–Laplace operator on forms and the Lichnerowicz formula for twisted Dirac operators. For quaternionic Kähler manifolds this leads to simple proofs of eigenvalue estimates for Dirac and Laplace operators. Moreover, it enables us to determine which representations can contribute to harmonic forms. As a corollary we prove the vanishing of certain odd Betti numbers on compact quaternionic Kähler manifolds of negative scalar curvature. We simplify the proofs of several related results in the positive case.

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Contents

1 A Prelude on Weitzenböck Formulas

Since decades the Weitzenböck formulas for Dirac operators on Clifford bundles have inspired intensive and important research. Beautiful results can be proved elegantly using the full power of the Weitzenböck machinery. The basic example of a Clifford bundle is the bundle

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of exterior forms $\Lambda^\bullet T^*M$ endowed with the scalar product induced by the metric on M and Clifford multiplication with tangent vectors

$$\star : \; T_pM \times \Lambda^\bullet T^*_pM \;\longrightarrow\; \Lambda^\bullet T^*_pM, \qquad (X,\,\omega) \;\longmapsto\; X \star \omega$$

defined by $X \star \omega := X^\sharp \wedge \omega \,-\, X \lrcorner\, \omega$. The Levi–Civita–connection induces a connection ∇ on $\Lambda^\bullet T^*M$ and an associated second order elliptic differential operator $\nabla^*\nabla \;:=\; -\sum_i \nabla^2_{E_i,E_i}$ where $\nabla^2_{X,Y} \;:=\; \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$ and the sum is over a local orthonormal base $\{E_i\}$. On the other hand we have the exterior differential d and its formal adjoint d^* as natural first order differential operators on $\Lambda^\bullet T^*M$ linked to $\nabla^*\nabla$ by the classical Weitzenböck formula

$$\Delta \quad := \quad (d + d^*)^2 \quad = \quad \nabla^*\nabla \;+\; \frac{1}{2}\sum_{ij} E_i \star E_j \star R_{E_i,E_j} \tag{1}$$

where $R_{X,Y}$ is the curvature endomorphism of $\Lambda^\bullet T^*_pM$. However the connection on $\Lambda^\bullet T^*M$ is induced by a connection on TM and consequently the curvature endomorphism $R_{X,Y}$ is just the curvature endomorphism of T_pM in a different representation, namely the representation

$$\bullet : \; \mathfrak{so}(T_pM) \times \Lambda^\bullet T^*_pM \;\longrightarrow\; \Lambda^\bullet T^*_pM, \qquad (X,\,\omega) \;\longmapsto\; X \bullet \omega$$

of the Lie algebra $\mathfrak{so}(T_pM)$ of $\mathbf{SO}\,(T_pM)$ on the exterior algebra induced by its representation on T_pM. The canonical identification of $\mathfrak{so}(T_pM)$ with the bivectors characterized by

$$\Lambda^2 T_pM \;\xrightarrow{\cong}\; \mathfrak{so}(T_pM), \qquad \langle\,(X \wedge Y) \bullet A,\; B\,\rangle \;:=\; \langle\, X \wedge Y,\; A \wedge B\,\rangle$$

reads $(X \wedge Y) \bullet A \;:=\; \langle X, A\rangle\, Y \;-\; \langle Y, A\rangle\, X$ and defines a unique bivector $R(X \wedge Y)$ via:

$$\langle R(X \wedge Y) \bullet Z, W\rangle \quad := \quad \langle R_{X,Y}Z, W\rangle \qquad\qquad R(X \wedge Y) \quad = \quad \frac{1}{2}\sum_i E_i \wedge R_{X,Y}E_i$$

In the spirit of this identification the representation of $\mathfrak{so}(T_pM)$ on $\Lambda^\bullet T^*_pM$ is given by $(X \wedge Y)\,\bullet \;=\; Y^\sharp \wedge X \lrcorner - X^\sharp \wedge Y \lrcorner$. In particular, the classical Weitzenböck formula becomes

$$\begin{aligned} \Delta \;&=\; \nabla^*\nabla \;+\; \frac{1}{2}\sum_{ij} (E_i^\sharp \wedge E_j^\sharp \wedge \,-\, E_i \lrcorner\, E_j^\sharp \wedge \,-\, E_i^\sharp \wedge E_j \lrcorner + E_i \lrcorner\, E_j \lrcorner)\; R(E_i \wedge E_j)\,\bullet \\ &=\; \nabla^*\nabla \;+\; \frac{1}{2}\sum_{ij} (E_i \wedge E_j) \bullet R(E_i \wedge E_j)\,\bullet \end{aligned}$$

because both potentially troublesome inhomogeneous terms cancel by the first Bianchi identity leaving us with a curvature term depending linearly on the curvature tensor:

$$R \quad := \quad \frac{1}{4}\sum_{ij} (E_i \wedge E_j) \;\cdot\; R(E_i \wedge E_j) \qquad \in \quad \mathrm{Sym}^2(\Lambda^2 T_pM)\;.$$

It will be convenient to compose the identification $\Lambda^2 T_pM \;\xrightarrow{\cong}\; \mathfrak{so}(T_pM)$ with the quantization map $q : \mathrm{Sym}^2\mathfrak{so}(T_pM) \;\longrightarrow\; \mathcal{U}\,\mathfrak{so}(T_pM)$, $X^2 \;\longmapsto\; X^2$, into the universal enveloping algebra of $\mathfrak{so}(T_pM)$ to get an element $q(R) \in \mathcal{U}\,\mathfrak{so}(T_pM)$ with:

$$\Delta \quad = \quad \nabla^*\nabla \;+\; 2\,q(R) \tag{2}$$

What is the advantage of writing the well known classical Weitzenböck formula (1) in this fancy way? Well, the Weitzenböck formula (2) brings the holonomy group of the underlying manifold into play. Recall that the holonomy group $\mathrm{Hol}_p M \subset \mathbf{O}(T_pM)$ is the closure of the group of all parallel transports along piecewise smooth loops in $p \in M$. We will assume throughout that M is connected so that the holonomy groups in different points p and $\tilde{p}$ are conjugated by parallel transport $T_pM \longrightarrow T_{\tilde{p}}M$. Choosing a suitable representative $\mathrm{Hol} \subset \mathbf{O}_n\mathbb{R}$ with $n := \dim M$ of their common conjugacy class acting on the abstract vector space $\mathbb{R}^n$ we can define the holonomy bundle of M:

$$\mathrm{Hol}(M) \quad := \quad \{\, f : \mathbb{R}^n \longrightarrow T_pM \mid p \in M \text{ and } f \text{ isometry with } f(\mathrm{Hol}) = \mathrm{Hol}_p M \,\} .$$

The holonomy bundle is a reduction of the orthonormal frame bundle $\mathbf{O}(M)$ to a principal bundle with structure group Hol, which is stable under parallel transport. Consequently the Levi–Civita connection is tangent to $\mathrm{Hol}(M)$ and descends to a connection on $\mathrm{Hol}(M)$.

The associated fibre bundle $\mathrm{Hol}(M) \times_{\mathrm{Hol}} \mathbf{O}_n\mathbb{R}$ is canonically diffeomorphic to the full orthonormal frame bundle $\mathbf{O}(M)$. This construction provides an explicit foliation of $\mathbf{O}(M)$ into mutually equivalent principal subbundles stable under parallel transport. Choosing a leaf different from the distinguished leaf $\mathrm{Hol}(M)$ amounts to choosing a different representative for the conjugacy class of $\mathrm{Hol} \subset \mathbf{O}_n\mathbb{R}$. In particular every principal subbundle of $\mathbf{O}(M)$ stable under parallel transport is a union of leaves and is characterized by a subgroup of $\mathbf{O}_n\mathbb{R}$ containing a representative of the conjugacy class of the holonomy group Hol.

With the Levi–Civita connection being tangent to the holonomy bundle $\mathrm{Hol}(M)$ its curvature tensor R takes values in the holonomy algebra $\mathfrak{hol}_p M$ at every point $p \in M$, so that $R \in \mathrm{Sym}^2\mathfrak{hol}_p M \subset \mathrm{Sym}^2\Lambda^2 T_pM$ and $q(R) \in \mathcal{U}\,\mathfrak{hol}_p M$. However by definition every point $f \in \mathrm{Hol}(M)$ identifies $\mathfrak{hol}_p M$ with $\mathfrak{hol}$ making $q(R)$ a $\mathcal{U}\,\mathfrak{hol}$–valued function on $\mathrm{Hol}(M)$:

$$q(R) \quad \in \quad C^\infty(\,\mathrm{Hol}(M), \mathcal{U}\,\mathfrak{hol}\,)^{\mathrm{Hol}} \;\cong\; \Gamma(\mathrm{Hol}(M) \times_{\mathrm{Hol}} \mathcal{U}\,\mathfrak{hol})$$

For an arbitrary irreducible complex representation π of Hol the associated vector bundle $\pi(M) := \mathrm{Hol}(M) \times_{\mathrm{Hol}} \pi$ over M is endowed with the connection induced from the Levi–Civita connection. Moreover there is a canonical second order differential operator defined on sections of $\pi(M)$:

$$\Delta_\pi \quad := \quad \nabla^*\nabla \;+\; 2\,q(R) \tag{3}$$

It is evident from the Weitzenböck formula (1) written as in (2) that the diagram

$$\begin{array}{ccc} \pi(M) & \xrightarrow{\Delta_\pi} & \pi(M) \\ F\big\downarrow & & \big\downarrow F \\ \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\Delta} & \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

commutes for any $F \in \mathrm{Hom}_{\mathrm{Hol}}(\pi, \Lambda^\bullet\mathbb{C}^{n*})$ or equivalently for any globally parallel embedding $F : \pi(M) \longrightarrow \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$. Hence the pointwise decomposition of $\Lambda^\bullet T^*_pM \otimes_{\mathbb{R}} \mathbb{C}$

into irreducible complex representations of $\mathrm{Hol}_p M$ becomes a global decomposition of any eigenspace of Δ, e. g. we have for its kernel:

$$H^\bullet_{dR}(M,\,\mathbb{C}) \;=\; \bigoplus_\pi \mathrm{Hom}_{\,\mathrm{Hol}}(\,\pi,\,\Lambda^\bullet\mathbb{C}^{n*}\,)\,\otimes\,\mathrm{Kern}\;\Delta_\pi$$

The same kind of reasoning is possible for the Dirac operator on spinors, assuming the manifold M to be spin and taking $\mathrm{Hol}_p M$ to be its spin holonomy group. Ignoring for the moment the Lichnerowicz result that the curvature term reduces to multiplication by the scalar curvature and employing the formula $(X\wedge Y)\bullet \;:=\; \frac{1}{2}(\,X\star Y\star+\langle X,Y\rangle\,)$ for the representation of $\mathfrak{so}(T_pM)$ on the spinor bundle $\mathbf{S}\,(M)$ we can proceed from (1) directly to:

$$D^2 \;=\; \nabla^*\nabla \;+\; 4\,q(R)\,. \tag{4}$$

In particular, all eigenspaces of D^2 decompose globally according to the pointwise decomposition of the spinor bundle under the spin holonomy group $\mathrm{Hol}_p M$. Whereas the change of the factor of $q(R)$ from 2 to 4 is certainly puzzling, there can be no doubt however that equation (4) is true. In fact from Lichnerowicz's result we already know that $q(R)$ acts by scalar multiplication with $\frac{\kappa}{16}$ on $\mathbf{S}\,(M)$, where κ is the scalar curvature of M. Hence we can read equation (4) as

$$D^2\Big|_\pi \;=\; \Delta_\pi \;+\; \frac{\kappa}{8}$$

where the restriction to π is a short hand notation for any globally parallel embedding $F:\;\pi(M)\;\longrightarrow\;\mathbf{S}\,(M)$ induced by some non–trivial $F\;\in\;\mathrm{Hom}_{\,\mathrm{Hol}}(\pi,\mathbf{S}\,)$. Written in this way formula (4) is seen to be a generalization of the Partharasathy formula for the Dirac square D^2 on a symmetric space G/K of compact type, because in this case the operators Δ_π defined above on sections of $\pi(M)$ all become the Casimir of G.

At this point the reader may argue that these results are not too surprising because intrinsically defined differential operators are restricted to parallel subbundles. However the main point is that Δ and D^2 do not only respect some decomposition into parallel subbundles, but that their restrictions to these subbundles are completely independent of the embedding. Counterexamples to the idea that intrinsically defined differential operators always enjoy these two properties are easily found among twisted Dirac operators.

Consider therefore a geometric vector bundle $\mathcal{R}(M)\;:=\;\mathrm{Hol}\,(M)\times_{\mathrm{Hol}}\mathcal{R}$ associated to the holonomy bundle via some not necessarily irreducible representation $\mathcal{R}$ of the holonomy group. The Levi–Civita connection on $\mathrm{Hol}\,(M)$ defines a geometric connection on this vector bundle, whose curvature endomorphism is still given through the representation

$$\bullet:\;\;\mathfrak{hol}_{\,p}M\times\mathcal{R}_p(M)\;\longrightarrow\;\mathcal{R}_p(M)$$

of the Lie algebra $\mathfrak{hol}_{\,p}M$ on $\mathcal{R}_p(M)$ by the formula $R^{\mathcal{R}}_{X,Y}\;=\;R(X\wedge Y)\bullet$. The twisted Dirac operator $D_{\mathcal{R}}$ is a first order differential operator acting on sections of the vector bundle $(\mathbf{S}\otimes\mathcal{R})(M)$. It satisfies a twisted Weitzenböck formula derived from (1):

$$D^2_{\mathcal{R}} \;=\; \nabla^*\nabla\;+\;\frac{1}{2}\sum_{ij}\Big(E_i\star E_j\star R(E_i\wedge E_j)\bullet\otimes\mathrm{id}_{\,\mathcal{R}}\;+\;E_i\star E_j\star\otimes R(E_i\wedge E_j)\bullet\Big) \tag{5}$$

This formula has an apparent asymmetry between the spinor bundle and the twist. However, we still have the formula $(X \wedge Y) \bullet \;=\; \frac{1}{2}(\, X \star Y \star + \langle X, Y \rangle\,)$ for the representation of $\mathfrak{so}(T_pM)$ on the fibre $\mathbf{S}_p(M)$ of the spinor bundle and we may try to balance this asymmetry to cast equation (5) into a form similar to (4). This is most easily achieved by rewriting the action of $q(R)$ on the tensor product $\mathbf{S} \otimes \mathcal{R}$ in the following asymmetric way:

$$q(R) \;=\; \frac{1}{2} \sum_{ij} \Big(\, (E_i \wedge E_j) \bullet R(E_i \wedge E_j) \bullet \otimes \mathrm{id}_{\mathcal{R}} \;+\; (E_i \wedge E_j) \bullet \otimes R(E_i \wedge E_j) \bullet \,\Big) \qquad (6)$$
$$-\; q(R) \otimes \mathrm{id}_{\mathcal{R}} \;+\; \mathrm{id}_{\mathbf{S}} \otimes q(R)$$

With Lichnerowicz's result $q(R) \;=\; \frac{\kappa}{16}$ for the spinor representation $\mathbf{S}$ equation (5) becomes

$$D^2_{\mathcal{R}} \;=\; \Delta_{\mathbf{S} \otimes \mathcal{R}} \;+\; \frac{\kappa}{8} \otimes \mathrm{id}_{\mathcal{R}} \;-\; \mathrm{id}_{\mathbf{S}} \otimes 2\, q(R) \qquad (7)$$

In conclusion, the squares $D^2_{\mathcal{R}}$ of twisted Dirac operators will in general not respect the decomposition of $(\mathbf{S} \otimes \mathcal{R})(M)$ into parallel subbundles because of the critical summand $\mathrm{id}_{\mathbf{S}} \otimes 2\, q(R)$. Nevertheless, if $q(R)$ acts by scalar multiplication not only on $\mathbf{S}$ but on $\mathcal{R}$, too, the global decomposition of the eigenspaces of $D^2_{\mathcal{R}}$ according to the pointwise decomposition of $\mathbf{S} \otimes \mathcal{R}$ is restored.

Equation (7) is the key relation of this article and forms the cornerstone and motivation of all statements and calculations to come. In fact, we can take advantage of equation (7) even if the manifold in question is not spin, because the twisted Dirac operator may be well defined on the vector bundle $(\mathbf{S} \otimes \mathcal{R})(M)$ although M is neither spin nor $\mathbf{S}(M)$ or $\mathcal{R}(M)$ are well defined vector bundles. The only thing that really matters is whether the representation $\mathbf{S} \otimes \mathcal{R}$ is defined for the holonomy group Hol itself or only for some covering group.

2 Quaternionic Kähler Holonomy

In this section we introduce the main notions of quaternionic Kähler holonomy based on the group $\mathrm{Hol} \;=\; \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ with $n \;\geq\; 2$. Very few examples of compact manifolds with this particular holonomy group are known, and it is a deep result that in every quaternionic dimension n there are up to isometry only finitely many of these manifolds with positive scalar curvature $\kappa \;>\; 0$ ([LeBSa94]). In fact, the only known examples with $\kappa \;>\; 0$ are symmetric spaces, the so-called Wolf spaces.

In order to introduce quaternionic Kähler holonomy we return for a moment to a point we glossed over in the definition of the holonomy bundle. There we had to choose a suitable representative $\mathrm{Hol} \;\subset\; \mathbf{O}_{4n}\mathbb{R}$ in the conjugacy class of the holonomy groups acting on an abstract vector space $\mathbb{R}^{4n}$. This abstract vector space has no meaning in itself but plays the role of the tangent representation of Hol just as T_pM is the tangent representation of Hol_pM. Instead of really choosing a representative $\mathrm{Hol} \;\subset\; \mathbf{O}_{4n}\mathbb{R}$ it is always better to start with specifying this tangent representation. Let us begin with an abstract complex vector space $E \;\cong\; \mathbb{C}^{2n}$ endowed with a symplectic form $\sigma \;\in\; \Lambda^2 E^*$ and an adapted, positive quaternionic structure J, i. e., a conjugate linear map $J:\; E \longrightarrow E$ satisfying

$$J^2 \;=\; -1 \qquad\qquad \sigma(Je_1,\, Je_2) \;=\; \overline{\sigma(e_1,\, e_2)} \qquad\qquad \sigma(e,\, Je) \;>\; 0$$

for all $e_1,\, e_2 \in E$ and $e \neq 0$. Such a set of structures is consistent and can be defined on the underlying complex vector space of $\mathbb{H}^n$. One merit of this explicit construction is that the group of all symplectic transformations of E commuting with J agrees in this picture with the quaternionic unitary group $\mathbf{Sp}\,(n) \;:=\; \{\, A \in \mathrm{M}_{n,n}\mathbb{H}$ such that $\overline{A}^t A = 1\,\}$. The symplectic form σ induces mutually inverse isomorphisms $\sharp : \; E \longrightarrow E^*,\; e \longmapsto \sigma(e,\cdot)$ and $\flat : \; E^* \longrightarrow E$. Similar to the representation of $\Lambda^2 T_pM$ on T_pM considered in the first section there is an action

$$\bullet :\;\; \mathrm{Sym}^2 E \times E \;\longrightarrow\; E, \qquad (e_1 e_2,\, e) \;\longmapsto\; (e_1 e_2) \bullet e \;:=\; \sigma(e_1,\, e) e_2 \,+\, \sigma(e_2,\, e) e_1$$

of the second symmetric power $\mathrm{Sym}^2 E$ on E. This action is skew symplectic and commutes with J for all real elements of $\mathrm{Sym}^2 E$. It identifies this real subspace with the Lie algebra $\mathfrak{sp}(n)$ of $\mathbf{Sp}\,(n)$ and makes $\bullet$ not only an action but a representation.

Let $H \;\cong\; \mathbb{C}^2$ be another abstract vector space with the same structures, a symplectic form $\sigma \;\in\; \Lambda^2 H^*$ and an adapted, positive quaternionic structure J. The tensor product $H \otimes E$ of these two vector spaces carries a real structure $J \otimes J$ and a complex bilinear symmetric form $\langle\,,\,\rangle \;:=\; \sigma \otimes \sigma$, which is positive definite on the real subspace. In this way the group $\mathbf{O}\,(H \otimes E)$ of all complex linear isometries of $H \otimes E$ commuting with $J \otimes J$ is isomorphic to $\mathbf{O}_{\,4n}\mathbb{R}$ and has a distinguished subgroup $\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n) \;:=\; \mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n)/\mathbb{Z}_2$ preserving the tensor product structure of $H \otimes E$:

Definition 2.1 *(Quaternionic Kähler Manifolds)*
A quaternionic Kähler manifold M is a Riemannian manifold of dimension $4n$, $n \;\geq\; 2$, endowed with a reduction of the frame bundle $\mathbf{O}\,(M)$ to a principal $\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n)$–bundle $\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(M)$ stable under parallel transport. Such a reduction exists if and only if the holonomy group Hol of M is conjugated to a subgroup of $\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n) \;\subset\; \mathbf{O}\,(H \otimes E)$ and in case of equality it may be defined as:

$$\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(M) := \{\, f : H \otimes E \longrightarrow T_pM \otimes_{\mathbb{R}} \mathbb{C} \,|\, f \textit{ isometry and } f(\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n)) = \mathrm{Hol}_{\,p}M \,\}\,.$$

If the holonomy group of a quaternionic Kähler manifold M is conjugated to a proper subgroup of $\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n)$, then M is necessarily locally symmetric and its universal covering is a Wolf space.

There are a few remarks to make on this definition. First of all we insist on $n \;\geq\; 2$, because taking this definition as it stands it applies to every oriented Riemannian manifold M of dimension 4. In addition a quaternionic Kähler manifold with vanishing scalar curvature $\kappa \;=\; 0$ is locally hyperkähler, its universal cover thus hyperkähler, and we will usually exclude these manifolds from consideration. In general, however, a quaternionic Kähler manifold with non–vanishing scalar curvature is despite nomenclature not Kähler.

In order to justify terminology after all these negative remarks and to get into contact with a more common definition of quaternionic Kähler manifolds we recall that $\mathrm{Sym}^2 H$ acts via $(h_1 h_2) \bullet h \;:=\; \sigma(h_1,\, h) h_2 + \sigma(h_2,\, h) h_1$ on H. For a normed real element $i\, h\, Jh \in \mathrm{Sym}^2 H$ with $\sigma(h,\, Jh) = 1$ the action on H commutes with J and satisfies:

$$(i\, h\, Jh) \bullet (i\, h\, Jh) \bullet \quad = \quad -\mathrm{id}\;\; .$$

This follows from the fundamental identity $\sigma(h_1, h)h_2 - \sigma(h_2, h)h_1 = \sigma(h_1, h_2)h$ for 2–dimensional symplectic vector spaces and hence does not work for E. Extending this action from H to the tangent representation $H \otimes E$ we conclude that normed real local sections of the parallel subbundle $\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(M) \times_{\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)} \mathrm{Sym}^2 H$ of the complexified endomorphism bundle $\mathrm{End}\,(TM \otimes_{\mathbb{R}} \mathbb{C})$ act as local complex structures on the tangent bundle TM. Choosing in this way three local complex structures I, J and K satisfying $IJ = K$ we define the canonical quaternionic orientation of M by declaring every base of the form $X_1, IX_1, JX_1, KX_1, \ldots, X_n, IX_n, JX_n, KX_n$ to be positively oriented. Alternatively the canonical quaternionic orientation is induced by the n–th power of the parallel Kraines form $\Omega \in \Lambda^4(H \otimes E)$ defined in ([Kra66]).

A rather subtle remark concerns the two representations H and E, which do not factor through the projection $\mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n) \longrightarrow \mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n)$. Although we may think of the complex tangent bundle as a tensor product of two complex vector bundles H and E, these vector bundles are not well defined and in general exist only locally. In passing from representation theory to geometry we always have to check, whether the representations factor through the projection $\mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n) \longrightarrow \mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n)$. Things get actually simpler in some respect, as the spinor representation $\mathbf{S}$ of $\mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n)$ factors through to a representation of $\mathbf{Sp}\,(1) \cdot \mathbf{Sp}\,(n)$ whenever n is even. Thus all quaternionic Kähler manifolds of even quaternionic dimension n are spin:

Proposition 2.2 *(The Signed Spinor Representation ([BaS83], [Wan89]))*
The spinor representation $\mathbf{S}$ of $\mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n)$ decomposes into the direct sum

$$\mathbf{S} \quad = \quad \bigoplus_{r=0}^{n} \mathbf{S}_r \quad := \quad \bigoplus_{r=0}^{n} \mathrm{Sym}^r H \otimes \Lambda^{n-r}_{\circ} E \tag{8}$$

where $\Lambda^{n-r}_{\circ} E$ is the kernel of the contraction $\sigma :\ \Lambda^{n-r} E \longrightarrow \Lambda^{n-r-2} E$ with the symplectic form. For the canonical quaternionic orientation of $H \otimes E$ the half spin representations are given by:

$$\mathbf{S}^+ \quad := \quad \bigoplus_{r \equiv n\,(2)} \mathbf{S}_r \qquad\qquad \mathbf{S}^- \quad := \quad \bigoplus_{r \not\equiv n\,(2)} \mathbf{S}_r\,.$$

The delicate point in an explicit proof of this proposition avoiding representation theory is the choice of Clifford multiplication $\star :\ (H \otimes E) \times \mathbf{S} \longrightarrow \mathbf{S}$. Besides the Clifford identity

$$(h_1 \otimes e_1) \star\, (h_2 \otimes e_2) \star\, + (h_2 \otimes e_2) \star\, (h_1 \otimes e_1) \star \quad = \quad -\,2\,\sigma(h_1, h_2)\,\sigma(e_1, e_2) \tag{9}$$

which has to be satisfied, there is another crucial property of this multiplication, namely the compatibility condition with the action of the Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ on $\mathbf{S}$. The representation $\bullet$ of the complexified Lie algebra $\mathrm{Sym}^2 H \oplus \mathrm{Sym}^2 E$ of the group $\mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n)$ on $\mathbf{S}$ has to agree with the representation implicitly defined by Clifford multiplication via $(X \wedge Y) \bullet := \frac{1}{2}(X \star Y \star + \langle X, Y \rangle)$. This condition depends on the correct formulation of the embedding $\mathrm{Sym}^2 H \oplus \mathrm{Sym}^2 E \longrightarrow \Lambda^2(TM \otimes_{\mathbb{R}} \mathbb{C})$. Choosing dual pairs of bases $\{de_\mu\}$, $\{e_\nu\}$ for E^*, E with $\langle de_\mu, e_\nu \rangle = \delta_{\mu\nu}$ and $\{dh_\alpha\}$, $\{h_\beta\}$ for H^*, H we can check that

$$(e\,\tilde{e}) \;\longmapsto\; \sum_\alpha (dh^\flat_\alpha\otimes e)\wedge(h_\alpha\otimes\tilde{e}) \qquad\qquad (h\,\tilde{h}) \;\longmapsto\; \sum_\mu (h\otimes de^\flat_\mu)\wedge(\tilde{h}\otimes e_\mu) \tag{10}$$

is the correct choice intertwining the representations of $\mathrm{Sym}^2 H$, $\mathrm{Sym}^2 E$ and $\Lambda^2(TM\otimes_{\mathbb{R}}\mathbb{C})$ on $H\otimes E = TM\otimes_{\mathbb{R}}\mathbb{C}$. Consequently the following two operator identities on the spinor representation $\mathbf{S}$ are at the heart of Proposition 2.2:

$$(e\,\tilde{e})\bullet \;=\; \frac{1}{2}\sum_\alpha\Big((dh^\flat_\alpha\otimes e)\star(h_\alpha\otimes\tilde{e})\star+\sigma(e,\,\tilde{e})\Big) \tag{11}$$

$$(h\,\tilde{h})\bullet \;=\; \frac{1}{2}\sum_\mu\Big((h\otimes de^\flat_\mu)\star(\tilde{h}\otimes e_\mu)\star+\sigma(h,\,\tilde{h})\Big) \tag{12}$$

We will not go into the details of this construction given in [KSW97a], but will take Proposition 2.2 as the assertion that a Clifford multiplication $\star:\ (H\otimes E)\times\mathbf{S}\longrightarrow\mathbf{S}$ with the properties (11) and (12) exists satisfying the Clifford identity (9).

The most important point in our present discussion of quaternionic Kähler holonomy is of course the discussion of the curvature tensor of a quaternionic Kähler manifold and of the associated element $q(R)$ in the universal enveloping algebra of the Lie algebra $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$ of the holonomy group $\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)$. In fact compared to other holonomy groups quaternionic Kähler holonomy is rather rigid. This is mainly due to the fact that the curvature tensor of a quaternionic Kähler manifold has to satisfy very stringent constraints and can be described completely by the scalar curvature κ and a section $\mathfrak{R}$ of $\mathrm{Sym}^4 E$. This decomposition was first derived by D. V. Alekseevskii (cf.:[Al68] or[Sal82]) and can be made explicit in the following way:

Lemma 2.3 *(The Curvature Tensor)*
A quaternionic Kähler manifold M is Einstein with constant scalar curvature κ. Its curvature tensor depends only on κ and a section $\mathfrak{R}$ of $\mathrm{Sym}^4 E$, this dependence reads

$$R \;=\; -\frac{\kappa}{8n(n+2)}(R^H+R^E)+R^{hyper} \tag{13}$$

where the endomorphism valued two forms R^H, R^E and R^{hyper} are defined by:

$$\begin{aligned}
R^H_{h_1\otimes e_1,h_2\otimes e_2} &= \sigma_E(e_1,e_2)(h_1h_2\bullet\otimes\mathrm{id}_E)\\
R^E_{h_1\otimes e_1,h_2\otimes e_2} &= \sigma_H(h_1,h_2)(\mathrm{id}_H\otimes e_1e_2\bullet)\\
R^{hyper}_{h_1\otimes e_1,h_2\otimes e_2} &= \sigma_H(h_1,h_2)(\mathrm{id}_H\otimes(e_2^\sharp\lrcorner\, e_1^\sharp\lrcorner\,\mathfrak{R})\bullet)
\end{aligned} \tag{14}$$

We will give a short sketch of the proof of this lemma, but refrain from giving all the details which again can be found in [KSW97a]. The essential point is to show that the linear space of $\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)$–curvature tensors, i. e., the intersection of $\mathrm{Sym}^2\mathfrak{hol}\subset\mathrm{Sym}^2\Lambda^2TM$ with the kernel of the Bianchi identity $\wedge:\ \mathrm{Sym}^2\Lambda^2TM\longrightarrow\Lambda^4TM$, is isomorphic to $\mathbb{C}\oplus\mathrm{Sym}^4E$. Consequently our ansatz for R as a linear combination of R^H+R^E and R^{hyper}

is justified since $R^H + R^E$ and R^{hyper} separately satisfy the first Bianchi identity. Note that $(e_2^\sharp \lrcorner e_1^\sharp \lrcorner \mathfrak{R}) \bullet e = \mathfrak{R}(e_1^\sharp, e_2^\sharp, e^\sharp, \cdot)$ is symmetric in e_1, e_2 and e. In order to determine the curvature tensor R completely, it is convenient to calculate the Ricci curvature of M, given by the trace $\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \longmapsto R_{Z,X}Y)$ of the endomorphism $Z \longmapsto R_{Z,X}Y$. The different tensors contribute to this endomorphism in the following way:

$$\begin{aligned}
R^H_{h\otimes e,\, h_1\otimes e_1}\, h_2 \otimes e_2 &= (\,\sigma(h,\, h_2)h_1 + \sigma(h_1,\, h_2)h\,) \otimes \sigma(e,\, e_1)e_2 \\
R^E_{h\otimes e,\, h_1\otimes e_1}\, h_2 \otimes e_2 &= \sigma(h,\, h_1)h_2 \otimes (\,\sigma(e,\, e_2)e_1 + \sigma(e_1,\, e_2)e\,) \\
R^{hyper}_{h\otimes e,\, h_1\otimes e_1}\, h_2 \otimes e_2 &= \sigma(h,\, h_1)h_2 \otimes \mathfrak{R}(e^\sharp,\, e_1^\sharp,\, e_2^\sharp,\, \cdot)
\end{aligned}$$

Note that all these endomorphisms preserve the tensor product structure. Hence their traces are the product of the partial traces in each tensor factor. However, $e \longmapsto \mathcal{R}(e_1^\sharp, e_2^\sharp, e^\sharp, \cdot)$ is induced by an element of $\mathrm{Sym}^2 E$ and hence trace–free, which rules out contributions from R^{hyper} to the Ricci curvature. As the trace of the endomorphism $e \longmapsto \sigma(e, e_2)e_1$ is $\sigma(e_1, e_2)$ the trace of $e \longmapsto \sigma(e, e_2)e_1 + \sigma(e_1, e_2)e$ is given by $(2n+1)\,\sigma(e_1, e_2)$. Similar remarks apply to H and we are left with:

$$(\mathrm{Ric}^H + \mathrm{Ric}^E)(\,h_1 \otimes e_1,\, h_2 \otimes e_2\,) \quad = \quad -\,(\,2n + 4\,)\,\sigma(h_1,\, h_2)\,\sigma(e_1,\, e_2) \tag{15}$$

The Ricci curvature being a multiple of the metric the quaternionic Kähler manifold M is Einstein, a fortiori the scalar curvature κ is constant on M and equation (15) fixes the coefficient of $R^H + R^E$ in R via $\mathrm{Ric}(X,\, Y) = \frac{\kappa}{4n}\langle X,\, Y\rangle$.

At the end of this section we want to describe the action of the element $q(R)$ of the universal enveloping algebra $\mathcal{U}(\,\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\,)$ on some representations. In particular we will see that for a large class of representations of $\mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n)$ the element $q(R)$ acts by scalar multiplication, because the contributions from the hyperkähler part R^{hyper} of the curvature tensor drop out. Observe first that $q(R)$ depends linearly on R:

$$q(R) \quad = \quad -\frac{\kappa}{8n(n+2)}\,\big(q(R^H) + q(R^E)\big) + q(R^{hyper})$$

Using equation (10) we can write down the terms appearing in this sum more explicitly:

Lemma 2.4

$$\begin{aligned}
q(R^H) &= \frac{1}{4}\sum_{\alpha\beta}(dh^\flat_\alpha\, dh^\flat_\beta) \bullet (h_\alpha\, h_\beta) \bullet \\
q(R^E) &= \frac{1}{4}\sum_{\mu\nu}(de^\flat_\mu\, de^\flat_\nu) \bullet (e_\mu\, e_\nu) \bullet \\
q(R^{hyper}) &= \frac{1}{4}\sum_{\mu\nu}(de^\flat_\mu\, de^\flat_\nu) \bullet (e^\sharp_\mu \lrcorner e^\sharp_\nu \lrcorner \mathfrak{R}) \bullet
\end{aligned}$$

Proof: Converting the sum over a local orthonormal base $\{E_i\}$ into the sum

$$\sum_i E_i \otimes E_i \quad = \quad \sum_{\alpha\mu}(dh^\flat_\alpha \otimes de^\flat_\mu) \otimes (h_\alpha \otimes e_\mu)$$

over dual pairs $\{de_\mu\}$, $\{e_\mu\}$ and $\{dh_\alpha\}$, $\{h_\alpha\}$ of bases we calculate say for $q(R^{hyper})$

$$\begin{aligned}
\frac{1}{4}\sum_{ij}(E_i\wedge E_j)\bullet R^{hyper}_{E_i,E_j} &= \frac{1}{4}\sum_{\alpha\beta\mu\nu}(dh^\flat_\alpha\otimes de^\flat_\mu\,\wedge\,dh^\flat_\beta\otimes de^\flat_\nu)\bullet\sigma(h_\alpha,\,h_\beta)(e^\sharp_\mu\lrcorner\, e^\sharp_\nu\lrcorner\,\mathfrak{R})\bullet \\
&= \frac{1}{4}\sum_{\alpha\mu\nu}(dh^\flat_\alpha\otimes de^\flat_\mu\,\wedge\,h_\alpha\otimes de^\flat_\nu)\bullet(e^\sharp_\mu\lrcorner\, e^\sharp_\nu\lrcorner\,\mathfrak{R})\bullet
\end{aligned}$$

which is equivalent to the stated equality in view of equation (10). □

Evidently $2q(R^H)$ and $2q(R^E)$ respectively are the Casimir operators for $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ in σ–normalization, i. e., when the defining invariant symmetric form on the Lie algebra Sym^2H or Sym^2E is not the Killing form itself but the natural extension of σ to the second symmetric powers using Gram's permanent. For some simple irreducible representations it is easy to calculate the Casimir eigenvalues of $q(R^H)$ and $q(R^E)$ directly. Strictly speaking this procedure is unnecessary because the general formula for these eigenvalues in terms of the highest weight is simple enough. In this way, however, we get all the Casimir eigenvalues we will need below and the precise relations to the Casimirs in Killing normalization:

Lemma 2.5 *(Casimir Eigenvalues)*
For the irreducible representations Sym^lE *and* $\Lambda^d_\circ E$ *the Casimir eigenvalues for* $q(R^E)$ *are:*

$$q(R^E)_{\mathrm{Sym}^lE} \;=\; -\,l\,(\,n+\frac{l}{2}\,) \qquad\qquad q(R^E)_{\Lambda^d_\circ E} \;=\; -\,d\,(\,n-\frac{d}{2}+1\,)$$

Proof: Both calculations are very similar, for the symmetric power Sym^lE we get:

$$\begin{aligned}
\frac{1}{4}\sum_{\mu\nu}(de^\flat_\mu\,de^\flat_\nu)\bullet(e_\mu\,e_\nu)\bullet &= \frac{1}{4}\sum_{\mu\nu}(de^\flat_\nu\cdot de_\mu\lrcorner+de^\flat_\mu\cdot de_\nu\lrcorner)\,(e_\nu\cdot e^\sharp_\mu\lrcorner+e_\mu\cdot e^\sharp_\nu\lrcorner) \\
&= \frac{1}{2}\sum_{\mu\nu}(de^\flat_\nu\cdot\delta_{\mu\nu}\,e^\sharp_\mu\lrcorner+de^\flat_\nu\cdot e^\sharp_\nu\lrcorner+de^\flat_\nu\cdot e_\mu\cdot de_\mu\lrcorner\, e^\sharp_\nu\lrcorner) \\
&= \frac{1}{2}(\,-\,l\,-\,2nl\,-\,l(l\,-\,1)\,) \;=\; -\,l\,(n+\frac{l}{2}) \quad \square
\end{aligned}$$

The eigenvalues of $q(R^H)$ are given by the same formulas with $n=1$. Setting $l=2$ we get the Casimir eigenvalues for $q(R^E)$ and $q(R^H)$ in the adjoint representations Sym^2E and Sym^2H of $\mathfrak{sp}(n)$ and $\mathfrak{sp}(1)$. Since by definition the Casimir eigenvalue of the adjoint representation is always one for Casimirs in the Killing normalization we get:

$$q(R^E) \;=\; -\,2\,(n+1)\,\mathrm{Cas}_{\mathfrak{sp}(n)} \qquad\qquad q(R^H) \;=\; -\,4\,\mathrm{Cas}_{\mathfrak{sp}(1)}$$

Now we claim that the hyperkähler contribution $q(R^{hyper})$ to the element $q(R)$ acts trivially on every irreducible representation occurring in the representation $\Lambda\,E$, i. e., on all representations $\Lambda^d_\circ E$ with $d=0,\,\ldots,\,n$. Because $q(R^{hyper})$ depends linearly on $\mathfrak{R}\in\mathrm{Sym}^4E$ we are allowed to expand $\mathfrak{R}$ into a sum of fourth powers $\frac{1}{24}e^4$, $e\in E$, to calculate $q(R^{hyper})$.

It is thus sufficient to prove that the action of $q(\frac{1}{24}e^4)$ on ΛE is trivial for all $e \in E$. According to Lemma 2.4 the element $q(\frac{1}{24}e^4)$ acts on ΛE as:

$$q(\frac{1}{24}e^4) \;=\; \frac{1}{2}(\frac{1}{2}e^2)\bullet(\frac{1}{2}e^2)\bullet \;=\; \frac{1}{2}(e\wedge e^\sharp \lrcorner)(e\wedge e^\sharp \lrcorner) \;=\; -\frac{1}{2}e\wedge e\wedge e^\sharp \lrcorner\, e^\sharp \lrcorner \;=\; 0\,.$$

Consequently the curvature tensor $q(R)$ will act by scalar multiplication on all representations $\mathcal{R}^{l,\,d} := \mathrm{Sym}^{\,l}H\otimes\Lambda^{\,d}_{\circ}E$. From equation (7) we conclude that the squares $D^2_{\mathcal{R}^{l,\,d}}$ of the twisted Dirac operators with these particular twists have properties similar to the Hodge–Laplacian Δ and the square D^2 of the untwisted Dirac operator:

Proposition 2.6 *(Global Decomposition Principle)*
The restriction $D^2_{\mathcal{R}^{l,\,d}}|_\pi$ of the square of a twisted Dirac operator $D^2_{\mathcal{R}^{l,\,d}}$ with twisting bundle $\mathcal{R}^{l,\,d} := (\mathrm{Sym}^{\,l}H\otimes\Lambda^{\,d}_{\circ}E)(M)$ to a parallel subbundle $\pi(M)\subset(\mathbf{S}\otimes\mathcal{R}^{l,\,d})(M)$ does not depend on the specific embedding of this subbundle and equation (7) becomes in this case:

$$\Delta_\pi \;=\; D^2_{\mathcal{R}^{l,\,d}}\Big|_\pi \;+\; \frac{\kappa}{8n(n+2)}(\,l\,+\,d\,-\,n\,)(\,l\,-\,d\,+\,n\,+\,2\,)$$

3 Classification of Minimal and Maximal Twists

In this section we will focus attention on the technicalities necessary to draw conclusions from Proposition 2.6. The irreducible representations occurring in the twisted spinor representations $\mathbf{S}\otimes\mathcal{R}^{l,\,d}$ are all of the form $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,\,b}_{\mathrm{top}}E$, where $\Lambda^{a,\,b}_{\mathrm{top}}E$ is the irreducible representation of highest weight in the tensor product $\Lambda^{\,a}_{\circ}E\otimes\Lambda^{\,b}_{\circ}E$. Alternatively we see from Weyl's construction of the irreducible representations of the classical matrix groups that $\Lambda^{a,\,b}_{\mathrm{top}}E$ is the common kernel of the diagonal contraction with the symplectic form $\sigma:\ \Lambda^{\,a}_{\circ}E\otimes\Lambda^{\,b}_{\circ}E \longrightarrow \Lambda^{\,a-1}_{\circ}E\otimes\Lambda^{\,b-1}_{\circ}E$ and the Plücker differential:

$$\sum_\mu e_\mu\wedge\otimes de_\mu\lrcorner:\quad \Lambda^{\,a}_{\circ}E\,\otimes\,\Lambda^{\,b}_{\circ}E \;\longrightarrow\; \Lambda^{\,a+1}E\,\otimes\,\Lambda^{\,b-1}_{\circ}E$$

In particular, we will characterize the twists $\mathcal{R}^{l,\,d}$ with $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,\,b}_{\mathrm{top}}E\subset\mathbf{S}\otimes\mathcal{R}^{l,\,d}$. Moreover, for each representation $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,\,b}_{\mathrm{top}}E$ in this class and will classify the special twists maximizing the curvature expression

$$-\frac{\kappa}{8n(n+2)}\,(\,l\,+\,d\,-\,n\,)(\,l\,-\,d\,+\,n\,+\,2\,)$$

of Proposition 2.6 for $\kappa > 0$ and $\kappa < 0$. This classification is the most important step used in the applications of the ideas encoded in Proposition 2.6. Global questions are postponed to the next sections. Hence, we will deal with representations of $\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)$ only.

Theorem 3.1 *(Characterization of Admissible Twists)*
A representation $\mathcal{R}^{l,\,d} := \mathrm{Sym}^{\,l}H\otimes\Lambda^{\,d}_{\circ}E$ with $l\geq 0$ and $n\geq d\geq 0$ is called an admissible twist

for the irreducible representation $\mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E$*, if there exists a non–trivial, equivariant homomorphism from* $\mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E$ *to the twisted spinor representation* $\mathbf{S} \otimes \mathcal{R}^{l,\,d}$*:*

$$\mathrm{Hom}_{\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)}(\,\mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E,\ \mathbf{S} \otimes \mathcal{R}^{l,\,d}\,) \quad \neq \quad \{0\}$$

A twist $\mathcal{R}^{l,\,d}$ *is admissible in this sense if and only if* $k + a + b \equiv n + l + d \bmod 2$ *and:*

$$\begin{aligned} b &\leq d && (16)\\ |\,k - l\,| + |\,a - d\,| &\leq n - b && (17)\\ |\,n - a + b - d\,| &\leq k + l && (18) \end{aligned}$$

A simple consequence of Theorem 3.1 is that every irreducible representation $\mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E$ occurs in a twisted spinor representation, e. g. in $\mathbf{S} \otimes \mathcal{R}^{k+n-b,\,a}$ and $\mathbf{S} \otimes \mathcal{R}^{|n-a-k|,\,b}$. In fact for the twist $\mathcal{R}^{k+n-b,\,a}$ inequality (17) is trivial and (18) needs $|n - 2a + b| \leq |n - a| + |a - b|$. For the second twist $\mathcal{R}^{|n-a-k|,\,b}$ inequality (17) follows from the distance decreasing property $||x| - |y|| \leq |x - y|$ of the absolute value via $||-k| - |n - a - k|| \leq n - a$, whereas (18) reduces to $|n - a| \leq \max\{n - a,\ 2k - n + a\} = k + |n - a - k|$. These two twists are the prototype examples of maximal and minimal twists to be defined below.

Proof: For the proof we recall a well–known fusion rule for the tensor product $\Lambda^c_\circ E \otimes \Lambda^d_\circ E$ of the two irreducible $\mathbf{Sp}\,(n)$–representations $\Lambda^c_\circ E$ and $\Lambda^d_\circ E$ (cf. [OnVi90]):

$$\Lambda^c_\circ E \otimes \Lambda^d_\circ E \quad = \quad \bigoplus_{\substack{a+b\,\equiv\, c+d \bmod 2\\ a+b\,\leq\, c+d\\ |c-d|\,\leq\, a-b\,\leq\, 2n-c-d}} \Lambda^{a,b}_{\mathrm{top}} E$$

Note in particular that each irreducible representation $\Lambda^{a,b}_{\mathrm{top}} E$ occurs at most once in the tensor product $\Lambda^c_\circ E \otimes \Lambda^d_\circ E$. Using this fusion rule together with the Clebsch–Gordan formula for irreducible $\mathbf{Sp}\,(1)$–representations and the decomposition of the spinor representation $\mathbf{S}$ under $\mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n)$ given in Proposition 2.2 we can formally write down the decomposition

$$\bigoplus_{c=0}^{n} (\mathrm{Sym}^{n-c} H \otimes \Lambda^c_\circ E) \otimes (\mathrm{Sym}^l H \otimes \Lambda^d_\circ E) \quad = \sum_{\substack{k\geq 0\\ n\geq a\geq b\geq 0}} \sharp\, \mathfrak{M}_{k,\,a,\,b}(l,\,d) \cdot \mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E \quad (19)$$

of $\mathbf{S} \otimes \mathcal{R}^{l,\,d}$, where $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is the set of all $n \geq c \geq 0$ satisfying the set of constraints:

$$\begin{array}{rcl} k &\equiv& n + c + l \mod 2\\ k &\leq& n - c + l\\ k &\geq& |n - c - l| \end{array} \qquad \begin{array}{rcl} a + b &\equiv& c + d \mod 2\\ a + b &\leq& c + d\\ a - b &\geq& |c - d|\\ a - b &\leq& 2n - c - d \end{array} \qquad (20)$$

It is clear from these constraints that $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is empty unless $k + a + b \equiv n + l + d \bmod 2$ reflecting in a way the consistency of the action of $(-1,\,-1) \in \mathbf{Sp}\,(1) \times \mathbf{Sp}\,(n)$. In particular, $k + a + b \equiv n + l + d \bmod 2$ is a necessary condition for the twist $\mathcal{R}^{l,\,d}$ to be admissible.

In view of this congruence we can drop one of the two constraints $a+b \equiv c+d \bmod 2$ or $k \equiv n+c+l \bmod 2$ and solve the inequalities (20) for c to arrive after a little manipulation at an equivalent description of $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ as the set of all $c \equiv a+b+d \bmod 2$ satisfying:

$$\max\{\,b+|a-d|, n-k-l\,\} \;\le\; c \;\le\; n \;-\; \max\{\,|k-l|, |n-a+b-d|\} \tag{21}$$

Under the standing hypothesis $k+a+b \equiv n+l+d \bmod 2$ we evidently have

$$\max\{\,b+|a-d|, n-k-l\,\} \;\equiv\; a\;+\;b\;+\;d \;\equiv\; n \;-\; \max\{\,|k-l|, |n-a+b-d|\} \quad \bmod 2$$

so that $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ will be non–empty if and only if the inequality (21) is consistent, because the congruence $c \equiv a+b+d \bmod 2$ will be fulfilled by either end of the resulting interval. However, the consistency condition for (21) is given by four inequalities in l, d depending of course on k, a, b. The first $n-k-l \;\le\; n-|k-l|$ is trivial for $k,\,l \ge 0$ and the next two become inequalities (17) and (18), whereas the last $b+|a-d| \;\le\; n-|n-a+b-d|$ is equivalent to inequality (16) for all $b \le a \le n$ and $d \le n$. $\square$

Note that if the set $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is non-empty all its elements will have the same parity as $a+b+d$. Of course their number $\sharp\,\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is just the multiplicity of the representation $\mathrm{Sym}^{\,k}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$ in $\mathbf{S} \otimes \mathcal{R}^{l,\,d}$, which we will need below as index multiplicity:

Definition 3.2 *(The Index of an Admissible Twist)*
The index of an admissible twist $\mathcal{R}^{l,\,d}$ for an irreducible representation $\mathrm{Sym}^{\,k}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$ is the index multiplicity of $\mathrm{Sym}^{\,k}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$ in the twisted spinor representation $\mathbf{S}^{\,\pm} \otimes \mathcal{R}^{l,\,d}$:

$$\begin{aligned}\mathrm{index}\,(k,a,b;\,l,d) \;&:=\; \dim\;\mathrm{Hom}_{\,\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)}(\mathrm{Sym}^{\,k}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E,\;\mathbf{S}^{\,+} \otimes \mathcal{R}^{l,\,d})\\ &\qquad -\;\dim\;\mathrm{Hom}_{\,\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)}(\mathrm{Sym}^{\,k}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E,\;\mathbf{S}^{\,-} \otimes \mathcal{R}^{l,\,d})\end{aligned}$$

From the proof of Theorem 3.1 we can easily read off an explicit formula for this index:

$$\begin{aligned}&\mathrm{index}\,(k,a,b;\,l,d) \;:=\\ &\quad\frac{(-1)^{a+b+d}}{2}\Big(n\;+\;2\;-\;\max\{|k-l|, |n-a+b-d|\}\;-\;\max\{b+|a-d|, n-k-l\}\Big)\end{aligned}$$

Although we have calculated the index multiplicity of the representation $\mathrm{Sym}^{\,k}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$ for an arbitrary twisted spinor representation $\mathbf{S} \otimes \mathcal{R}^{l,\,d}$, it will turn out below that only very few representations actually contribute to the index of a particular twisted Dirac operator. These representations are characterized by the following extremality condition:

Definition 3.3 *(Minimal and Maximal Twists)*
An admissible twist $\mathcal{R}^{l,\,d} := \mathrm{Sym}^{\,l}H \otimes \Lambda^{\,d}_{\circ}E$ for the irreducible representation $\mathrm{Sym}^{\,k}H \otimes \Lambda^{a,b}_{\mathrm{top}}E$ is called a minimal or maximal twist, if the curvature term of Proposition 2.6, or equivalently the function $\phi(\tilde{l},\tilde{d}) \;:=\; (\tilde{l}\,+\,\tilde{d}\,-\,n)\,(\tilde{l}\,-\,\tilde{d}\,+\,n\,+\,2)$, assumes its minimum or maximum among all admissible twists $\mathcal{R}^{\tilde{l},\,\tilde{d}}$ in the twist $\mathcal{R}^{l,\,d}$.

To determine the index of a twisted Dirac operator in terms of the dimension of the eigenspaces of the operators Δ_π, all we will further need is a classification of all minimal twists for negative scalar curvature $\kappa < 0$ and similarly of all maximal twists for $\kappa > 0$:

Theorem 3.4 *(Classification of Maximal Twists)*
All representations $\mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E$ *with* $k > 0$ *or* $a > b$ *have unique maximal twists:*

$$\mathcal{R}^{k+n-b,\,a} \quad = \quad \mathrm{Sym}^{\,k+n-b} H \otimes \Lambda^{\,a}_{\circ} E \qquad\qquad \mathrm{index}\,(k, a, b;\; k+n-b, a) \;=\; (-1)^b$$

For the special representations $\Lambda^{a,a}_{\mathrm{top}} E$ *with* $k = 0$ *and* $a = b$ *all admissible twists* $\mathcal{R}^{n-d,d}$ *with* $d = a, \ldots, n$ *have* $\phi(n-d, d) = 0$ *and are thus automatically maximal and minimal:*

$$\mathcal{R}^{n-d,\,d} \quad = \quad \mathrm{Sym}^{\,n-d} H \otimes \Lambda^{\,d}_{\circ} E \qquad\qquad \mathrm{index}\,(0, a, a;\; n-d, d) \;=\; (-1)^d$$

The classification of all minimal twists splits into more cases:

Theorem 3.5 *(Classification of Minimal Twists)*
According to their minimal twists the irreducible representations $\mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E$ *are divided into four classes. In the first class we have* $k > (n-a) + (n-b)$ *and a unique minimal twist:*

$$\mathcal{R}^{k-n+b,\,a} \quad = \quad \mathrm{Sym}^{\,k-n+b} H \otimes \Lambda^{\,a}_{\circ} E \qquad\qquad \mathrm{index}\,(k, a, b;\; k-n+b, a) \;=\; (-1)^b$$

In the second class with $k = (n-a) + (n-b)$ *the minimal twist is no longer unique. All minimal twists for representations in this class are given by*

$$\mathcal{R}^{n-d,\,d} \quad = \quad \mathrm{Sym}^{\,n-d} H \otimes \Lambda^{\,d}_{\circ} E \qquad\qquad \mathrm{index}\,(k, a, b;\; n-d, d) \;=\; (-1)^{k+d}$$

with $d = b, \ldots, a$. *The special representations* $\Lambda^{a,a}_{\mathrm{top}} E$ *with* $k = 0$ *and* $a = b$ *form the third class overlapping in* $k = 0$ *and* $a = b = n$ *with the second. All admissible twists* $\mathcal{R}^{n-d,\,d}$ *with* $d = a, \ldots, n$ *for these special representations are minimal and maximal at the same time:*

$$\mathcal{R}^{n-d,\,d} \quad = \quad \mathrm{Sym}^{\,n-d} H \otimes \Lambda^{\,d}_{\circ} E \qquad\qquad \mathrm{index}\,(0, a, a;\; n-d, d) \;=\; (-1)^d$$

The remaining representations are characterized by $k < (n-a) + (n-b)$ *and* $k + (a-b) > 0$. *The minimal twists of the representations in this fourth class are all unique:*

$$\mathcal{R}^{|n-a-k|,\,b} \quad = \quad \mathrm{Sym}^{\,|n-a-k|} H \otimes \Lambda^{\,b}_{\circ} E \qquad\qquad \mathrm{index}\,(k, a, b;\; |n-a-k|, b) \;=\; (-1)^a$$

Before proceeding to the actual proofs of Theorem 3.4 and Theorem 3.5 let us agree on some geometric terms in order to help intuition. The set of solutions to the inequality (17) in (l, d)–space is a ball in L^1–norm, i. e. a diamond, with center (k, a) and radius $n - b$. Its right and left corner are thus $(k \pm (n-b), a)$ with $(k, a \pm (n-b))$ being its top and bottom corner. On the other hand the set of solutions to the inequality (18) is the cone $\{\, (l, d) \mid l + d \geq -k + n - a + b \text{ and } l - d \geq -k - n + a - b \,\}$ opening diagonally to the right from its vertex in the point $(-k, n - a + b)$.

In particular the set of solutions to both inequalities (17) and (18) is always a rectangle in (l, d)–space, which may degenerate into a straight line but always contains at least the points $(k+n-b,\, a)$ and $(|n-a-k|,\, b)$. Note that all corners of the diamond as well as the vertex of the cone and the corners of the resulting intersection rectangle satisfy the congruence condition $l+d \equiv n+k+a+b$, which consequently will care for itself below.

Finally the level sets of the function $\phi(l,\, d) \;=\; (l{+}d{-}n)\,(l{-}d{+}n{+}2)$, which we are going to extremize, are hyperbolas with two diagonal axes $l+d \;=\; n$ and $l-d \;=\; -n-2$ dividing $(l,\, d)$–space into four quadrants. In the first quadrant with $l+d \;\geq\; n,\; l-d \;\geq\; -n-2$ the function $\phi \geq 0$ is positive, whereas it is negative in the second $l+d \;\leq\; n,\; l-d \;\geq\; -n-2$. Eventually we only care for points $l \geq 0$ and $n \geq d \geq 0$ in these two quadrants.

Proof of Theorem 3.4: We already know that the right corner $(k{+}n{-}b,\, a)$ of the diamond always corresponds to the admissible twist $\mathcal{R}^{k+n-b,\, a}$ since $|n{-}2a{+}b| \;\leq\; |n{-}a|{+}|a{-}b| \;=\; n{-}b$. If this right corner of the diamond lies in the strict interior of the first quadrant, then it will be the unique point, where ϕ assumes its maximum on the diamond, tacitly ignoring of course third and fourth quadrant. In particular the twist $\mathcal{R}^{k+n-b,\, a}$ will be the unique maximal twist as soon as $k+n-b+a \;>\; n$, equivalently $k > 0$ or $a > b$.

Assuming now $k = 0$ and $a = b$ we see that the top corner $(0,\, n)$ of the diamond coincides with the vertex of the cone. Thus the intersection rectangle degenerates into the face of the diamond running from its top corner $(0,\, n)$ to its right corner $(n-a,\, a)$. Consequently the admissible twists are exactly the twists $\mathcal{R}^{n-d,\, d}$ with $d \;=\; a,\, \ldots,\, n$ and $\phi(n{-}d,\, d) \;=\; 0$. The calculation of the index multiplicities is left to the reader. $\square$

Proof of Theorem 3.5: We first concentrate on the case $k \;>\; (n-a)+(n-b)$ or equivalently $k-n+b+a \;>\; n$, where the diamond lies completely in the strict interior of the first quadrant since its left corner does. With the axes of the level sets of ϕ running parallel to the faces of the diamond ϕ assumes a unique minimum on the diamond in its left corner. Consequently we are done once we have checked that $\mathcal{R}^{k-n+b,\, a}$ is an admissible twist. However inequality (18) immediately follows from $|n-2a+b| \;\leq\; n-b \;<\; k$, which is needed for calculating the index multiplicity, too.

Assuming next that $k \;=\; (n-a)+(n-b)$ the left corner of the diamond is the point $(n-a,\, a)$ in the first quadrant. Hence, all of the diamond lies in the first quadrant $\phi \geq 0$ with $\phi \;=\; 0$ only on the face from its left to its bottom corner $(2n-a-b,\, a-n+b)$. Note that the bottom corner fails to satisfy inequality (16) and that inequality (18) is satisfied by the left corner $(n-a,\, a)$ due to $|n-2a+b| \;\leq\; n-b \;\leq\; k$. Taking this into account the only admissible twists satisfying $\phi \;=\; 0$ are exactly the twists $\mathcal{R}^{n-d,\, d}$ with $d \;=\; b,\, \ldots,\, a$.

The admissible twists for the special representations $\Lambda^{a,\,a}_{\mathrm{top}}E$ with $k \;=\; 0$ and $a \;=\; b$ are exactly the twists $\mathcal{R}^{n-d,\, d}$ with $d \;=\; a,\, \ldots,\, n$, because the top corner $(0,\, n)$ of the diamond coincides with the vertex of the cone. As all these admissible twists have $\phi(n-d,\, d) \;=\; 0$, they are all both minimal and maximal.

Recall now that $\mathcal{R}^{|n-a-k|,\, b}$ is an admissible twist, because $|n-a| \;\leq\; k+|n-a-k|$ and $||-k|-|n-a-k|| \;\leq\; n-a$ by distance decrease. Turning to geometry we see that the bottom corner of the intersection rectangle of cone and diamond will be either $(k,\, a{-}n{+}b)$ for $k \;\geq\; n{-}a$ or $(n{-}a,\, b{-}k)$ for $k \;\leq\; n{-}a$, i. e. whatever point has larger l and d–coordinate. In particular this bottom corner fails in general to satisfy inequality (16) chopping off a triangle from the rectangle. The resulting face runs from the point $(|n-a-k|,\, b)$ to $(n-a+k,\, b)$ independent of whether $k \;\geq\; n-a$ or $k \;\leq\; n-a$. Note that the geometry may become even more complicated, but we already know that the twist $\mathcal{R}^{|n-a-k|,b}$ is admissible, which fixes this problem as far as we need it.

In order to classify the minimal twists of the remaining representations characterized by $k < (n-a)+(n-b)$ and $k+(a-b) > 0$ we observe that these two assumptions together are equivalent to $|n-a-k|+b < n$, so that the point $(|n-a-k|,\, b)$ will lie in the strict interior of the second quadrant. From the geometric discussion above we conclude that ϕ assumes a unique minimum in this point, because the tangents to the level surfaces of ϕ are never diagonal and horizontal only for $l = -1 < |n-a-k|$. $\Box$

4 Eigenvalue estimates

The potential applications of Proposition 2.6 include eigenvalue estimates for the Laplace and for twisted Dirac operators. The general procedure is described in this section and carried out in some particularly interesting cases. Our first example are the irreducible $\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)$–representations $\mathrm{Sym}^r H \otimes \Lambda^r_\circ E$ defining parallel subbundles in the bundle of r–forms (cf. [Sal86]). On these parallel subbundles we have the following lower bound for the spectrum of the Laplace operator.

Proposition 4.1 *(Eigenvalue Estimate on* $\mathrm{Sym}^r H \otimes \Lambda^r_\circ E$*)*
Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of positive scalar curvature $\kappa > 0$. Then any eigenvalue λ of the Laplace operator restricted to $\mathrm{Sym}^r H \otimes \Lambda^r_\circ E$ satisfies

$$\lambda \quad \geq \quad \frac{r(n+1)}{2n(n+2)}\,\kappa\,.$$

Proof: It follows from Theorem 3.4 that $\mathrm{Sym}^{n+r} H \otimes \Lambda^r_\circ E$ is a maximal twist for the representation $\mathrm{Sym}^r H \otimes \Lambda^r_\circ E$. Using Proposition 2.6 with $l = n+r$ and $d = r$ we obtain:

$$\Delta_{\mathrm{Sym}^r H\otimes\Lambda^r_\circ E} \quad = \quad D^2_{\mathcal{R}^{n+r,r}}\Big|_{\mathrm{Sym}^r H\otimes\Lambda^r_\circ E} \;+\; \frac{r(n+1)}{2n(n+2)}\,\kappa \quad \geq \quad \frac{r(n+1)}{2n(n+2)}\,\kappa\,. \quad \Box$$

An interesting special case is $H \otimes E = TM \otimes_{\mathbb{R}} \mathbb{C}$ for $r = 1$, leading to an eigenvalue estimate for the Laplace operator on 1–forms. In particular, the first Betti number has to vanish. Since the differential of any eigenfunction of the Laplace operator is an eigenform for the same eigenvalue we also obtain an estimate on functions (cf. [AlMa95] and [LeB95]):

Corollary 4.2 *(Vanishing of the First Betti Number for Positive Scalar Curvature)*
Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of positive scalar curvature $\kappa > 0$. Any eigenvalue λ of the Laplace operator on non–constant functions or 1–forms satisfies

$$\lambda \quad \geq \quad \frac{n+1}{2n(n+2)}\,\kappa\,.$$

Replacing maximal by minimal twists to compensate the sign of the scalar curvature the same argument provides eigenvalue estimates on $\mathrm{Sym}^r H \otimes \Lambda^r_\circ E$ on manifolds with $\kappa < 0$:

Proposition 4.3 *(Vanishing of the First Betti Number for Negative Scalar Curvature)* *Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of negative scalar curvature $\kappa < 0$. Then any eigenvalue λ of the Laplace operator on 1–forms satisfies:*

$$\lambda \quad \geq \quad \frac{|\kappa|}{2(n+2)}\,.$$

In particular the first Betti number has to vanish even in the case of negative scalar curvature.

Proof: Recall that we excluded the case $n = 1$ from the very beginning in Definition 2.1. Since $n \geq 2$ and $r = 1$ we are in the fourth case of Theorem 3.5. The unique minimal twist for $H \otimes E$ is thus $\mathrm{Sym}^{\,n-2}H$ and we can apply Proposition 2.6 with $l = n - 2$ and $d = 0$ to obtain:

$$\Delta_{H\otimes E} \;=\; D^2_{\mathcal{R}^{n-2,0}}\Big|_{H\otimes E} \;-\; \frac{\kappa}{2(n+2)} \;\geq\; \frac{|\kappa|}{2(n+2)}\,. \quad \square$$

The vanishing of the first Betti number in the case of negative scalar curvature was also proved in [Ho96]. In Proposition 5.8 we will prove a stronger vanishing result for the odd Betti numbers.

As an other application we consider the Laplace operator on 2–forms $\Lambda^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}$, which decompose into $\mathrm{Sym}^{\,2}H \oplus (\mathrm{Sym}^{\,2}H \otimes \Lambda^2_\circ E) \oplus \mathrm{Sym}^{\,2}E$. In the next section we will see that the Laplace operator may have a kernel in the sections of the parallel subbundle $\mathrm{Sym}^{\,2}E$. Nevertheless we have a positive lower bound on the other two parallel subbundles:

Proposition 4.4 *(Eigenvalue Estimates on 2–forms)* *Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of positive scalar curvature κ. Then all eigenvalues λ of the Laplace operator on 2–forms in $\mathrm{Sym}^{\,2}H$ or $\mathrm{Sym}^{\,2}H \otimes \Lambda^2_\circ E$ satisfy*

$$\lambda(\Delta_{\mathrm{Sym}^{\,2}H}) \quad \geq \quad \frac{\kappa}{2n} \qquad \textit{and} \qquad \lambda(\Delta_{\mathrm{Sym}^{\,2}H\otimes\Lambda^2_\circ E}) \quad \geq \quad \frac{n+1}{n(n+2)}\,\kappa\,.$$

The estimate for the Laplace operator on $\mathrm{Sym}^{\,2}H \subset \Lambda^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}$ was proved for the first time in [AlMa98]. Again we have similar results in the case of negative curvature. In particular, the lower bound for $\Delta_{\mathrm{Sym}^{\,2}H}$ is the same as in Proposition 4.3.

Our next aim is to derive properties of twisted Dirac operators. For doing so we make the following crucial observation. If π is any representation with admissible twists $\mathcal{R}^{l,d}$ and $\mathcal{R}^{\tilde{l},\tilde{d}}$ then we can apply Proposition 2.6 twice to obtain

$$D^2_{\mathcal{R}^{l,d}}\Big|_{\pi} \;=\; D^2_{\mathcal{R}^{\tilde{l},\tilde{d}}}\Big|_{\pi} \;+\; \frac{\kappa}{8n(n+2)}\,\Big(\phi(\tilde{l},\,\tilde{d}) \;-\; \phi(l,\,d)\Big)\,, \tag{22}$$

with $\phi(l,\,d) = (l + d - n)(l - d + n + 2)$. We first use this observation to give a short proof of the eigenvalue estimate for the untwisted Dirac operator:

Proposition 4.5 *(Eigenvalue Estimate for the Untwisted Dirac Operator [KSW97a])* *Let $(M^{4n},\, g)$ be a compact quaternionic Kähler spin manifold of positive scalar curvature κ. Then any eigenvalue λ of the untwisted Dirac operator satisfies*

$$\lambda^2 \quad \geq \quad \frac{n+3}{n+2}\,\frac{\kappa}{4}\,.$$

Proof: According to Proposition 2.2 the spinor bundle decomposes into the parallel subbundles $\mathbf{S} = \oplus_{r=0}^{n} \mathbf{S}_r$ with $\mathbf{S}_r = \mathrm{Sym}^r H \otimes \Lambda_\circ^{n-r} E$. To estimate the square of the Dirac operator on $\mathrm{Sym}^r H \otimes \Lambda_\circ^{n-r} E$ we observe that the unique maximal twist for $\mathrm{Sym}^r H \otimes \Lambda_\circ^{n-r} E$ is $\mathcal{R}^{n+r,n-r}$ and for $l = d = 0$ and $\tilde{l} = n + r$, $\tilde{d} = n - r$ equation (22) reads:

$$D^2\Big|_{\mathbf{S}_r} \;=\; D^2_{\mathcal{R}^{n+r,n-r}}\Big|_{\mathbf{S}_r} \;+\; \frac{\kappa}{8n(n+2)}\Big(n(2r+n+2) \;+\; n(n+2)\Big) \;\geq\; \frac{n+2+r}{n+2}\,\frac{\kappa}{4}\,.$$

Consequently some hypothetical eigenspinor $\phi \in \Gamma(\mathbf{S})$ of D^2 with eigenvalue $\lambda^2 < \frac{n+3}{n+2}\frac{\kappa}{4}$ would have to be localized in the subbundle $\mathbf{S}_0 \subset \mathbf{S}$. But the Dirac operator on a manifold of positive scalar curvature has trivial kernel so that $D\phi \in \Gamma(\mathbf{S}_1)$ would be a nontrivial eigenspinor for D^2 again with eigenvalue λ^2 in contradiction to the estimate for $\mathbf{S}_1$. □

We now use equation (22) for describing the kernels of twisted Dirac operators in the case of positive scalar curvature. If π is any representations which contributes to the kernel of $D^2_{\mathcal{R}^{l,d}}$ then $\mathcal{R}^{l,d}$ has to be a maximal twist for π. In fact equation (22) implies that $D^2_{\mathcal{R}^{l,d}}$ is positive on π as soon as there is another admissible twist $\mathcal{R}^{\tilde{l},\tilde{d}}$ for π with $\phi(\tilde{l},\tilde{d}) > \phi(l,d)$. From this remark and Proposition 2.6 we conclude in the case of positive scalar curvature

$$\ker(D^2_{\mathcal{R}^{l,d}}) \;=\; \bigoplus_{\pi} \ker\left(\Delta_\pi \;-\; \frac{\kappa}{8n(n+2)}\phi(l,d)\right) \tag{23}$$

where the sum is over all representations π for which $\mathcal{R}^{l,d}$ is a maximal twist. Since $\frac{\kappa}{8n(n+2)}\,\phi(l,d)$ is the smallest possible eigenvalue of the operator Δ_π equation (23) is in essence a decomposition of $\ker(D^2_{\mathcal{R}^{l,d}})$ into a sum of minimal eigenspaces for the operators Δ_π.

If $\mathcal{R}^{l,d}$ is a maximal twist for a representation π then Theorem 3.4 also provides us with the information whether π occurs in $\mathbf{S}^+ \otimes \mathcal{R}^{l,d}$ or in $\mathbf{S}^- \otimes \mathcal{R}^{l,d}$. Hence a corollary of equation (23) is a formula for the index of the twisted Dirac operator $D_{\mathcal{R}^{l,d}}$ in terms of dimensions of certain minimal eigenspaces. We will describe this in two examples:

Proposition 4.6
Let (M^{4n}, g) be a compact quaternionic Kähler manifold of positive scalar curvature $\kappa > 0$, then:

$$\ker\left(D^2_{\mathcal{R}^{l,d}}\right) \;=\; \{0\} \qquad \textit{for} \qquad l + d < n\,.$$

Proof: All maximal twists $\mathcal{R}^{l,d}$ satisfy $l + d \geq n$ by Theorem 3.4. □

An immediate consequence of this proposition is the vanishing of the index $\mathrm{ind}\,(D_{\mathcal{R}^{l,d}})$ for $l+d < n$. This was also proved in [LeBSa94] by using the Akizuki–Nakano vanishing theorem on the twistor space. For the second example we consider the twisted Dirac operator $D_{\mathcal{R}^{n+2,0}}$. It easily follows from Theorem 3.4 that $\mathrm{Sym}^2 H$ is the only representation with maximal twist $\mathcal{R}^{n+2,0}$:

Proposition 4.7 *(Killing Vector Fields)*
On every compact quaternionic Kähler manifold (M^{4n}, g) of positive scalar curvature κ we have:

$$\ker\left(D^2_{\mathcal{R}^{n+2,0}}\right) \;=\; \ker\left(\Delta_{\mathrm{Sym}^2 H} \;-\; \frac{\kappa}{2n}\right)\,.$$

The index of $D_{\mathcal{R}^{n+2,0}}$ equals the dimension of the isometry group of (M, g) (cf. [Sal82]). But since $\mathrm{Sym}^2 H$ is the only representation contributing to $\ker(D^2_{\mathcal{R}^{n+2,0}})$ the index is just the dimension of the minimal eigenspace of $\Delta_{\mathrm{Sym}^2 H}$. In fact, there is an explicit isomorphism from the space of Killing vector fields to $\mathrm{Sym}^2 H$ (cf. [AlMa98]). It is given by projecting the covariant derivative of a Killing vector field onto its component in $\mathrm{Sym}^2 H \subset \Lambda^2 T^* M \otimes_{\mathbb{R}} \mathbb{C}$.

5 Harmonic forms and Betti numbers

This section contains the most important application of Proposition 2.6. We will determine which parallel subbundles of the differential forms may carry harmonic forms and thus prove vanishing theorems for Betti numbers both for positive and negative scalar curvature. These results will lead to quaternionic Kähler analogues of the weak and strong Lefschetz theorem in Kähler geometry. Recall that the weak Lefschetz theorem for Kähler manifolds M states the inequality $b_k \leq b_{k+2}$ of the Betti numbers for $k < \frac{1}{2}\dim M$, whereas the strong Lefschetz theorem asserts that the wedge product with the parallel 2–form descends to an injective map of the cohomology $H^k(M, \mathbb{R}) \longrightarrow H^{k+2}(M, \mathbb{R})$.

Proposition 5.1 *(Representations and Harmonic Forms)*
Let (M^{4n}, g) be a compact quaternionic Kähler manifold of scalar curvature $\kappa \neq 0$ and let π be an irreducible representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ occurring in the forms $\Lambda^\bullet(H \otimes E)$:

$$\mathrm{Hom}_{\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)}(\pi, \Lambda^\bullet(H \otimes E)) \neq \{0\}$$

If the scalar curvature is positive then $\ker(\Delta_\pi) = \{0\}$ unless $\pi = \Lambda^{a,a}_{\mathrm{top}} E$ for some a with $n \geq a \geq 0$. Similarly if the scalar curvature is negative then $\ker(\Delta_\pi) = \{0\}$ unless either $\pi = \Lambda^{a,a}_{\mathrm{top}} E$ as before or π is a representation of the form $\pi = \mathrm{Sym}^{2n-a-b} H \otimes \Lambda^{a,b}_{\mathrm{top}} E$ with $n \geq a \geq b \geq 0$.

Although the representations $\mathrm{Sym}^{2n-a-b} H \otimes \Lambda^{a,b}_{\mathrm{top}} E$ form a larger class of representations they are still rather special among all the representations occurring in the forms. The appearance of these exceptional representations potentially carrying harmonic forms could have been foreseen from the difficulties encountered in the attempt to push Kraines original strong Lefschetz theorem ([Kra66]) for quaternionic Kähler manifolds beyond degree n. In higher degrees the given proofs fail precisely for these representations. It follows from Proposition 5.1 that this problem is absent in the positive scalar curvature case.

Proof: For any manifold of even dimension the bundle of exterior forms is the tensor product of the spinor bundle with itself. The decomposition of $\mathbf{S}$ given in Proposition 2.2 implies:

$$\Lambda^\bullet(H \otimes E) \quad = \quad \mathbf{S} \otimes \mathbf{S} \quad = \quad \bigoplus_{r=0}^{n} \mathbf{S} \otimes \mathcal{R}^{r,n-r}.$$

In particular, a representation π occurs in the forms if and only if it occurs in a twisted spinor bundle $\mathbf{S} \otimes \mathcal{R}^{r,n-r}$ for some r with $n \geq r \geq 0$. It is consequently of the form

$\pi = \mathrm{Sym}^k H \otimes \Lambda^{a,b}_{\mathrm{top}} E$ for suitable $k \geq 0$ and $n \geq a \geq b \geq 0$. In this situation Proposition 2.6 becomes:

$$\Delta\Big|_\pi \quad = \quad \Delta_\pi \quad = \quad D^2_{\mathcal{R}^{r,\,n-r}}\Big|_\pi$$

A harmonic form in the parallel subbundle determined by π is thus identified with an harmonic twisted spinor for the twist $\mathcal{R}^{r,\,n-r}$. However, we have already expressed the kernel of the twisted Dirac operators $D^2_{\mathcal{R}^{r,\,n-r}}$ in formula (23) at least for positive scalar curvature.

The point in this formula is of course that only those representations π may contribute to the kernel of the twisted Dirac operator $D^2_{\mathcal{R}^{r,\,n-r}}$, for which the twist $\mathcal{R}^{r,\,n-r}$ is a maximal twist. Replacing maximal by minimal twists the same argument applies in the case of negative scalar curvature and we conclude that a representation π may carry harmonic forms in the case of negative or positive scalar curvature if and only if it has a minimal or maximal twist respectively of the form $\mathcal{R}^{r,\,n-r}$ for some r with $n \geq r \geq 0$. A look at the classification of maximal and minimal twists in Theorems 3.4 and 3.5 completes the proof. □

We now want to point out a remarkable property of minimal and maximal twists: If a twist $\mathcal{R}^{l,\,d}$ is minimal or maximal for a representation π then π always occurs with multiplicity one in the twisted spinor representation $\mathbf{S} \otimes \mathcal{R}^{l,\,d}$. Although this property seems very natural it is obtained only as a corollary of the calculation of the index multiplicities in Theorems 3.4 and 3.5 using all the rather technical calculations of that section. Surely it is tempting to search for a direct argument providing better insight into the nature of this property.

For us this property is very convenient counting the total multiplicity of those representations π in the differential forms, which may carry harmonic forms. In fact for any representation π this total multiplicity is given by:

$$\dim \mathrm{Hom}_{\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)}(\,\pi,\,\Lambda^\bullet(H\otimes E)\,) \quad = \quad \sum_{r=0}^{n} \dim \mathrm{Hom}_{\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)}(\,\pi,\,\mathbf{S}\otimes\mathcal{R}^{r,\,n-r}\,)\;. \quad (24)$$

However, in the course of the proof of Theorem 5.1 we characterized the representations π potentially carrying harmonic forms in negative or positive scalar curvature by their property of having a minimal or maximal twist respectively of the form $\mathcal{R}^{r,\,n-r}$, $n \geq r \geq 0$. For such a representation π a twist of the form $\mathcal{R}^{\tilde{r},\,n-\tilde{r}}$ is minimal or maximal respectively if and only if it is admissible, because in this case $\phi(r, n-r) \;=\; 0 \;=\; \phi(\tilde{r}, n-\tilde{r})$.

Consequently for any representation π which may carry harmonic forms the summands on the right hand side of equation (24) are all either 0 or 1 and the total multiplicity of π in the differential forms is just the number of different minimal or maximal twists respectively. This number is easily read off from Theorems 3.4 and 3.5 and is part of the following lemma:

Lemma 5.2 *(Embeddings of Harmonic Forms)*
The representation $\pi = \Lambda^{a,a}_{\mathrm{top}} E$, $n \geq a \geq 0$, occurs $n-a+1$ times in the forms: it occurs with multiplicity one in the forms of degree $2a$, $2a+4$, $2a+8$, $\ldots$, $4n-2a$. Similarly the representation $\pi = \mathrm{Sym}^{2n-a-b} H \otimes \Lambda^{a,b}_{\mathrm{top}} E$, $n \geq a \geq b \geq 0$, occurs in the forms of degree $2n-a+b$, $2n-a+b+2$, $2n-a+b+4$, $\ldots$, $2n+a-b$ with multiplicity 1 and $a-b+1$ times in total.

Proof: We have already calculated the total multiplicity of the representations $\Lambda^{a,\,a}_{\rm top}E$ and $\mathrm{Sym}^{\,2n-a-b}H\otimes\Lambda^{a,\,b}_{\rm top}E$ in the differential forms so that it is sufficient to prove the existence of embeddings of these representations into the forms of the claimed degrees. First let us recall the well known general decomposition of the exterior forms $\Lambda^{\,k}(\,H\otimes E\,)$ into Schur functors

$$\Lambda^{\,k}(\,H\otimes E\,)\quad=\quad\bigoplus_{\mathfrak{Y}}\mathrm{Schur}_{\,\mathfrak{Y}}H\otimes\mathrm{Schur}_{\,\overline{\mathfrak{Y}}}E$$

where the sum is over all Young tableaus $\mathfrak{Y}$ of size $|\mathfrak{Y}|=k$ and $\overline{\mathfrak{Y}}$ denotes the conjugated Young tableau ([FuHa]). All Schur functors have two preferred realizations as the images of Schur symmetrizers in iterated tensor products. Specifying the Young tableau $\mathfrak{Y}$ either by the length of its rows $(r_1,\,r_2,\,\ldots,r_{c_1})$ or of its columns $(c_1,\,c_2,\,\ldots,\,c_{r_1})$ satisfying $r_1\geq r_2\geq\ldots\geq r_{c_1}$ and $c_1\geq c_2\geq\ldots\geq c_{r_1}$ these two preferred realizations of the Schur functors

$$\begin{array}{rcl}\mathrm{Schur}_{\,\mathfrak{Y}}H & \subset & \Lambda^{\,c_1}H\ \otimes\ \Lambda^{\,c_2}H\ \otimes\ \ldots\ \otimes\ \Lambda^{\,c_{r_1}}H\\ \mathrm{Schur}_{\,\mathfrak{Y}}E & \subset & \mathrm{Sym}^{\,r_1}E\otimes\mathrm{Sym}^{\,r_2}E\otimes\ \ldots\ \otimes\mathrm{Sym}^{\,r_{c_1}}E\end{array}$$

are given as the intersection of the kernels of all possible Plücker differentials. In our case all Schur functors in H corresponding to Young tableaus of more than two rows vanish and since $\Lambda^{\,2}H\cong\mathbb{C}$ is trivial the Schur functor in H for the Young tableau of size k with two rows $(k-s,\,s)$ is equivalent to $\mathrm{Sym}^{\,k-2s}H$:

$$\Lambda^{\,k}(H\otimes E)\quad=\quad\bigoplus_{s=0}^{\lfloor\frac{k}{2}\rfloor}\ \mathrm{Sym}^{\,k-2s}H\otimes\mathrm{Schur}_{\,\overline{(k-s,\,s)}}E\,.$$

Conjugation of Young tableaus is defined by exchanging rows and columns. Conjugated to the Young tableau with two rows $(k-s,\,s)$ is the tableau with two columns $\overline{(k-s,\,s)}$. Thus $\mathrm{Schur}_{\,\overline{(k-s,\,s)}}E$ can be defined as the kernel of the Plücker differential:

$$\sum_{\mu}e_\mu\wedge\otimes de_\mu\,\lrcorner:\quad\Lambda^{\,k-s}E\otimes\Lambda^{\,s}E\ \longrightarrow\ \Lambda^{\,k-s+1}E\otimes\Lambda^{\,s-1}E\,.$$

From Weyl's construction of the representation $\Lambda^{a,\,b}_{\rm top}E$ as the intersection of the kernel of the Plücker differential $\Lambda^{\,a}_{\,\circ}E\otimes\Lambda^{\,b}_{\,\circ}E\longrightarrow\Lambda^{\,a+1}E\otimes\Lambda^{\,b-1}_{\,\circ}E$ with the kernel of the diagonal contraction with the symplectic form we see that $\Lambda^{a,\,a}_{\rm top}E\subset\mathrm{Schur}_{\,\overline{(a,\,a)}}E$. Consider now the map

$$\Omega:\quad\Lambda^{\,a}E\otimes\Lambda^{\,b}E\ \longrightarrow\ \Lambda^{\,a+2}E\otimes\Lambda^{\,b+2}E$$

defined by

$$\Omega:=\sum_{\mu,\nu}\big(\,de^\flat_\mu\wedge de^\flat_\nu\wedge\otimes e_\mu\wedge e_\nu\wedge\ +de^\flat_\mu\wedge e_\mu\wedge\otimes de^\flat_\nu\wedge e_\nu\wedge\big)\ ,$$

which curiously enough commutes with the Plücker differential. Consequently we may extend the above embedding to a chain of $\mathbf{Sp}\,(n)$–equivariant linear maps:

$$\Lambda^{a,\,a}_{\rm top}E\ \longrightarrow\ \mathrm{Schur}_{\,\overline{(a,\,a)}}E\ \stackrel{\Omega}{\longrightarrow}\ \mathrm{Schur}_{\,\overline{(a+2,\,a+2)}}E\ \stackrel{\Omega}{\longrightarrow}\ \ldots\ \stackrel{\Omega}{\longrightarrow}\ \mathrm{Schur}_{\,\overline{(2n-a,\,2n-a)}}E\,.$$

Explicit calculation shows that $\Omega^{n-a} = (2n-2a+1)!\,(\star\otimes\star)$ on $\Lambda^{a,\,a}_{\mathrm{top}}E$, where $\star$ denotes the Hodge isomorphism $\Lambda^{a}E \longrightarrow \Lambda^{2n-a}E$. Hence $\Lambda^{a,\,a}_{\mathrm{top}}E$ embeds into all the Schur functors $\mathrm{Schur}_{\overline{(a+2s,\,a+2s)}}E$ with $n-a\geq s\geq 0$ and further into the forms $\Lambda^{2a+4s}(\,H\otimes E\,)$ of degree $2a+4s$ with $n-a\geq s\geq 0$. The appearance of the map Ω is by no means an accident, it can be shown that it corresponds exactly to the wedge product with the parallel Kraines form Ω on the level of forms.

The construction of the different embeddings of the representations $\mathrm{Sym}^{2n-a-b}H\otimes\Lambda^{a,\,b}_{\mathrm{top}}E$ is simpler, although it is a dead end to start with the inclusion $\Lambda^{a,\,b}_{\mathrm{top}}E\subset\mathrm{Schur}_{\overline{(a,\,b)}}E$. Instead we have to use the Hodge isomorphism $(\star\otimes 1):\ \Lambda^{a}E\otimes\Lambda^{b}E\longrightarrow\Lambda^{2n-a}E\otimes\Lambda^{b}E$, which interchanges in a sense the roles of the Plücker differential and the diagonal contraction with the symplectic form. The Hodge isomorphism can be extended to a chain of maps

$$\Lambda^{a,\,b}_{\mathrm{top}}E\ \longrightarrow\ \Lambda^{2n-a}E\otimes\Lambda^{b}E\ \xrightarrow{\ \sigma\ }\ \Lambda^{2n-a+1}E\otimes\Lambda^{b+1}E\ \xrightarrow{\ \sigma\ }\ \ldots\ \xrightarrow{\ \sigma\ }\ \Lambda^{2n-b}E\otimes\Lambda^{a}E$$

using the diagonal multiplication σ with the symplectic form. Since diagonal contraction and multiplication with the symplectic form generate an $\mathfrak{sl}_2$–algebra of operators the final map $\Lambda^{a,\,b}_{\mathrm{top}}E\longrightarrow\Lambda^{2n-b}E\otimes\Lambda^{a}E$ is injective and maps into the kernel of σ. In addition the commutator relations between the Plücker differential and σ imply that $\Lambda^{a,\,b}_{\mathrm{top}}E$ is mapped into the kernel $\mathrm{Schur}_{\overline{(2n-a+s,\,b+s)}}E$ of the Plücker differential at each step, so that

$$\mathrm{Sym}^{2n-a-b}H\otimes\Lambda^{a,\,b}_{\mathrm{top}}E\ \longrightarrow\ \mathrm{Sym}^{2n-a-b}H\otimes\mathrm{Schur}_{\overline{(2n-a+s,\,b+s)}}E\ \xrightarrow{\ \subset\ }\ \Lambda^{2n-a+b+2s}(\,H\otimes E\,)$$

embeds into the forms of degree $2n-a+b+2s$ for all $a-b\geq s\geq 0$. $\square$

Remark 5.3 *(Strong Lefschetz Theorems)*
In the course of the proof of Lemma 5.2 we have sketched a proof of the strong Lefschetz Theorem for quaternionic Kähler manifolds of positive scalar curvature. The wedge product with the parallel Kraines form Ω is injective on the forms of type $\Lambda^{a,\,a}_{\mathrm{top}}E$ in all degrees $k<\frac{1}{2}\dim M$ and hence descends to an injective map of the cohomology $H^{k}(M,\,\mathbb{R})\longrightarrow H^{k+4}(M,\,\mathbb{R})$.

A completely different argument can be given to show that the wedge product with the Kraines form is injective on forms of type $\mathrm{Sym}^{2n-a-b}H\otimes\Lambda^{a,\,b}_{\mathrm{top}}E$ in degrees $k<\frac{1}{2}\dim M-1$, too. In contrast to the positive scalar curvature case however, the decomposition of the cohomology given in Proposition 5.1 for quaternionic manifolds of negative scalar curvature is finer than the decomposition into primitive cohomologies with respect to the Kraines form.

The weak Lefschetz Theorem for quaternionic Kähler manifolds of positive scalar curvature was proved by S. Salamon (cf. [Sal82]) by analyzing the cohomology of the twistor space. Applying Proposition 5.1 in combination with Lemma 5.2 we get a more explicit version of this result:

Proposition 5.4 *(Weak Lefschetz Theorem for Positive Scalar Curvature)*
Let $(M^{4n},\,g)$ be a compact quaternionic Kähler manifold of positive scalar curvature $\kappa>0$. Its Betti numbers b_k satisfy for all $0\leq k\leq n$ the following relations:

$$\begin{aligned}
b_{2k+1} &= 0\,,\\
b_{2k} &= \sum_{\nu=0}^{\lfloor\frac{k}{2}\rfloor}\dim\left(\ker\Delta_{\Lambda^{k-2\nu,\,k-2\nu}_{\mathrm{top}}E}\right),\\
b_{2k}-b_{2k-4} &= \dim\left(\ker\Delta_{\Lambda^{k,\,k}_{\mathrm{top}}E}\right)\ \geq\ 0\,.
\end{aligned}$$

Proof: For a compact quaternionic Kähler manifold of positive scalar curvature it follows from Proposition 5.1 that the only representations potentially carrying harmonic forms are $\Lambda^{a,\,a}_{\mathrm{top}}E$ with $n \geq a \geq 0$. But according to Lemma 5.2 all these representations embed into forms of even degree, i. e. all odd Betti numbers necessarily vanish. Moreover the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$ occur in the forms of degree $2k$ if and only if $a = k,\, k-2,\, \ldots$ and in this case they occur with multiplicity one. $\square$

Remark 5.5 *(Associated Twistor Space and 3–Sasakian Manifold [GaSa96])*
Let $\mathcal{S}$ be the 3–Sasakian manifold and $\mathcal{Z}$ the twistor space associated with the quaternionic Kähler manifold M^{4n}. The dimension of $\ker \Delta_{\Lambda^{k,\,k}_{\mathrm{top}}E}$ can be reinterpreted as the dimension of the cohomology of $\mathcal{S}$ and as the dimension of the primitive cohomology group of $\mathcal{Z}$:

$$\dim(\ker \Delta_{\Lambda^{k,\,k}_{\mathrm{top}}E}) \;=\; b_{2k}(\mathcal{S}) \;=\; b_{2k}(\mathcal{Z}) \;-\; b_{2k-2}(\mathcal{Z}) \qquad\qquad k \leq n \;\;.$$

As an immediate consequence of Proposition 5.4 we obtain a result of S. Salamon and C. LeBrun (cf. [LeBSa94]) on the index of the twisted Dirac operator $D_{\mathcal{R}^{l,\,d}}$ with $l + d \,=\, n$:

Corollary 5.6 *(Index of Twisted Dirac Operators and Betti Numbers)*
Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of positive scalar curvature $\kappa > 0$:

$$\begin{aligned}
\ker(D^2_{\mathcal{R}^{n-d,\,d}}) \quad &= \quad \textstyle\bigoplus_{a\leq d} \ker(\Delta_{\Lambda^{a,\,a}_{\mathrm{top}}E})\,,\\
\dim\ker(D^2_{\mathcal{R}^{n-d,\,d}}) \quad &= \quad b_{2d} \;+\; b_{2d-2}\,,\\
\mathrm{ind}\,(D_{\mathcal{R}^{n-d,\,d}}) \quad &= \quad (-1)^d(b_{2d} \;+\; b_{2d-2})\,.
\end{aligned}$$

Proof: We already observed in formula (23) that in the case of positive scalar curvature a representation π may contribute to the kernel of a twisted Dirac operator $D^2_{\mathcal{R}^{l,\,d}}$ only if the twist $\mathcal{R}^{l,\,d}$ is maximal for π. On the other hand the twisted spinor representation $\mathbf{S} \otimes \mathcal{R}^{n-d,\,d}$ occurs in the forms so that a representation π contributes to the kernel of $D^2_{\mathcal{R}^{n-d,\,d}}$ if and only if it carries harmonic forms, i. e. π must be one of the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$ for some a with $n \geq a \geq 0$. From equation (16) of Theorem 3.1 it is evident that $\pi = \Lambda^{a,\,a}_{\mathrm{top}}E$ occurs in $\mathbf{S} \otimes \mathcal{R}^{n-d,\,d}$ if and only if $a \leq d$. Consequently Proposition 5.4 provides the expression for the dimension of the kernel of $D_{\mathcal{R}^{n-d,\,d}}$ in terms of Betti numbers. $\square$

In dealing with quaternionic Kähler manifolds of negative scalar curvature it is convenient to decompose their cohomology into two direct summands with quite different behavior:

Definition 5.7 *($\mathfrak{sp}(1)$–Invariant and Exceptional Cohomology)*
Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of negative scalar curvature. According to Proposition 5.1 two different series of representations contribute to harmonic forms on M, namely $\Lambda^{a,\,a}_{\mathrm{top}}E$, $n \geq a \geq 0$ and $\mathrm{Sym}^{\,2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$, $n \geq a \geq b \geq 0$. In particular the de Rham cohomology of M splits into the direct sum

$$H^\bullet_{dR}(\,M,\,\mathbb{C}\,) \quad = \quad H^\bullet_{\mathfrak{sp}(1)}(\,M,\,\mathbb{C}\,) \,\oplus\, H^\bullet_{\mathrm{expt}}(\,M,\,\mathbb{C}\,)$$

of its $\mathfrak{sp}(1)$–invariant cohomology $H^\bullet_{\mathfrak{sp}(1)}(\,M,\,\mathbb{C}\,)$, which is the sum of all isotypical components corresponding to the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$, $n \geq a \geq 0$, and its exceptional cohomology $H^\bullet_{\mathrm{expt}}(\,M,\,\mathbb{C}\,)$, which is the direct sum of all isotypical components corresponding to the remaining representations $\mathrm{Sym}^{\,2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$, $n \geq a \geq b \geq 0$, $b \neq n$.

Because the curvature tensor of M is $\mathfrak{sp}(1)$–invariant the same is true for all its characteristic classes. Moreover $H^{\bullet}_{\mathfrak{sp}(1)}(M, \mathbb{C})$ is closed under multiplication and the decomposition of the de Rham–cohomology into $\mathfrak{sp}(1)$–invariant and exceptional cohomology is respected by the induced modul structure. A deeper analysis of the ring structure of the cohomology ring of M will be given in a forthcoming paper (cf. [Wei00]).

As a final application of the ideas developed in this article we combine Proposition 5.1 and Lemma 5.2 to obtain new information on the Betti numbers of compact quaternionic Kähler manifolds of negative scalar curvature.

Proposition 5.8 *(Weak Lefschetz Theorem for Negative Scalar Curvature)*
Let (M^{4n}, g) be a compact quaternionic Kähler manifold of negative scalar curvature $\kappa < 0$. Its $\mathfrak{sp}(1)$–invariant and exceptional Betti numbers $b_{\mathfrak{sp}(1),\,k}$ and $b_{\mathrm{expt},\,k}$ satisfy:

$$\begin{array}{rcll} b_{\mathfrak{sp}(1),\,k} & = & 0 & \textit{for } k \textit{ odd}\,, \\ b_{\mathrm{expt},\,k} & = & 0 & \textit{for } k \leq n-1\,, \\ b_{\mathfrak{sp}(1),\,k} & \leq & b_{\mathfrak{sp}(1),\,k+4} & \textit{for } k \leq 2n-2\,, \\ b_{\mathrm{expt},\,k} & \leq & b_{\mathrm{expt},\,k+2} & \textit{for } k \leq 2n-1\,. \end{array}$$

In particular, its Betti numbers $b_k = b_{\mathfrak{sp}(1),\,k} + b_{\mathrm{expt},\,k}$ satisfy:

$$\begin{array}{rcll} b_{2k+1} & = & 0 & \textit{for } 2k+1 \leq n-1\,, \\ b_k & \leq & b_{k+2} & \textit{for odd } k \leq 2n-1\,, \\ b_k & \leq & b_{k+4} & \textit{for } \quad k \leq 2n-2\,. \end{array}$$

Proof: Since the $\mathfrak{sp}(1)$–invariant Betti numbers correspond by definition to the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$, $n \geq a \geq 0$, they have the same properties as Betti numbers of a quaternionic Kähler manifolds of positve scalar curvature given in Proposition 5.4.

It follows from Lemma 5.2 that the remaining representations $\mathrm{Sym}^{\,2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$ with $n \geq a \geq b \geq 0$ and $b \neq n$ corresponding to the exceptional Betti numbers embed into forms of degree $2n-a+b,\ 2n-a+b+2,\ \ldots,\ 2n+a-b$. For $a \not\equiv b \bmod 2$ these embeddings give rise to harmonic forms of odd degree. Nevertheless the odd Betti numbers of degree less than n have to vanish because of $2n-a+b \geq n$. $\square$

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