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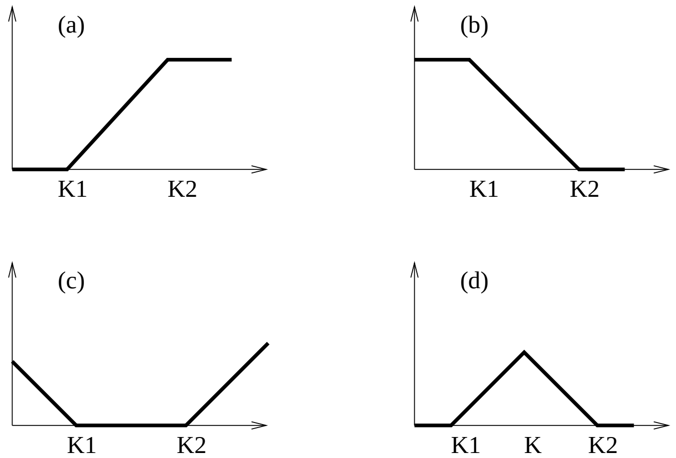
Computational Finance
Set of Exercises

Exercise 1 Portfolios

The figure sketches some payoffs over S: (a) bull spread, (b) bear spread, (c) strangle, (d) butterfly spread. For each of these payoffs, construct portfolios out of two or three vanilla options (same underlying, same expiry) such that the portfolio meets the payoff.



Exercise 2 Bounds and Arbitrage

Using arbitrage arguments, show the following bounds for the values V_C of vanilla call options:

a) $0 \le V_C$

b) $(S-K)^+ \le V_C^{Am} \le S$

Exercise 3 Put-Call Parity

Consider a portfolio consisting of three positions related to the same asset, namely, one share (price S), one European put (value V_P), plus a short position of one European call (value V_C). Put and call have the same expiration date T, and no dividends are paid.

a) Assume a no-arbitrage market without transaction costs. Show

$$S + V_P - V_C = K e^{-r(T-t)}$$

for all t, where K is the strike and r the risk-free interest rate.

b) Use the put-call parity to realize

$$V_C(S,t) \ge S - K e^{-r(T-t)}$$
$$V_P(S,t) \ge K e^{-r(T-t)} - S\,.$$

Exercise 4 Standard Normal Distribution Function

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2})\,\mathrm{d}t$$

a) Use the *error function*

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-t^2)\,\mathrm{d}t$$

to calculate $F(x)$.

b) Establish an algorithm for the evaluation of F that uses a trapezoidal sum with n subintervals as quadrature method. Use the error formula of the trapezoidal sum to guess on a proper n that matches an absolute error of 10^{-8}. Write a computer program.

Hint: This way of calculating F is not very efficient. For comparison with other methods see Topic 10 in the *Topics in Computational Finance.*

c) Write another computer program that implements the Black–Scholes formula (see the end of Chapter 1 in the Course Notes). This program may call the F of b).

d) Evaluate European put and call using the parameters $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S_0 = 5$, $\delta = 0$.

Exercise 5 Anchoring the Binomial Grid at the Strike

The equation $ud = 1$ has established a kind of symmetry for the grid of a binomial tree. As an alternative, one may anchor the grid by requiring (for even M)

$$S_0 u^{M/2} d^{M/2} = K\,.$$

Give a geometrical interpretation. Use $ud = \gamma$ for a proper value of γ and derive the relevant formula

$$u = \beta + \sqrt{\beta^2 - \gamma} \quad \text{for} \quad \beta := \frac{1}{2}(\gamma \mathrm{e}^{-r\Delta t} + \mathrm{e}^{(r+\sigma^2)\Delta t})\,.$$

Exercise 6 Wiener Process

In the definition of a Wiener process the requirement (a) $W_0 = 0$ is dispensable. Then the requirement (b) reads $W_t - W_0 \sim \mathcal{N}(0, t)$. Use this relation to deduce for $t > s$

$$\mathsf{E}(W_t - W_s) = 0\,, \quad \mathsf{Var}((W_t - W_s)) = t - s\,.$$

Hint: $(W_t - W_s)^2 = (W_t - W_0)^2 + (W_s - W_0)^2 - 2(W_t - W_0)(W_s - W_0)$

Exercise 7 Negative Prices

Assume $Z \sim \mathcal{N}(0, 1)$, $S > 0$, $\sigma > 0$, and a step $(t, S) \to (t + \Delta t, S + \Delta S)$ of the discretized GBM

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma Z \sqrt{\Delta t}\,.$$

What is the probability that the resulting price $S + \Delta S$ is negative?

Exercise 8 Transforming the Black–Scholes Equation

Show that the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{BS}$$

for $V(S,t)$ with constant σ and r is equivalent to the equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

for $y(x,\tau)$. For proving this, you may proceed as follows:

a) Use the transformation $S = K\mathrm{e}^x$ and a suitable transformation $t \leftrightarrow \tau$ to show that (BS) is equivalent to
$$-\dot{V} + V'' + \alpha V' + \beta V = 0$$
with $\dot{V} = \frac{\partial V}{\partial \tau}$, $V' = \frac{\partial V}{\partial x}$, α, β depending on r and σ.

b) The next step is to apply for suitable γ, δ a transformation of the type
$$V = K \exp(\gamma x + \delta\tau)\, y(x,\tau)\,.$$

c) Transform the terminal condition of the Black–Scholes equation accordingly.

Exercise 9 Price Evolution of the Binomial Method

For β from Exercise 5 and $u = \beta + \sqrt{\beta^2 - \gamma}$ with $\gamma = 1$ analyze possible cancellation, and show
$$u = \exp\left(\sigma\sqrt{\Delta t}\right) + O\left(\sqrt{(\Delta t)^3}\right)\,.$$

Exercise 10 Implementing a Binomial Method

Write a computer program for calculating the value $V^{(M)}$ of a European or American option. Use the *tilted version* following Exercise 5.

INPUT: r, σ, T, K, S, the choices *put* or *call*, *European* or *American*, and an (initial) M.

Control the mesh size $\Delta t = T/M$ adaptively. For example, a crude startegy calculates V for $M = 8$ and $M = 16$ and in case of a significant change in V use $M = 32$ and possibly $M = 64$.

Test examples:

a) put, European, $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 5$

b) call, otherwise as in a)

c) put, American, $S = 9$, otherwise as in a)

Exercise 11 Analytical Solution of Special SDEs

Apply Itô's lemma with suitable functions g and $Y_t = g(X_t, t)$ to show

a) $X_t = \exp\left(\lambda W_t - \frac{1}{2}\lambda^2 t\right)$ solves $\mathrm{d}X_t = \lambda X_t \,\mathrm{d}W_t$

b) $X_t = \exp\left(2W_t - t\right)$ solves $\mathrm{d}X_t = X_t \,\mathrm{d}t + 2X_t \,\mathrm{d}W_t$

Hint: In a) start with $X_t = W_t$ and $g(x,t) = \exp(\lambda x - \frac{1}{2}\lambda^2 t)$.

Exercise 12 Ornstein–Uhlenbeck Process

An Ornstein–Uhlenbeck process is defined as solution of the SDE

$$\mathrm{d}X_t = -\alpha X_t \,\mathrm{d}t + \gamma \,\mathrm{d}W_t \;, \quad \alpha > 0 \,.$$

a) Show

$$X_t = \mathrm{e}^{-\alpha t}\left(X_0 + \gamma \int_0^t \mathrm{e}^{\alpha s} \mathrm{d}W_s\right)$$

b) Suppose the volatility σ_t is an Ornstein–Uhlenbeck process. Show that the variance $v_t := \sigma_t^2$ follows a Cox–Ingersoll–Ross process, namely,

$$\mathrm{d}v_t = \kappa(\theta - v_t)\,\mathrm{d}t + \sigma_{\mathrm{v}}\sqrt{v_t}\,\mathrm{d}W_t \;.$$

Exercise 13 Moments of the Lognormal Distribution

For the density function

$$f_{\mathrm{GBM}}(S, t - t_0;\, S_0, \mu, \sigma) := \frac{1}{S\sigma\sqrt{2\pi(t-t_0)}} \exp\left\{-\frac{\left(\log(S/S_0) - \left(\mu - \frac{\sigma^2}{2}\right)(t-t_0)\right)^2}{2\sigma^2(t-t_0)}\right\}$$

show

$\int_0^\infty S^2 f(S; t - t_0, S_0)\,\mathrm{d}S = S_0^2 \mathrm{e}^{(\sigma^2 + 2\mu)(t-t_0)}$

Hint: Set $y = \log(S/S_0)$ and transform the argument of the exponential function to a squared term.

Exercise 14 General Black–Scholes Equation

Assume a portfolio

$$\Pi_t = \alpha_t S_t + \beta_t B_t$$

consisting of α_t units of a stock S_t and β_t units of a bond B_t, which obey

$$\mathrm{d}S_t = \mu(S_t, t)\,\mathrm{d}t + \sigma(S_t, t)\,\mathrm{d}W_t$$
$$\mathrm{d}B_t = r(t) B_t \,\mathrm{d}t$$

The functions μ, σ, and r are assumed to be known, and $\sigma > 0$. Further assume the portfolio is *self-financing* in the sense

$$\mathrm{d}\Pi_t = \alpha_t \,\mathrm{d}S_t + \beta_t \,\mathrm{d}B_t \,,$$

and *replicating* such that Π_T equals the payoff of a European option. (Then Π_t equals the price of the option for all t.) Derive the Black–Scholes equation for this scenario, assuming

$\Pi_t = g(S_t, t)$ with g sufficiently often differentiable.
Hint: coefficient matching of two versions of $d\Pi_t$

Exercise 15 Deficient Random Number Generator

Show for the generator

$$N_i = aN_{i-1} \bmod M, \quad \text{with } a = 2^{16} + 3,\ M = 2^{31}$$

and the the sequence $U_i := N_i/M$ that

$$U_{i+2} - 6U_{i+1} + 9U_i \ \text{ is integer.}$$

Exercise 16 Coarse Approximation of Normal Deviates

Let $U_1, U_2, \ldots$ be independent random numbers $\sim \mathcal{U}[0,1]$, and

$$X_k := \sum_{i=k}^{k+11} U_i - 6\,.$$

Calculate mean and variance of the X_k.

Exercise 17 Cauchy-Distributed Random Numbers

A Cauchy-distributed random variable has the density function

$$f_c(x) := \frac{c}{\pi}\frac{1}{c^2 + x^2}\,.$$

Show that its distribution function F_c and its inverse F_c^{-1} are

$$F_c(x) = \frac{1}{\pi}\arctan\frac{x}{c} + \frac{1}{2}\,, \quad F_c^{-1}(y) = c\tan(\pi(y - \frac{1}{2}))\,.$$

How can this be used to generate Cauchy-distributed random numbers out of uniform deviates?

Exercise 18 Inverting the Normal Distribution

Suppose $F(x)$ is the standard normal distribution function. Construct a rough approximation $G(u)$ to $F^{-1}(u)$ for $0.5 \le u < 1$ as follows:

a) Construct a rational function $G(u)$ with correct asymptotic behavior, point symmetry with respect to $(u, x) = (0.5, 0)$, using only one parameter.
b) Fix the parameter by interpolating a given point $(x_1, F(x_1))$.
c) What is a simple criterion for the error of the approximation?

Exercise 19 Uniform Distribution

For the uniformly distributed random variables (V_1, V_2) on the unit disk consider the transformation

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} V_1^2 + V_2^2 \\ \frac{1}{2\pi}\arg((V_1, V_2)) \end{pmatrix}$$

where $\arg((V_1, V_2))$ denotes the corresponding angle. Show that (X_1, X_2) is distributed uniformly.

Exercise 20 Random Number Generators

a) Implement the *linear congruential generator* given by

$$N_i = (aN_{i-1} + b) \mod M \quad \text{with} \;\; a = 1366,\; b = 150889,\; M = 714025.$$

The seed N_0 should be the input value. Use your program to compute 10000 pairs (U_{i-1}, U_i) in the unit square and plot them.

b) Implement the *Fibonacci generator* given by

$$U_i := U_{i-17} - U_{i-5}\,, \qquad U_i := U_i + 1 \quad \text{if} \;\; U_i < 0.$$

Calculate $U_1, \ldots, U_{17}$ with the linear congruential generator of a).

Use your program to compute 10000 pairs (U_{i-1}, U_i) in the unit square and plot them.

c) Implement the *polar method of Marsaglia.* Calculate the input values with the Fibonacci generator of b).

Use your program to compute 10000 standard normally distributed numbers and plot them in two dimensions by separating them vertically with distance 10^{-4}. Furthermore, divide the x-axis into subintervals having the same length and count the computed numbers in each subinterval. Then set up the corresponding histogram.

Exercise 21 Monte Carlo for Option Pricing

Implement a Monte Carlo method for single-asset European options, based on the Black–Scholes model. Perform experiments with various values of N (see below) and a random number generator of your choice. To obtain values for S_T, use the analytic solution formula for S_t and also alternatively Euler's discretization. (Compare the different results.)

Input values: S_0, number of simulations (trajectories) N, payoff function $\Psi(S)$, risk-neutral interest rate r, volatility σ, time to maturity T, strike K.

Output value: approximated value of the option $V_0^{(N)}$.

Compute approximations $V_0^{(N)}$ for $N = 1, 10, 100, 1000, 10000$ for the following option prices at time $t = 0$:

(i) European put with $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 5$, $\delta = 0$,

(ii) European call with the same parameters,

(iii) binary call with the same parameters.

For i) and ii), compare your results with those obtained via the Black–Scholes formula.

Exercise 22

Use the inversion method and uniformly distributed $U \sim \mathcal{U}[0, 1]$ to calculate a stochastic variable X with distribution

$$F(x) = 1 - \mathrm{e}^{-2x(x-a)}\;,\; x \geq a\,.$$

Exercise 23 Error of Biased Monte Carlo

Assume

$$\text{MSE} = \zeta(h,N) := \alpha_1^2 h^{2\beta} + \frac{\alpha_2}{N}$$

as error model of a Monte Carlo simulation with sample size N, based on a discretization of an SDE with stepsize h, where α_1, α_2 are two constants.

a) Argue why for some constant α_3

$$C(h,N) := \alpha_3 \frac{N}{h}$$

is a reasonable model for the costs of the MC simulation.

b) Minimize $\zeta(h,N)$ with respect to h, N subject to the side condition

$$\alpha_3 N/h = C$$

for given budget C.

c) Show that for the optimal h, N

$$\sqrt{\text{MSE}} = \alpha_4 C^{-\frac{\beta}{1+2\beta}} \,.$$

Exercise 24 Integration by Parts

Let X_t, Y_t be Itô processes in $\mathbb{R}$. Prove that

$$\mathrm{d}(X_t Y_t) = X_t \,\mathrm{d}Y_t + Y_t \,\mathrm{d}X_t + \mathrm{d}X_t \cdot \mathrm{d}Y_t \,.$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s \,\mathrm{d}Y_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s \,\mathrm{d}X_s - \int_0^t \mathrm{d}X_s \cdot \mathrm{d}Y_s \,.$$

Hint: Use the general multidimensional Itô Lemma.

Appendix:

Multidimensional Itô-Lemma

Let X_t be an n-dimensional Itô process, i.e.

$$\mathrm{d}X_t = a(X_t,t)\,\mathrm{d}t + b(X_t,t)\,\mathrm{d}W_t$$

where

$$X_t = \begin{pmatrix} X_t^{(1)} \\ \vdots \\ X_t^{(n)} \end{pmatrix}, \quad W_t = \begin{pmatrix} W_t^{(1)} \\ \vdots \\ W_t^{(m)} \end{pmatrix}, \quad a(X_t,t) = \begin{pmatrix} a_1(X_t^{(1)},\dots,X_t^{(n)},t) \\ \vdots \\ a_n(X_t^{(1)},\dots,X_t^{(n)},t) \end{pmatrix},$$

and

$$b(X_t,t) = ((b_{ik}(X_t,t)))_{i=1,\dots,n}^{k=1,\dots,m} \,.$$

Further let $g : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^p$ be a C^2 map.

Then the process

$$Y_t = g(X_t, t)$$

is again an Itô process, whose component with number $k \in \{1,\dots,p\}$, $Y_t^{(k)}$, is given by

$$\mathrm{d}Y_t^{(k)} = \frac{\partial g_k}{\partial t}(X_t,t)\,\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(X_t,t)\,\mathrm{d}X_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(X_t,t)\,\mathrm{d}X_t^{(i)}\mathrm{d}X_t^{(j)}$$

where $\mathrm{d}W_t^{(i)}\mathrm{d}W_t^{(j)} = \delta_{ij}\,\mathrm{d}t,\ \mathrm{d}t\,\mathrm{d}t = \mathrm{d}W_t^{(i)}\mathrm{d}t = \mathrm{d}t\,\mathrm{d}W_t^{(i)} = 0.$

Exercise 25 Integration by Parts for Itô Integrals

a) Show

$$\int_{t_0}^{t} s\,\mathrm{d}W_s = tW_t - t_0 W_{t_0} - \int_{t_0}^{t} W_s\,\mathrm{d}s$$

Hint: Start with the Wiener process $X_t = W_t$ and apply the Itô Lemma with the transformation $y = g(x,t) := tx$.

b) Denote $\Delta Y := \int_{t_0}^{t}\int_{t_0}^{s} \mathrm{d}W_z\,\mathrm{d}s$ and $\Delta t := t - t_0$. Show by using a) that

$$\int_{t_0}^{t}\int_{t_0}^{s} \mathrm{d}z\,\mathrm{d}W_s = \Delta W \Delta t - \Delta Y\,.$$

Exercise 26 Control Variates

Let $\widehat{V}, \widehat{V}^*$ be two random variables, and $V^* := \mathsf{E}(\widehat{V}^*)$. For a free parameter α define the controlled variable

$$V_{\mathrm{CV}}^{\alpha} := \widehat{V} + \alpha(V^* - \widehat{V}^*)\,.$$

a) Show

$$\mathsf{Var}(V_{\mathrm{CV}}^{\alpha}) = \mathsf{Var}(\widehat{V}) + \alpha^2 \mathsf{Var}(\widehat{V}^*) - 2\alpha \mathsf{Cov}(\widehat{V}, \widehat{V}^*)\,.$$

b) Determine the parameter α_0 for which $\mathsf{Var}(V_{\mathrm{CV}}^{\alpha})$ is minimal.

c) The optimal ratio of the variance of the controlled variable to that of the uncontrolled variable is $q_0 := \mathsf{Var}(V_{\mathrm{CV}}^{\alpha_0})/\mathsf{Var}\widehat{V}$. How does q_0 depend on the *correlation* $\rho_{\widehat{V},\widehat{V}^*}$ between $\widehat{V}$ and $\widehat{V}^*$? What is q_0 for $\rho = 0.95$, $\rho = 0.8$ and $\rho = 0.5$?

Exercise 27 Discrete Dividend Payment

Assume that a stock pays a dividend D at ex-dividend date t_D, with $0 < t_D < T$.

a) Assume that a known dividend is paid once per year. Calculate a corresponding continuous dividend rate δ under the assumptions

$$\dot{S} = (\mu - \delta)S\,,\ \mu = 0, \quad S(1) = S(0) - D > 0\,.$$

Generalize the result to general growth rates μ and arbitrary t_D. (To apply for options, note that this assumes $T = 1$.)

b) Define for an American put with strike K

$$\tilde{t} := t_D - \frac{1}{r} \log \left(\frac{D}{K} + 1 \right) .$$

Assume $r > 0$, $D > 0$, and a time instant t in $\tilde{t} < t < t_D$. Argue that instead of exercising early it is reasonable to wait for the dividend.
Note: For $\tilde{t} > 0$, depending on S, early exercise may be reasonable for $0 \leq t < \tilde{t}$.

Exercise 28 Stability of the Fully Implicit Method

The backward-difference method is defined via the solution of the equation

$$A_{\text{impl}}\, w^{(\nu+1)} = w^{(\nu)} , \quad A_{\text{impl}} = \text{tridiag}(-\lambda, 1 + 2\lambda, -\lambda) .$$

Prove the stability.

Exercise 29 Crank–Nicolson Order

Let the function $y(x, \tau)$ solve the equation

$$y_\tau = y_{xx}$$

and be sufficiently smooth. With the difference quotient

$$\delta_{xx} w_{i,\nu} := \frac{w_{i+1,\nu} - 2w_{i,\nu} + w_{i-1,\nu}}{\Delta x^2}$$

the local discretization error ϵ of the Crank–Nicolson method is defined

$$\epsilon := \frac{y_{i,\nu+1} - y_{i,\nu}}{\Delta \tau} - \frac{1}{2} \left(\delta_{xx} y_{i,\nu} + \delta_{xx} y_{i,\nu+1} \right) .$$

Show

$$\epsilon = O(\Delta \tau^2) + O(\Delta x^2) .$$

Exercise 30 Implied Volatility (programming exercise)

For European options we take the valuation formula of Black and Scholes of the type $V = v(S, \tau, K, r, \sigma)$, where τ denotes the time to maturity, $\tau := T - t$, and the function v denotes the Black–Scholes formula. [This formula can be found in the Supplements to Chapter 1 in the Course Notes.] If actual market data V^{mar} of the price are known, then one of the parameters considered known so far can be viewed as unknown and fixed via the implicit equation

$$V^{\text{mar}} - v(S, \tau, K, r, \sigma) = 0 . \qquad (*)$$

In this *calibration* approach the unknown parameter is calculated iteratively as solution of equation $(*)$. Consider σ to be in the role of the unknown parameter. The volatility σ determined in this way is called *implied volatility* and is zero of $f(\sigma) := V^{\text{mar}} - v(S, \tau, K, r, \sigma)$.

Assignment:

a) Design, implement and test an algorithm to calculate the implied volatility of a call. Use Newton's method to construct a sequence $x_k \to \sigma$. The derivative $f'(x_k)$ can be approximated by the difference quotient

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} .$$

For the resulting *secant iteration* invent a stopping criterion that requires smallness of both $|f(x_k)|$ and $|x_k - x_{k-1}|$.

b) Calculate the implied volatilities for the market data

$$T - t = 0.32787,\ S_0 = 7133.06,\ r = 0.0487,$$

K	6400	6700	7000	7300	7600	7900	8200	8500	8800
V	934.0	690.0	469.0	283.0	145.0	62.0	22.0	7.5	2.1

For each calculated value of σ enter the point (K, σ) into a figure, joining the points with straight lines. (You will notice a convex shape of the curve. This shape has lead to call this phenomenon *volatility smile.*)

Exercise 31 Perpetual Put Option

For $T \to \infty$ it is sufficient to analyze the ODE

$$\frac{\sigma^2}{2} S^2 \frac{\mathrm{d}^2 V}{\mathrm{d}S^2} + (r - \delta) S \frac{\mathrm{d}V}{\mathrm{d}S} - rV = 0 .$$

Consider an American put with contact to the payoff $(K - S)^+$ at $S = S_\mathrm{f}$. Show:

a) Upon substituting the boundary condition for $S \to \infty$ one obtains

$$V(S) = c \left(\frac{S}{K} \right)^{\lambda_2} ,$$

where $\lambda_2 = \frac{1}{2}\left(1 - q_\delta - \sqrt{(q_\delta - 1)^2 + 4q}\right)$, $\quad q = \frac{2r}{\sigma^2}$, $\quad q_\delta = \frac{2(r-\delta)}{\sigma^2}$
and c is a positive constant.
Hint: Apply the transformation $S = K\mathrm{e}^x$. (The other root λ_1 drops out.)

b) V is convex.

For $S < S_\mathrm{f}$ the option is exercised; then its intrinsic value is $K - S$. For $S > S_\mathrm{f}$ the option is not exercised and has a value $V(S) > K - S$. The holder of the option decides when to exercise. This means, the holder makes a decision on the contact S_f such that the value of the option becomes maximal.

c) Show: $V'(S_\mathrm{f}) = -1$, if S_f maximizes the value of the option.
Hint: Determine the constant c such that $V(S)$ is continuous in the contact point.

Exercise 32 UL **decomposition**

Assume a square matrix A that is tridiagonal and strictly diagonally dominant. A system of equations $Ax = b$ is to be solved. Formulate the UL decomposition as an algorithm. ($UL = A$, where U is upper triangular, and L is lower triangular with normalized diagonals. Sometimes the matrix U is denoted R; so RL decomposition means the same.)

Exercise 33 Semidiscretization, Method of Lines

For a semidiscretization of the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

consider the semidiscretized domain

$$0 \le t \le T\ , \quad S = S_i := i\Delta S\ , \quad \Delta S := \frac{S_{\max}}{m}\ , \quad i = 0, 1, \ldots, m$$

for suitable values of $S_{\max} > K$ and m. On this set of lines parallel to the t-axis define for $\tau := T - t$ and $1 \le i \le m - 1$ functions $w_i(\tau)$ as approximation to $V(S_i, \tau)$.

a) Using the standard second-order difference schemes, derive the ODE system $\dot{w} = Bw$ that up to boundary conditions approximates the BS equation. Here w is the vector $(w_1, \ldots, w_{m-1})^{tr}$ and $\dot{w}$ denotes differentiation w.r.t. τ . Show that B is a tridiagonal matrix, and calculate its coefficients.

b) For a European option assume Dirichlet boundary conditions for $w_0(\tau)$ and $w_m(\tau)$ and set up a vector c such that

$$\dot{w} = Bw + c$$

realizes the ODE system with correct boundary conditions, and with initial conditions from the payoff.

Exercise 34 Backward Differentiation

(continues Exercise 33) The backward differentiation formula

$$f_i \approx \frac{4}{3}f_{i-1} - \frac{1}{3}f_{i-2} + \frac{2}{3}hf'(x_i) \qquad \text{(BDF2)}$$

(with the usual notations) is of second order, provided the function f is $\mathcal{C}^3$-smooth. Adapt this BDF2 scheme to the initial-value problem with system

$$\dot{w} = Bw + c$$

from Exercise 33 and a European call option.
(Notice that the i-indices of (BDF2) must be properly changed to indices of the ν/t-world.)

Exercise 35 Approximating the Free Boundary

Assume that after a finite-difference calculation of an American put three approximate values $V(S_i, t)$ are available, for a value of t and $i = k, k+1, k+2$. Assume further a k such that these three (S, V)-pairs are close to the free boundary $S_\mathrm{f}(t)$, and inside the continuation area.

a) Derive an approximation $\bar{S}_\mathrm{f}$ to $S_\mathrm{f}(t)$ based on the available data.

b) Discuss the error $O(\bar{S}_\mathrm{f} - S_\mathrm{f})$.

Hints: The derivative $\frac{\partial V}{\partial S}$ at S_f is -1. For b) assume an equidistant spacing of the S_i.

Exercise 36 Front-Fixing for American Options

Apply the transformation

$$\zeta := \frac{S}{S_\mathrm{f}(t)} \quad , \quad y(\zeta, t) := V(S, t)$$

to the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0 \,.$$

a) Show

$$\frac{\partial y}{\partial t} + \frac{\sigma^2}{2} \zeta^2 \frac{\partial^2 y}{\partial \zeta^2} + \left[(r - \delta) - \frac{1}{S_\mathrm{f}} \frac{\mathrm{d} S_\mathrm{f}}{\mathrm{d} t} \right] \zeta \frac{\partial y}{\partial \zeta} - ry = 0$$

b) Set up the domain for (ζ, t) and formulate the boundary conditions for an American call. (Assume $\delta > 0$.)

Most of the exercises are from [R.U. Seydel: Tools for Computational Finance], in other numbering.