Approximating Volatility Surfaces

When for a strike K and a time to maturity T a market price $\overline{C}$ of a European vanilla call is known, then the implied volatility σ with respect to the Black–Scholes model can be calculated by inverting the BS-formula. That is, $\sigma = \sigma(K, T)$. In practice, one is interested in this relation also for other (K, T), for which no market price is known. In fact, available market data are typically scattered, as in Figure 1. The aim is to approximate both the price function $C(K, T)$ of a call and the *implied volatility surface* (IVS) $\sigma(K, T)$ for *all* (K, T) in a larger domain (Figure 2).

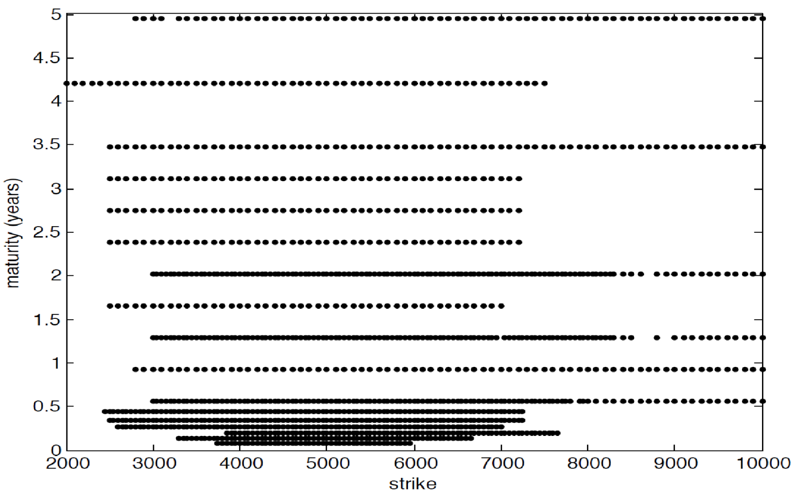

Figure 1: Prototypical distribution of data K_i, T_i, for which $\overline{C}_i := \overline{C}(K_i, T_i)$ was observed. (Call options on the DAX on $29th$ July 2009; $S_0 = 5268.51$ points; from http://www.x-markets.db.com/)

Let us denote the observed market data: K_i, T_i, $\overline{C}_i := \overline{C}(K_i, T_i)$, for $i = 1, \ldots, N$. Since market data are noisy, direct use of $(K_i, T_i, \overline{C}_i)$ for interpolation or BS-inversion $(\rightarrow \sigma)$ is questionable: We require some smoothing of the data. Further, a reasonable approximating surface $C(K, T)$ should reflect an arbitrage-free world.

The method of Glaser and Heider is based on a *moving least-squares* (MLS) approach. MLS approaches have been used to approximate multivariate arbitrarily spaced data. Typically, the approach minimizes over a set of polynomials p, evaluated at x. In our context, $x = (K, T)$, and the data are $x_i = (K_i, T_i)$, $\overline{C}_i$.

Moving Least Squares
For given x, minimize

$$\min_p \sum_{i \in I(x)} (p(x_i) - \overline{C}_i)^2 \, \phi_R(x - x_i)$$

where

$$I(x) := \{i \mid \|x - x_i\|_2 < R\}$$

$$\phi_R(x) := \begin{cases} \exp(-\|x\|_2^2/R^2) & \text{for } \|x\|_2 < R \\ 0 & \text{for } \|x\|_2 \geq R \end{cases}$$

The set $I(x)$ is the index set of neigbors nearest to x, and $\phi_R(x)$ is a weighting function for a suitably chosen radius R.

Backed by Dupire's results [Dupire(1994)] we define a

arbitrage-free function $C(K,T)$, if for all K,T:

$$C \geq 0\,, \qquad \frac{\partial C}{\partial K} \leq 0\,, \qquad \frac{\partial^2 C}{\partial K^2} \geq 0\,, \qquad \frac{\partial C}{\partial T} + (r-\delta)K\frac{\partial C}{\partial K} + \delta C \geq 0\,.$$

We now require that the function $C(K,T)$ satisfies these four inequalities. This leads to a minimization under constraints.

Minimization under Constraints: For a local ansatz polynomial

$$c(\kappa,\tau) = a_0 + a_1(\kappa - K) + a_2(\tau - T) + \frac{a_3}{2}(\kappa - K)^2$$

require

$$a_0 \geq 0, \quad a_1 \leq 0, \quad a_3 \geq 0, \quad \delta a_0 + (r-\delta)Ka_1 + a_2 \geq 0\,.$$

The resulting MLS method is efficient since it minimizes in four-dimensional space (unknowns are a_0, a_1, a_2, a_3) with four constraints, and the index set I is kept small. Denote the minimizer c^*, then we obtain a pointwise definition of the approximation by

$$C(K,T) := c^*(K,T) = a_0^*\,,$$

with properties

(a) There is a unique minimizer c^*.

(b) C is arbitrage-free.

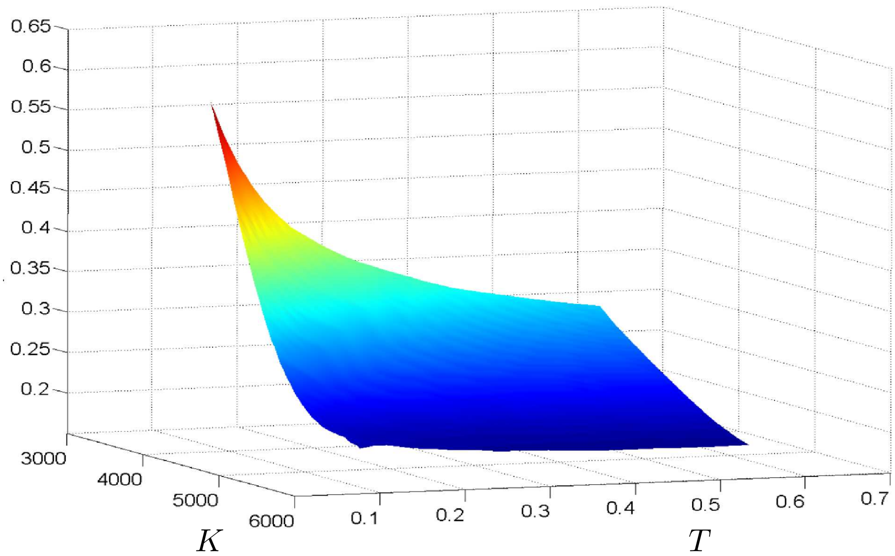

Figure 2: Implied volatility IVS σ over strike K and time to maturity T

Further Applications

1.) by-product partial derivatives: probability density $\mathrm{e}^{rT}\frac{\partial^2 C}{\partial K^2}(K,T) = \mathrm{e}^{rT}\,a_3$

2.) local volatility surface

$$\sigma_{\text{loc}}(K,T) = \sqrt{\frac{\frac{\partial C}{\partial T} + (r-\delta)K\frac{\partial C}{\partial K} + \delta C}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}} = \sqrt{\frac{a_2 + (r-\delta)Ka_1 + \delta a_0}{\frac{1}{2}K^2 a_3}}$$

3.) $C(K,T)$ and $P(K,T)$ for the put provide the basis of implied trees.

publication:

J. Glaser, P. Heider: Arbitrage-free approximation of call price surfaces and input data risk. Quantitative Finance **12** (2012) 61–73