High Contact for a Perpetual Option

For a perpetual put option (with time to maturity $T \to \infty$) there is an analytic solution for $t = 0$. In the Black–Scholes framework, this value function $V(S)$ solves the second-order ordinary-differential equation

$$\tfrac{1}{2}\sigma^2 S^2 V'' + rS V' - r V = 0 \,.$$

We study the contact with the payoff function $(K-S)^+$ at the point $S = \alpha$, with a free parameter α smaller than the strike K. That is, the boundary conditions at $S = \alpha$ and $S \to \infty$ are

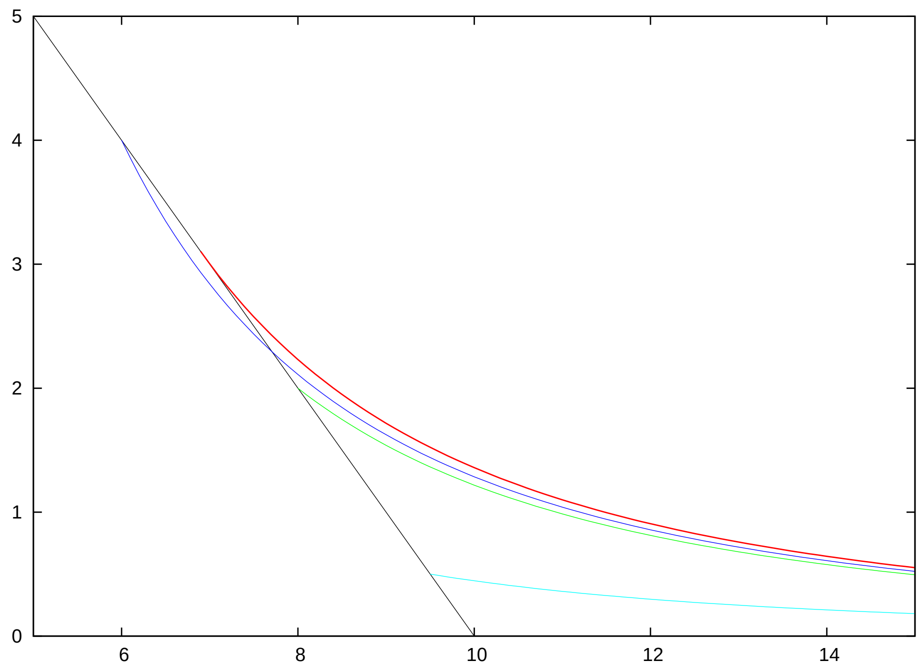
$$V(\alpha) = K - \alpha \qquad \text{and} \qquad V(S) \to 0 \text{ for } S \to \infty \,.$$

The solution for $\alpha \le S < \infty$ is

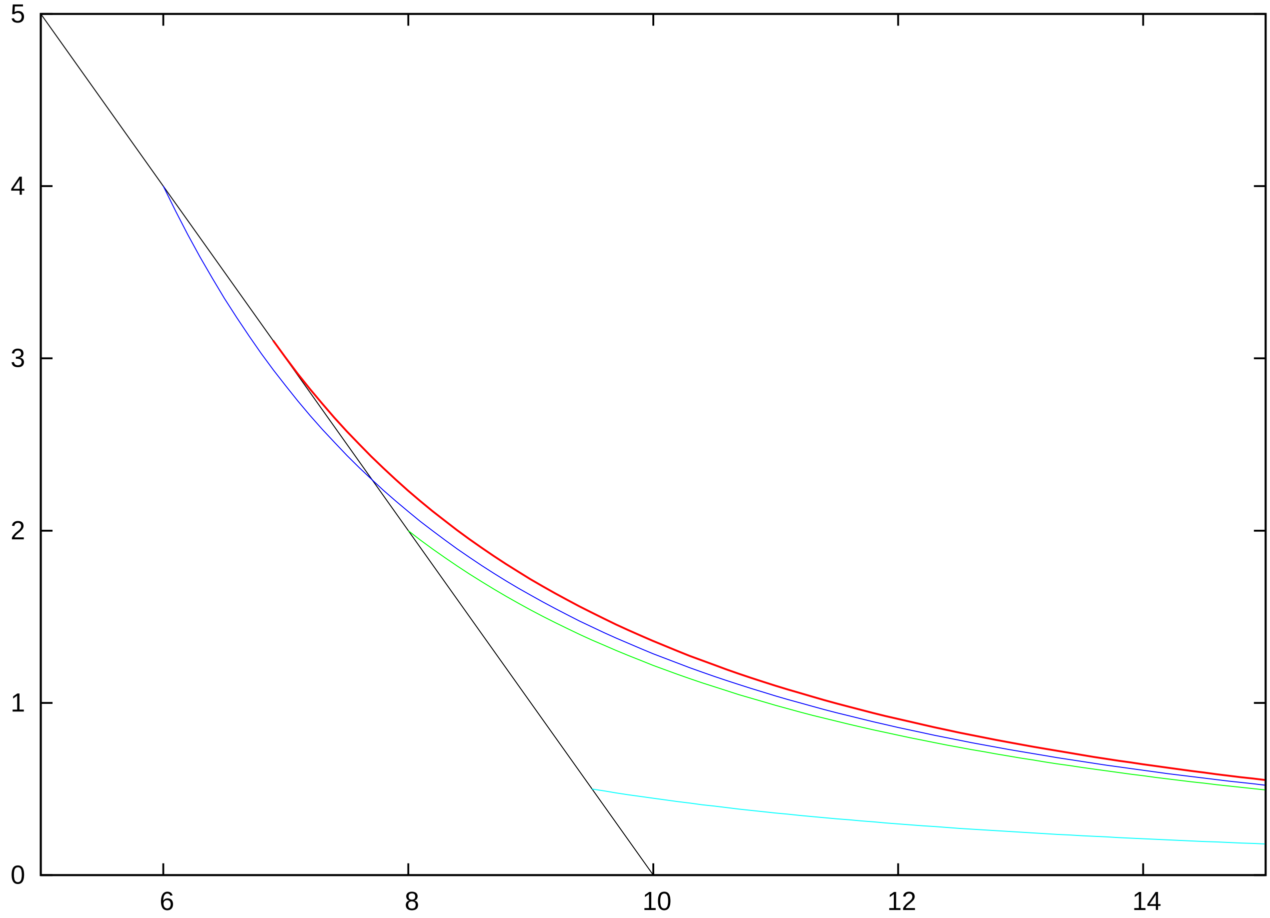
$$V(S;\, \alpha) := (K - \alpha)\left(\frac{S}{\alpha}\right)^{-q} ,$$

where $q := \frac{2r}{\sigma^2}$. This defines a family of functions, parameterized by the parameter α. (Exercise 4.8) Maximizing V with respect to α leads to the specific parameter $\alpha_0 := K\frac{q}{1+q}$. The specific function $V(S;\, \alpha_0)$ shows the *high contact* — that is, its derivative at the left boundary α_0 is that of the payoff, -1.

Example: $K = 10$, $\sigma = 0.3$, $r = 0.1$. Then, $q = 20/9$ and $\alpha_0 = 200/29 = 6.89....$ The figure displays the payoff (in black), and four solutions for the parameters α_0, 6, 8, 9.5. (continued on the back)



The figure a bit larger:



The figure depicts $V(S;\alpha)$ over S. The maximum function (for α_0, in red) exhibits the smooth contact to the payoff. Functions with parameter $\alpha < \alpha_0$ have an S-interval with $V(S)$ smaller than the payoff (blue curve, for $\alpha = 6$). This allows arbitrage and hence should not happen. And $\alpha > \alpha_0$ is not optimal.

The value α_0 has been denoted also S_f.