

Summer 2012
April, 10th

Computational Finance - 1st Assignment

Deadline: April, 18th

Exercise 1 (Continuous interest rate) (oral exercise)

Assume a time horizon of one year ($t \in [0, 1]$) and a discrete interest payment of $\frac{r}{m}$ at fixed dates $\frac{j}{m}$, $j \in \{1, \dots, m\}$, $m \in \mathbb{N}$, $r \in [0, 1]$. Then, investing an amount of K_0 at $t = 0$ would result in a return of

$$K_1 = K_0 \cdot \left(1 + \frac{r}{m}\right)^m$$

at $t = 1$. Explain the above formula, generalize it to any time horizon ($t \in [0, T]$) and derive a formula for continuous interest payments.

Exercise 2 (Hedging currency risks) (oral exercise)

An U.S. corporation will receive a fixed EUR-amount K at time $t = T$. For hedging its currency risk the corporation buys a currency forward, which offers a fixed EUR-USD exchange rate $F(0, T)$ (unit: [USD/EUR]) in $t = T$.

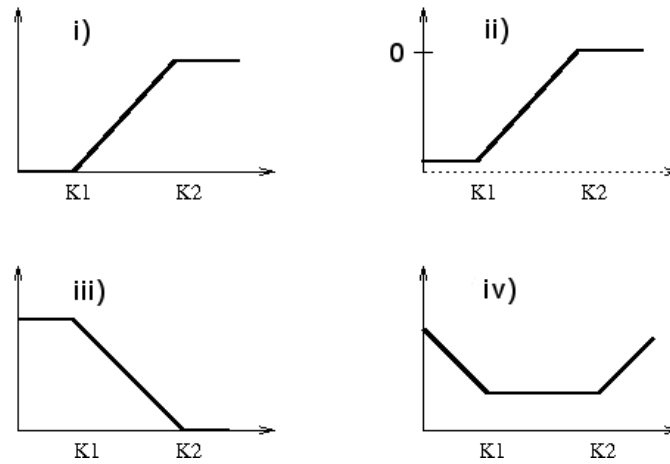
Assume risk free interest rates r_{USD} in the USD-zone and r_{EUR} in the EUR-zone and a today's USD-EUR exchange rate of $E(0)$ (unit: [USD/EUR]). What is the price of $F(0, T)$ at $t = 0$?

Exercise 3 (Portfolios) (5 + 5 points)

In a financial market the following assets are traded: a share S as well as put and call options (all related to S) with three strike prices $K_1 \leq K_2 \leq K_3$.

- a) Sketch the payoffs of the following portfolios. What is the maximum profit or loss in each case?
 - i) Put with strike K_1 as a short and as a long position respectively.
 - ii) Call with strike K_1 as a short and as a long position respectively.
 - iii) One call with strike K_1 (long), two calls with strike K_2 (short) and one call with strike K_3 (long).

- b) For each of the following payoffs, construct portfolios out of vanilla options such that the payoff is met. (In example ii) note that the S -axis is shifted.)



Exercise 4 (No-Arbitrage Principle, Put-Call Parity)

(3+7 points)

- a) Use arbitrage arguments to prove the following bounds on option prices:

i) $V_C \geq 0$

ii) $S \geq V_C^{\text{am}} \geq (S - K)^+$

- b) Consider a portfolio consisting of three positions related to the same asset, namely one share (price S), one European put (value V_P^{eur}), plus a short position of one European call (value V_C^{eur}). Put and call have the same maturity date T , and no dividends are paid.

- i) Show that the *put-call parity*

$$S + V_P^{\text{eur}} - V_C^{\text{eur}} = Ke^{-r(T-t)}$$

holds for all t , where K is the strike and r the risk-free interest rate.

- ii) Use the put-call parity to show

$$V_C^{\text{eur}}(S, t) \geq S - Ke^{-r(T-t)},$$

$$V_P^{\text{eur}}(S, t) \geq Ke^{-r(T-t)} - S.$$

Information:

- The first exercise course takes place on April, 18th.
- Please prepare the oral exercises for presentation in the first exercise course and additionally hand in the written exercises there.
- Additional information can be found under

Summer 2012
April, 18th

Computational Finance - 2nd Assignment

Deadline: April, 25th (written exercises)
May, 2nd (programming exercises)

Exercise 5 (European Options with Infinite Maturity) (3+2 points)

Calculate the following limits:

$$\text{a) } \lim_{T \rightarrow \infty} V_P^{\text{eur}}(S, t), \quad \text{b) } \lim_{T \rightarrow \infty} V_C^{\text{eur}}(S, t).$$

Hint: Consider options with finite maturity and use the bounds of exercise 4.

Exercise 6 (Black-Scholes Formula) (5+15P points)

For a European call the analytic solution of the Black-Scholes equation is given by

$$d_1 := \frac{\log \frac{S}{K} + \left(r - \delta + \frac{\sigma^2}{2}\right) (T - t)}{\sigma \sqrt{T - t}},$$
$$d_2 := d_1 - \sigma \sqrt{T - t} = \frac{\log \frac{S}{K} + \left(r - \delta - \frac{\sigma^2}{2}\right) (T - t)}{\sigma \sqrt{T - t}},$$
$$V_C(S, t) = S e^{-\delta(T-t)} F(d_1) - K e^{-r(T-t)} F(d_2),$$

where F denotes the standard normal cumulative distribution (compare a)), and δ is a continuous dividend yield. The value $V_P(S, t)$ of a European put is obtained by applying the put-call parity

$$V_P = V_C - S e^{-\delta(T-t)} + K e^{-r(T-t)},$$

from which

$$V_P = -S e^{-\delta(T-t)} F(-d_1) + K e^{-r(T-t)} F(-d_2)$$

follows.

a) Establish an algorithm (pseudocode) to calculate

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt.$$

Hint: Construct an algorithm to calculate the *error function*

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

and use $\text{erf}(x)$ to calculate $F(x)$. Use a trapezoidal sum as a quadrature method.

b) Write a computer program that calculates prices of European call and put options using the above Black-Scholes formula and the algorithm constructed in a). Then compute the following option prices at time $t = 0$:

i) European put with $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 5$, $\delta = 0$.

ii) European call with the same parameters.

(programming exercise)

Exercise 7 (Approximation Formula for F)

(5P+4 points)

The function

$$f(x) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

is the density function of the standard normal distribution. Then

$$F(x) := \int_{-\infty}^x f(t) dt$$

is the associated distribution function (compare exercise 6).

The following relation holds for $0 \leq x < \infty$:

$$F(x) = 1 - f(x)(a_1z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5) + \varepsilon(x),$$

where

$$z := \frac{1}{1 + 0.2316419x},$$

$$a_1 = 0.319381530, \quad a_2 = -0.356563782, \quad a_3 = 1.781477937,$$

$$a_4 = -1.821255978, \quad a_5 = 1.330274429$$

and the absolute error ε is bounded by

$$|\varepsilon(x)| < 7.5 \cdot 10^{-8}.$$

Hence we have the approximating formula

$$F(x) \approx 1 - f(x)z(((a_5z + a_4)z + a_3)z + a_2)z + a_1).$$

a) Use this approximation formula to compute the option prices of exercise 6 b). Compare these results with those obtained with your computer program (programming exercise).

b) How many subintervals do you need for the computation of $F(x)$ via a trapezoidal sum in order that the error is smaller than $7.5 \cdot 10^{-8}$? Use the error formula for the trapezoidal sum to answer this question.

Exercise 8 (Transforming the Black–Scholes Equation)

(5+4+3 points)

Show that the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (\text{BS})$$

for $V(S, t)$ with constant σ and r is equivalent to the equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

for $y(x, \tau)$. For proving this, you may proceed as follows:

- a) Use the transformation $S = Ke^x$ and a suitable transformation $t \leftrightarrow \tau$ to show that (BS) is equivalent to

$$-\dot{V} + V'' + \alpha V' + \beta V = 0$$

with $\dot{V} = \frac{\partial V}{\partial \tau}$, $V' = \frac{\partial V}{\partial x}$, α , β depending on r and σ .

- b) Next apply a transformation of the type

$$V = K \exp(\gamma x + \delta \tau) y(x, \tau)$$

for suitable γ , δ .

- c) Transform the terminal condition of the Black–Scholes equation accordingly.

Information:

- You are allowed to work on the programming exercises with a partner.
- The deadline for the programming exercises is May, 2nd. Please hand in a printed version of your code in your exercise group and additionally send it to `programme.nufi@googlemail.com`.
- Additional information can be found under

`www.mi.uni-koeln.de/~aschroet/vorlesung_ss_12` .

Summer 2012
April, 25th

Computational Finance - 3rd Assignment

Deadline: May, 2nd

Exercise 9 (On the Binomial Method)

(4+5+5 points)

a) For $ud = \gamma$ in the binomial method show

$$u = \beta + \sqrt{\beta^2 - \gamma}, \quad \text{where} \\ \beta := \frac{1}{2}(\gamma e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}).$$

b) For $\gamma = 1$ show

$$u = \exp(\sigma\sqrt{\Delta t}) + O(\sqrt{(\Delta t)^3}) \quad \text{and} \quad d = \exp(-\sigma\sqrt{\Delta t}) + O(\sqrt{(\Delta t)^3}).$$

c) Show that

$$\Delta_{Binom} := \frac{V^{(u)} - V^{(d)}}{S_0(u - d)},$$

where $V^{(u)} = V(uS_0, t + \Delta t)$ and $V^{(d)} = V(dS_0, t + \Delta t)$, fulfils

$$\Delta_{Binom} = \frac{\partial V}{\partial S} + O(\Delta t).$$

Exercise 10 (Anchoring the Binomial Grid at K)

(3+3 points)

The equation $ud = 1$ has established a kind of symmetry for the grid. As an alternative, one may anchor the grid in another way by choosing (for even M)

$$S_0 u^{M/2} d^{M/2} = K.$$

a) Give a geometrical interpretation.

b) Derive the relevant formula for u and d .

Hint: Use exercise 9a).

Exercise 11 (Limiting Case of the Binomial Model)

(3+4+3+6 points)

Consider a European call in the binomial model. Suppose the calculated value is $V_0^{(M)}$. In the limit $M \rightarrow \infty$ the sequence $V_0^{(M)}$ converges to the value $V_C(S_0, 0)$ of the continuous Black-Scholes model listed in exercise 6. To prove this, proceed as follows:

- a) Let j_K be the smallest index j with $S_{jM} \geq K$. Find an argument why

$$\sum_{j=j_K}^M \binom{M}{j} p^j (1-p)^{M-j} (S_0 u^j d^{M-j} - K)$$

is the expectation $E(V_T)$ of the payoff.

- b) The approximated value of the option is obtained by discounting: $V_0^{(M)} = e^{-rT} E(V_T)$. Show

$$V_0^{(M)} = S_0 B_{M, \tilde{p}}(j_K) - e^{-rT} K B_{M, p}(j_K).$$

Here $B_{M, p}(j)$ is defined by the binomial distribution, and $\tilde{p} := p e^{-r\Delta t}$.

- c) For large M the binomial distribution is approximated by the normal distribution with distribution $F(x)$. Show that $V_0^{(M)}$ is approximated by

$$S_0 F\left(\frac{M\tilde{p} - \alpha}{\sqrt{M\tilde{p}(1-\tilde{p})}}\right) - e^{-rT} K F\left(\frac{Mp - \alpha}{\sqrt{Mp(1-p)}}\right),$$

where

$$\alpha := -\frac{\log \frac{S_0}{K} + M \log d}{\log u - \log d}.$$

- d) Substitute p, u, d to show

$$\frac{Mp - \alpha}{\sqrt{Mp(1-p)}} \rightarrow \frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

for $M \rightarrow \infty$.

Hint: Use exercise 9b): Up to terms of high order the approximations $u \approx e^{\sigma\sqrt{\Delta t}}$, $d \approx e^{-\sigma\sqrt{\Delta t}}$ hold.

(The other argument of F can be analyzed in an analogous way.)

Summer 2012
May, 2nd

Computational Finance - 4th Assignment

**Deadline: May, 9th (written exercises)
May, 16th (programming exercise)**

Exercise 12 (Binomial Model) (20P points)

Write a computer program that calculates prices of options using the binomial model. Create a subroutine which consists of the algorithm of the lecture.

The subroutine should have the following input values:

r (interest rate), σ (volatility), T (maturity), K (strike), S_0 (value of the asset at $t = 0$),
 M (number of time steps), *type1* (European/American option), *type2* (put/call).

The today's value V_0 of the option should be the output value.

Calculate the following option values for $M = 8, 16, 32, 64$:

- European put with $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 5$.
- European call with the same parameters.
- American put with $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 9$.

Calculate the difference between V_C^{eur} and $V_0^{(M)}$, where V_C^{eur} is the value obtained via the Black-Scholes formula, for example b) and consecutive $M = 50 \dots 100$. Use $V_C^{\text{eur}} \approx 0.01282$ (programming exercise).

Exercise 13 (Properties of a Wiener Process) (5 points)

In the definition of a Wiener process the requirement (b) $W_0 = 0$ is dispensable. Then the requirement in (c) reads

$$W_t - W_0 \sim \mathcal{N}(0, t).$$

Use this relation to deduce for $t > s$:

$$\begin{aligned} \text{E}(W_t - W_s) &= 0, \\ \text{Var}(W_t - W_s) &= \text{E}((W_t - W_s)^2) = t - s. \end{aligned}$$

Hint: $(W_t - W_s)^2 = (W_t - W_0)^2 + (W_s - W_0)^2 - 2(W_t - W_0)(W_s - W_0)$

Exercise 14 (Lemma of Paragraph 1.4)

(4+3+1 points)

a) Suppose that a random variable X_t satisfies $X_t \sim \mathcal{N}(0, \sigma^2)$. Use

$$E(X) := \int_{-\infty}^{\infty} x f(x) dx$$

to show

$$E(X_t^4) = 3\sigma^4.$$

b) Apply a) to show the assertion

$$E\left(\left[\sum_j ((\Delta W_j)^2 - \Delta t_j)\right]^2\right) = 2 \sum_j (\Delta t_j)^2.$$

c) Deduce the lemma of paragraph 1.4 from b).

Exercise 15 (Vasicek Model)

(1+3+3 points)

The Vasicek model is used to describe the evolution of the interest rate r . In this model Vasicek assumes that the movement of the interest rate is only driven by one source of market risk. It is given by

$$dr_t = a \cdot (b - r_t) dt + \sigma dW_t$$

where a, b, σ are constants.

- Interpret the model: what are the meanings of a, b, σ ?
- Solve the stochastic differential equation by means of Itô's lemma.
Hint: Consider the process $e^{at} r_t$.
- Use the Itô isometry

$$E\left(\left[\int_b^a f(t, \omega) dW_t\right]^2\right) = \int_b^a E(f^2(t, \omega)) dt$$

to show

$$E(r_t) = r_0 e^{-at} + b(1 - e^{-at}) \quad \text{and}$$

$$\text{Var}(r_t) = E([r_t - E(r_t)]^2) = \frac{\sigma^2}{2a}(1 - e^{-2at})$$

and thus

$$\lim_{t \rightarrow \infty} E(r_t) = b, \quad \lim_{t \rightarrow \infty} \text{Var}(r_t) = \frac{\sigma^2}{2a}.$$

Exercise 16 (Moments of the Lognormal Distribution)

(4+4 points)

For the density function $f(S; t - t_0, S_0, \mu, \sigma)$ defined by

$$f(S; t - t_0, S_0, \mu, \sigma) := \frac{1}{S\sigma\sqrt{2\pi(t-t_0)}} \exp\left\{-\frac{\left(\log(S/S_0) - \left(\mu - \frac{\sigma^2}{2}\right)(t-t_0)\right)^2}{2\sigma^2(t-t_0)}\right\}$$

show

- $\int_0^\infty S f(S; t - t_0, S_0, \mu, \sigma) dS = S_0 e^{\mu(t-t_0)},$
- $\int_0^\infty S^2 f(S; t - t_0, S_0, \mu, \sigma) dS = S_0^2 e^{(\sigma^2 + 2\mu)(t-t_0)}.$

Hint: Set $y = \log(S/S_0)$ and transform the argument of the exponential function to a squared term.

Summer 2012
May, 9th

Computational Finance - 5th Assignment

Deadline: May, 16th

Exercise 17 (Analytical Solution of a Special SDE) (5 points)

Solve the stochastic differential equation

$$dY_t = Y_t^2[(4Y_t - 1) dt - 2 dW_t].$$

Hint: Start with $dX_t = a dt + b dW_t$ und use a suitable function g with $Y_t = g(X_t, t)$.

Exercise 18 (General Black–Scholes Equation) (5 points)

Assume a portfolio

$$\Pi_t = \alpha_t S_t + \beta_t B_t$$

consisting of a stock S_t and a bond B_t , which obey

$$\begin{aligned} dS_t &= \mu(S_t, t) dt + \sigma(S_t, t) dW_t \\ dB_t &= r(t) B_t dt. \end{aligned}$$

The functions μ , σ , and r are assumed to be known, and $\sigma > 0$. Further assume the portfolio is *self-financing* in the sense

$$d\Pi_t = \alpha_t dS_t + \beta_t dB_t,$$

and *replicating* such that Π_T equals the payoff of a European option. (Recall the consequence that Π_t equals the price of the option for all t .)

Derive the Black–Scholes equation for this scenario, assuming $\Pi_t = g(S_t, t)$ with g sufficiently often differentiable.

Hint: coefficient matching of two versions of $d\Pi_t$.

Exercise 19 (Integration by Parts)

(4 points)

The *multi-dimensional Itô lemma* is as follows:

Let X_t be an n -dimensional Itô process, i.e.

$$dX_t = a(X_t, t) dt + b(X_t, t) dW_t$$

where

$$X_t = \begin{pmatrix} X_t^{(1)} \\ \vdots \\ X_t^{(n)} \end{pmatrix}, \quad W_t = \begin{pmatrix} W_t^{(1)} \\ \vdots \\ W_t^{(m)} \end{pmatrix}, \quad a(X_t, t) = \begin{pmatrix} a_1(X_t^{(1)}, \dots, X_t^{(n)}, t) \\ \vdots \\ a_n(X_t^{(1)}, \dots, X_t^{(n)}, t) \end{pmatrix}$$

and

$$b(X_t, t) = ((b_{ik}(X_t, t)))_{i=1, \dots, n}^{k=1, \dots, m}.$$

Further let $g : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^p$ be a C^2 map.

Then the process

$$Y_t = g(X_t, t)$$

is again an Itô process, whose component number $k \in \{1, \dots, p\}$, $Y_t^{(k)}$, is given by

$$dY_t^{(k)} = \frac{\partial g_k}{\partial t}(X_t, t) dt + \sum_{i=1}^n \frac{\partial g_k}{\partial x_i}(X_t, t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 g_k}{\partial x_i \partial x_j}(X_t, t) dX_t^{(i)} dX_t^{(j)}$$

where $dW_t^{(i)} dW_t^{(j)} = \delta_{ij} dt$, $dt dt = dW_t^{(i)} dt = dt dW_t^{(i)} = 0$.

Use this multi-dimensional Itô lemma for the following assignment:

Let X_t, Y_t be Itô processes in \mathbb{R} . Prove that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t.$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s.$$

Summer 2012
May, 16th

Computational Finance - 6th Assignment

**Deadline: May, 23th (written exercises)
June, 4th (programming exercise)**

Exercise 20 (Analysis of a Random Number Generator) (5 points)

Consider the linear congruential generator

$$N_i = (455N_{i-1} + 23) \bmod 4096, \quad U_i = \frac{N_i}{4096}.$$

Construct a family of parallel straight lines containing all the points (U_{i-1}, U_i) so that only few of them cut the square $[0, 1]^2$. What is the distance between them?

Hint: Examine the condition $c \in \mathbb{Z}$ for the linear equation. For this purpose consider the quotient $\frac{M}{a}$.

Exercise 21 (Deficient Random Number Generator) (5 points)

For some time the generator

$$N_i = aN_{i-1} \bmod M \quad \text{with } a = 2^{16} + 3, \quad M = 2^{31}$$

was in wide use. Show for the sequence $U_i := N_i/M$:

$$U_{i+2} - 6U_{i+1} + 9U_i \text{ is integer!}$$

What does this imply for the distribution of the triples (U_i, U_{i+1}, U_{i+2}) in the unit cube?

Exercise 22 (Inverting the Normal Distribution) (3+1+2 points)

Suppose $F(x)$ is the standard normal distribution function. Construct a rough approximation $G(u)$ to $F^{-1}(u)$ for $0.5 \leq u < 1$ as follows:

- Construct a rational function $G(u)$ with correct asymptotic behavior, point symmetry with respect to $(u, x) = (0.5, 0)$, using only one parameter.
- Fix the parameter by interpolating a given point $(x_1, F(x_1))$.
- What is a simple criterion for the error of the approximation?

Exercise 23 (Uniform Distribution) (6 points)

For the uniformly distributed random variable (V_1, V_2) on $V_1^2 + V_2^2 < 1$ consider the transformation

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} V_1^2 + V_2^2 \\ \frac{1}{2\pi} \arg(V_1, V_2) \end{pmatrix}.$$

Show that (X_1, X_2) is distributed uniformly.

Exercise 24 (Implied Volatility)

(20P points)

For European options we take the valuation formula of Black and Scholes of the type $V = v(S, \tau, K, r, \sigma)$, where τ denotes the time to maturity, $\tau := T - t$. For the definition of the function v see exercise 6. If actual market data of the price V are known, then one of the parameters considered known so far can be viewed as unknown and fixed via the implicit equation

$$V - v(S, \tau, K, r, \sigma) = 0. \quad (*)$$

In this *calibration* approach the unknown parameter is calculated iteratively as solution of equation (*). Consider σ to be in the role of the unknown parameter. The volatility σ determined in this way is called *implied volatility* and is a zero of $f(\sigma) := V - v(S, \tau, K, r, \sigma)$.

Assignment:

- a) Design, implement and test an algorithm to calculate the implied volatility of a call. Use Newton's method to construct a sequence $x_k \rightarrow \sigma$. The derivative $f'(x_k)$ can be approximated by the difference quotient

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

For the resulting *secant iteration* invent a stopping criterion that requires smallness of both $|f(x_k)|$ and $|x_k - x_{k-1}|$.

- b) Consider the following market data for call options on the same underlying:

$$T - t = 0.32787, \quad S_0 = 7133.06, \quad r = 0.0487,$$

K	6400	6700	7000	7300	7600	7900	8200	8500	8800
V	934.0	690.0	469.0	283.0	145.0	62.0	22.0	7.5	2.1

Calculate the implied volatilities for these data. For each calculated value of σ enter the point (K, σ) into a figure and join the points with straight lines. (You will notice a convex shape of the curve. This shape has led to call this phenomenon *volatility smile*.)

(programming exercise)

Summer 2012
May, 23th

Computational Finance - 7th Assignment

Deadline: June, 6th

Exercise 25 (Integration by Parts for Itô Integrals) (2+3 points)

a) Show

$$\int_{t_0}^t s \, dW_s = tW_t - t_0W_{t_0} - \int_{t_0}^t W_s \, ds.$$

Hint: Start with the Wiener process $X_t = W_t$ and apply the Itô Lemma with the transformation $y = g(x, t) := tx$.

b) Denote $\Delta Y := \int_{t_0}^t \int_{t_0}^s dW_z ds$, $\Delta W := W_t - W_{t_0}$ and $\Delta t := t - t_0$. Show by using a) that

$$\int_{t_0}^t \int_{t_0}^s dz \, dW_s = \Delta W \Delta t - \Delta Y.$$

Exercise 26 (Integral Representation) (8 points)

For a European put with time to maturity $\tau := T - t$ prove that

$$\begin{aligned} [V(S_t, t)] &= e^{-r\tau} \int_0^\infty (K - S_T)^+ \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left\{ -\frac{[\ln(S_T/S_t) - (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau} \right\} dS_T \\ &= e^{-r\tau} KF(-d_2) - S_t F(-d_1), \end{aligned}$$

where F , d_1 and d_2 were defined in exercise 6.

Hint: Use $(K - S_T)^+ = 0$ for $S_T > K$, and get two integrals.

(please turn over)

Exercise 27 (Moments of Itô Integrals)

(3+7+3 points)

a) Use the Itô isometry

$$\mathbb{E} \left(\left[\int_a^b f(t, \omega) dW_t \right]^2 \right) = \int_a^b \mathbb{E} (f^2(t, \omega)) dt$$

to show its generalization

$$\mathbb{E}(I(f)I(g)) = \int_a^b \mathbb{E}(fg) dt, \quad \text{where } I(f) = \int_a^b f(t, \omega) dW_t.$$

Hint: $4fg = (f + g)^2 - (f - g)^2$ b) Show for ΔY , ΔW and Δt defined in exercise 25 by using a) and $\mathbb{E} \left(\int_a^b f(t, \omega) dW_t \right) = 0$ the following assertions for the moments:

$$\mathbb{E}(\Delta Y) = 0, \quad \mathbb{E}(\Delta Y^2) = \frac{\Delta t^3}{3}, \quad \mathbb{E}(\Delta Y \Delta W) = \frac{\Delta t^2}{2}, \quad \mathbb{E}(\Delta Y \Delta W^2) = 0.$$

c) By transformation of two independent standard normally distributed random variables $Z_i \sim \mathcal{N}(0, 1), i = 1, 2$, two new random variables are obtained by

$$\Delta \widehat{W} := Z_1 \sqrt{\Delta t}, \quad \Delta \widehat{Y} := \frac{1}{2}(\Delta t)^{3/2} \left(Z_1 + \frac{1}{\sqrt{3}} Z_2 \right).$$

Show that $\Delta \widehat{W}$, $\Delta \widehat{Y}$ and their corresponding products have the same moments like ΔW and ΔY in b).

Summer 2012
June, 6th

Computational Finance - 8th Assignment

**Deadline: June, 13th (written exercises)
June, 20th (programming exercise)**

Exercise 28 (Second-order Weakly Convergent Method) (3+2 points)

In addition to the moments of exercise 27b) further moments of the random variables ΔW and ΔY of exercise 25 and 27 are

$$E(\Delta W) = E(\Delta W^3) = E(\Delta W^5) = 0, \quad E(\Delta W^2) = \Delta t, \quad E(\Delta W^4) = 3\Delta t^2.$$

Assume a new random variable $\widetilde{\Delta W}$ satisfying

$$P\left(\widetilde{\Delta W} = \pm\sqrt{3\Delta t}\right) = \frac{1}{6}, \quad P\left(\widetilde{\Delta W} = 0\right) = \frac{2}{3}$$

and the additional random variable

$$\widetilde{\Delta Y} := \frac{1}{2}\widetilde{\Delta W}\Delta t.$$

- a) Show that the random variables $\widetilde{\Delta W}$ and $\widetilde{\Delta Y}$ have up to terms of order $\mathcal{O}(\Delta t^3)$ the same moments as ΔW and ΔY .
- b) Deduce that the method

$$y_{j+1} = y_j + a\Delta t + b\widetilde{\Delta W} + \frac{1}{2}bb' \left((\widetilde{\Delta W})^2 - \Delta t \right) \\ + \frac{1}{2} \left(a'b + ab' + \frac{1}{2}b^2b'' \right) \widetilde{\Delta W}\Delta t + \frac{1}{2} \left(aa' + \frac{1}{2}b^2a'' \right) \Delta t^2$$

is second-order weakly convergent.

Exercise 29 (Continuous Dividend Flow) (5 points)

Assume that a stock pays a dividend D once per year. Calculate a corresponding continuous dividend rate δ under the assumptions

$$\dot{S} = (\mu - \delta)S, \quad \mu = 0, \quad S(1) = S(0) - D > 0.$$

Generalize the result to general growth rates μ and arbitrary day t_D of dividend payment.

Exercise 30 (Mean Square Error)

(1+2+1 points)

For the mean square error the relation

$$\text{MSE}(\hat{x}) = (\text{bias}(\hat{x}))^2 + \text{Var}(\hat{x})$$

is valid. Let $\hat{x} := y_T^h$ be the result of a weakly convergent discretization scheme with order β and $g = \text{identity}$, where h denotes the step length. Then the bias is of the order β ,

$$\text{bias}(\hat{x}) = \alpha_1 h^\beta, \quad \alpha_1 \text{ a constant.}$$

Since the variance of Monte Carlo is of the order $\frac{1}{N}$ (N being the number of samples), we have

$$\zeta(h, N) := \text{MSE}(\hat{x}) = \alpha_1^2 h^{2\beta} + \frac{\alpha_2}{N}, \quad \alpha_2 \text{ another constant.}$$

- a) Argue why for some constant
- α_3

$$C(h, N) := \alpha_3 \frac{N}{h}$$

is a reasonable model for the costs of the MC simulation.

- b) Minimize
- $\zeta(h, N)$
- with respect to
- h, N
- subject to the side condition

$$\alpha_3 \frac{N}{h} = C$$

for given budget C .

- c) Show that for the optimal
- h, N

$$\sqrt{\text{MSE}(\hat{x})} = \alpha_4 C^{-\frac{\beta}{1+2\beta}}$$

for some constant α_4 .

(please turn over)

Exercise 31 (Random Number Generators and Monte Carlo)

(25P points)

a) Implement the *linear congruential generator* given by

$$N_i = (aN_{i-1} + b) \bmod M \quad \text{with } a = 1366, b = 150889, M = 714025.$$

The seed N_0 should be the input value.

Use your program to compute 10000 pairs (U_{i-1}, U_i) in the unit square and plot them.

b) Implement the *Fibonacci generator* given by

$$\begin{aligned} U_i &:= U_{i-17} - U_{i-5}, \\ U_i &:= U_i + 1 \quad \text{if } U_i < 0. \end{aligned}$$

Calculate U_1, \dots, U_{17} with the linear congruential generator of a).

Use your program to compute 10000 pairs (U_{i-1}, U_i) in the unit square and plot them.

c) Implement the *polar method of Marsaglia*. Calculate the initial values with the Fibonacci generator of b).

Use your program to compute 10000 standard normally distributed numbers and plot them in two dimensions by separating them vertically with distance 10^{-4} . Furthermore, divide the x -axis into subintervals having the same length and count the computed numbers in each subinterval. Then set up the corresponding histogram.

d) Implement a Monte Carlo method for single-asset European options, based on the Black-Scholes model. Perform experiments with various values of N (see below) and a random number generator of your choice. To obtain values for S_T , use the analytic solution formula for S_t and also alternatively Milstein's discretization. (Compare the different results.)

Input values: S_0 , number of simulations (trajectories) N , payoff function $\Lambda(S)$, risk-neutral interest rate r , volatility σ , time to maturity T , strike K .

Output value: approximated value of the option $V_0^{(N)}$.

Compute approximations $V_0^{(N)}$ for $N = 1, 10, 100, 1000, 10000$ for the following option prices at time $t = 0$:

- i) European put with $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 5$, $\delta = 0$,
- ii) European call with the same parameters,
- iii) binary call with the same parameters.

For i) and ii), compare your results with the values obtained via the Black-Scholes formula (programming exercise).

Computational Finance - 9th Assignment

Deadline: June, 20th

Exercise 32 (*LU Decomposition of a Special Tridiagonal Matrix*) (4 points)

A matrix of the form

$$A = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \dots & \dots & \dots & \\ & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & b_n & a_n \end{pmatrix} =: \text{tridiag}(b_\mu, a_\mu, c_\mu)$$

with

$$\begin{aligned} |a_1| &> |c_1| > 0, \\ |a_\mu| &\geq |b_\mu| + |c_\mu|, \quad b_\mu \neq 0, c_\mu \neq 0, \quad 2 \leq \mu \leq n-1, \\ |a_n| &\geq |b_n| > 0 \end{aligned}$$

is called an *irreducible diagonally dominant* tridiagonal matrix.

Show that an irreducible diagonally dominant tridiagonal matrix $A = \text{tridiag}(b_\mu, a_\mu, c_\mu)$ has an *LU* decomposition $A = LU$ with $L = \text{tridiag}(b_\mu, \alpha_\mu, 0)$ and $U = \text{tridiag}(0, 1, \gamma_\mu)$, where $\alpha_1 := a_1$, $\gamma_1 := c_1 \alpha_1^{-1}$ and

$$\begin{aligned} \alpha_\mu &:= a_\mu - b_\mu \gamma_{\mu-1}, \quad 2 \leq \mu \leq n, \\ \gamma_\mu &:= c_\mu \alpha_\mu^{-1}, \quad 2 \leq \mu \leq n-1. \end{aligned}$$

Exercise 33 (*Crank-Nicolson Order*) (8 points)

Let the function $y(x, \tau)$ solve the equation

$$y_\tau = y_{xx}$$

and be sufficiently smooth. With the difference quotient

$$\delta_{xx} w_{i,\nu} := \frac{w_{i+1,\nu} - 2w_{i,\nu} + w_{i-1,\nu}}{\Delta x^2}$$

the local discretization error ϵ of the Crank-Nicolson method is defined as

$$\epsilon := \frac{y_{i,\nu+1} - y_{i,\nu}}{\delta\tau} - \frac{1}{2} (\delta_{xx} y_{i,\nu} + \delta_{xx} y_{i,\nu+1}).$$

Show

$$\epsilon = O(\Delta\tau^2) + O(\Delta x^2).$$

Exercise 34 (Transformation of the Boundary Conditions of BS)

(3 points)

In exercise 8 the Black-Scholes equation is transformed into the equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}.$$

Transform the boundary conditions of the Black-Scholes equation accordingly.

Exercise 35 (Black-Scholes and Strange Payoff)

(3+2 points)

A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T .

- a) Calculate the price $V(S, t)$ of the security at time t in terms of the stock price S_t by solving the Black-Scholes differential equation with proper terminal conditions.
Hint: Make the ansatz $V(S, t) = S^2 \cdot f(t)$ with an unknown function f .
- b) Confirm your calculation by using the risk-neutral valuation, i.e.

$$V(S, 0) = e^{-rT} \mathbb{E}(\text{payoff}(S_T, T)).$$

Summer 2012
June, 20th

Computational Finance - 10th Assignment

**Deadline: June, 27th (written exercises)
July, 4th (programming exercise)**

Exercise 36 (Perpetual Put Option)

(4+2+4 points)

For $T \rightarrow \infty$ it is sufficient to analyze the ODE

$$\frac{\sigma^2}{2} S^2 \frac{d^2 V}{dS^2} + (r - \delta) S \frac{dV}{dS} - rV = 0.$$

Consider an American put with high contact to the payoff $V = (K - S)^+$ at $S = S_f$.
Show:

- a) Upon substituting the boundary condition for $S \rightarrow \infty$ one obtains

$$V(S) = c \left(\frac{S}{K} \right)^{\lambda_2},$$

where $\lambda_2 = \frac{1}{2} \left(1 - q_\delta - \sqrt{(q_\delta - 1)^2 + 4q} \right)$, $q = \frac{2r}{\sigma^2}$, $q_\delta = \frac{2(r-\delta)}{\sigma^2}$
and c is a positive constant.

Hint: Apply the transformation $S = Ke^x$. (The other root λ_1 drops out.)

- b) V is convex.

For $S < S_f$ the option is exercised; then its intrinsic value is $K - S$. For $S > S_f$ the option is not exercised and has a value $V(S) > K - S$. The holder of the option decides when to exercise. This means, the holder makes a decision on the high contact S_f such that the value of the option becomes maximal.

- c) Show: $V'(S_f) = -1$, if S_f maximizes the value of the option.

Hint: Determine the constant c such that $V(S)$ is continuous in the contact point.

Exercise 37 (Discrete Dividend Payment)

(3 points)

Assume that a stock pays a dividend D at ex-dividend date t_D , with $0 < t_D < T$. Define for an American put with strike K

$$\tilde{t} := t_D - \frac{1}{r} \log \left(\frac{D}{K} + 1 \right).$$

Assume $S = 0$, $r > 0$, $D > 0$ and a time instant t in $\tilde{t} < t < t_D$. Argue that instead of exercising early it is reasonable to wait for the dividend. Note: For $\tilde{t} > 0$, depending on S , early exercise may be reasonable for $0 \leq t < \tilde{t}$.

Exercise 38 (UL decomposition)

(3 points)

Assume a system of linear equations $Ax = b$ with irreducible diagonally dominant tridiagonal matrix A . Formulate the UL decomposition as an algorithm.

Exercise 39 (Brennan-Schwartz Algorithm)

(3+2 points)

Let A be a irreducible diagonally dominant tridiagonal matrix and b and g vectors. The system of equations $Aw = b$ is to be solved such that the side condition $w \geq g$ is obeyed componentwise. Assume for a put $w_i = g_i$ for $1 \leq i \leq i_f$ and $w_i > g_i$ for $i_f < i \leq n$, with unknown i_f .

- Formulate an algorithm that solves $Aw = b$ in the *backward/forward* approach. In the final forward loop, for each i the calculated candidate w_i is tested for $w_i \geq g_i$: In case $w_i < g_i$ the calculated value w_i is corrected to $w_i = g_i$.
- Apply the algorithm to the case of a put with relevant A, b, g .

Exercise 40 (Computation of American Options)

(20P points)

Implement an algorithm for the calculation of American-style options, following the prototype algorithm below. Use exercises 38 and 39. For this assignment, it is sufficient to implement the case of a put.

Test your program with the following example: $K = 10$, $r = 0.25$, $\sigma = 0.6$, $T = 1$, $\delta = 0.2$. Calculate approximations to $V_P(10, 0)$.

Algorithm (prototype algorithm)

Set up the function $g(x, \tau)$ listed in the summary below.

Choose θ ($\theta = 1/2$ for Crank-Nicolson).

Fix the discretization by choosing x_{\min} , x_{\max} , m , ν_{\max} (for example: $x_{\min} = -5$, $x_{\max} = 5$, $\nu_{\max} = m = 100$).

Calculate $\Delta x := (x_{\max} - x_{\min})/m$,

$$\Delta \tau := \frac{1}{2} \sigma^2 T / \nu_{\max},$$

$$x_i := x_{\min} + i \Delta x \text{ for } i = 0, \dots, m,$$

$$\lambda := \Delta \tau / \Delta x^2 \text{ and } \alpha := \lambda \theta.$$

Initialize the iteration vector w with

$$g^{(0)} = (g(x_1, 0), \dots, g(x_{m-1}, 0))^{tr}.$$

Arrange for the matrix A .

τ -loop: for $\nu = 0, 1, \dots, \nu_{\max} - 1$:

$$\tau_\nu := \nu \Delta \tau$$

initialize the vector b with

$$b_i := w_i + \lambda(1 - \theta)(w_{i+1} - 2w_i + w_{i-1}) \text{ for } 2 \leq i \leq m - 2,$$

$$b_1 := w_1 + \lambda(1 - \theta)(w_2 - 2w_1 + g_{0\nu}) + \alpha g_{0,\nu+1},$$

$$b_{m-1} := w_{m-1} + \lambda(1 - \theta)(g_{m\nu} - 2w_{m-1} + w_{m-2}) + \alpha g_{m,\nu+1}$$

subroutine for the LCP solution w , directly as in exercises 38 and 39

$$w^{(\nu+1)} = w$$

(please turn over)

Summary of American options, for a put ($r > 0$) or a call ($\delta > 0$), after transformation into (x, τ, y) -variables:

$$q := \frac{2r}{\sigma^2}; \quad q_\delta := \frac{2(r - \delta)}{\sigma^2}$$

$$\text{put: } g(x, \tau) := \exp\left\{\frac{1}{4}((q_\delta - 1)^2 + 4q)\tau\right\} \max\{e^{\frac{1}{2}(q_\delta - 1)x} - e^{\frac{1}{2}(q_\delta + 1)x}, 0\}$$

$$\text{call: } g(x, \tau) := \exp\left\{\frac{1}{4}((q_\delta - 1)^2 + 4q)\tau\right\} \max\{e^{\frac{1}{2}(q_\delta + 1)x} - e^{\frac{1}{2}(q_\delta - 1)x}, 0\}$$

$$\left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y - g) = 0$$

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0, \quad y - g \geq 0$$

$$y(x, 0) = g(x, 0), \quad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T$$

$$\lim_{x \rightarrow \pm\infty} y(x, \tau) = \lim_{x \rightarrow \pm\infty} g(x, \tau)$$

(programming exercise)

Summer 2012
June, 27th

Computational Finance - 11th Assignment

Deadline: July, 4th

Exercise 41 (Stability of the Fully Implicit Method) (4 points)

The backward-difference method is defined via the solution of the equation

$$A_{\text{impl}} w^{(\nu)} = w^{(\nu-1)} \quad \text{with} \quad A_{\text{impl}} = \text{tridiag}(-\lambda, 2\lambda + 1, -\lambda).$$

Prove the stability.

Hint: Use $w^{(\nu)} = A_{\text{impl}}^{-1} w^{(\nu-1)}$.

Exercise 42 (Front-Fixing for American Options) (4+2 points)

Apply the transformation

$$\zeta := \frac{S}{S_f(t)}, \quad y(\zeta, t) := V(S, t)$$

to the Black-Scholes equation.

a) Show

$$\frac{\partial y}{\partial t} + \frac{\sigma^2}{2} \zeta^2 \frac{\partial^2 y}{\partial \zeta^2} + \left[(r - \delta) - \frac{1}{S_f} \frac{dS_f}{dt} \right] \zeta \frac{\partial y}{\partial \zeta} - ry = 0.$$

b) Set up the domain for (ζ, t) and formulate the boundary conditions for an American call (Assume $\delta > 0$).

Exercise 43 (Upwind Scheme) (5 points)

Apply von Neumann's stability analysis to

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = b \frac{\partial^2 u}{\partial x^2}, \quad b > 0$$

using the FTBS upwind scheme for the left-hand side and the centered second-order difference quotient for the right-hand side.