Monte Carlo: Regression for American-Style Options

The American-style option is approximated by a **Bermudan option**, which replaces the time interval $0 \le t \le T$ by M subintervals, with $\Delta t := T/M$, $t_i := i\,\Delta t$ $(i = 0, \ldots, M)$, and exercising is only allowed at t_i. Let $\Psi(S)$ denote the payoff function, S the price of the underlying, and $V_i(S)$ be the approximation of the option-value function $V(S,t)$ at t_i. The dynamic programming procedure amounts to set

$$V_i(S) = \max\{\, \Psi(S)\,,\ C_i(S)\}\,,$$

where the continuation value C is defined by the conditional expectation

$$C_i(x) := \mathrm{e}^{-r\Delta t}\, \mathsf{E}(V(S_{t_{i+1}}, t_{i+1}) \mid S_{t_i} = x)$$

with respect to the risk-neutral measure. These functions C_i are approximated recursively backwards for $i = M-1, M-2, \ldots$, starting with $V_M = \Psi$.

Basic Procedure

For each i, the procedure is as follows: For given $S_{t_i} = x$, determine a value for $S_{t_{i+1}}$, thereby observing the underlying law. In this way, a value $V(S_{t_{i+1}}, t_{i+1})$ can be made available for each x. Doing this for numerous x gives numerous points $(x, \mathrm{e}^{-r\Delta t}V)$. Based on these points a least-squares regression is applied to approximate C by a model function $\widehat{C}$.

Figure 1 illustrates the (S,V)-plane of a vanilla put (payoff in black): the curve of one such $\widehat{C}$ (in red), and (in yellow) the many (x,V)-points on which the approximation is based.

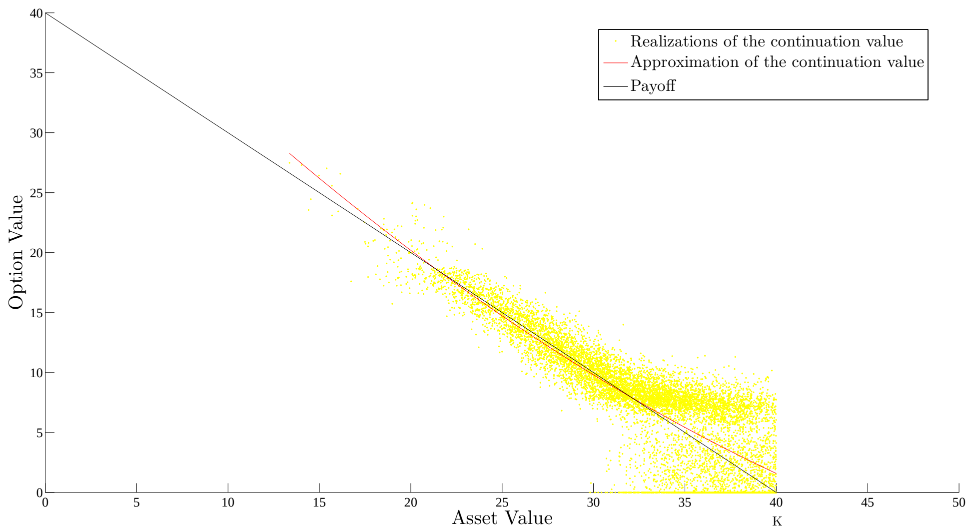

Figure 1: Approximation $\widehat{C}$ of the continuation value via least squares at a fixed time date.

How is this organized?

The above describes only the principle approach. Each of the three tasks, calculating the paths to generate the $S_{t_{i+1}}$, evaluating V at these points, and approximating C, is highly expensive. To obtain a powerful algorithm, many more ideas are needed.

The easiest to organize is the calculation of the paths, which is done beforehand. This simulates N paths S_t for $0 \le t \le T$, thereby storing the values of the $S(t_i)$. For each t_i this provides N candidates $x_{n,i}$ for x, for $n = 1, \ldots, N$. The x-grid is stochastic. Following Longstaff and Schwartz

(2001), each path is furnished with an own stopping time:

$$\tau_M^n = M,$$
$$\tau_i^n = \begin{cases} i & , \ \Psi(x_{n,i}) \geq \widehat{C}_i(x_{n,i}) \\ \tau_{i+1}^n & , \ \text{otherwise} \end{cases} \quad , \ i = M-1, ..., 1;$$

For $i = 1, ..., M-1$ the model function $\widehat{C}_i$ for the continuation value is obtained by least squares on the points $(x_{n,i}, \mathrm{e}^{-r(\tau_{i+1}^n - i)\triangle t}\Psi(x_{n,\tau_{i+1}^n}))$, $n = 1, ..., N$. In order to get a better approximation of the continuation value, Longstaff and Schwartz suggest to consider only paths in-the-money, see **Figure 1**.

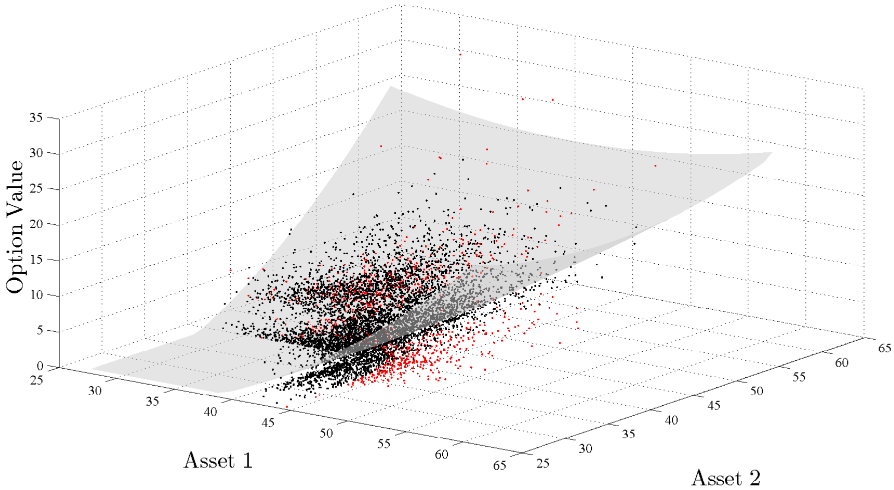

Figure 2: Approximation of the continuation value by least squares for an American call option on the maximum of two assets; red points denote outliers.

Several further ideas by Jonen improve the algorithm significantly. As we can see in **Figure 2**, there are some points, namely the red points, which are really far away from the light gray surface showing the continuation value for an American max call option on two assets (compare also **Topic 7** for the early-exercise structure of this option type). The key idea of the Jonen regression method is to fit the continuation value at every exercise point by *robust regression* rather than by ordinary least squares; in so doing we obtain a more accurate approximation of the continuation value as robust regression is able to handle outliers. Besides an efficient implementation of the Robust Regression Monte Carlo method, Jonen (2011) suggests variance reduction techniques to tackle the problem of developing efficient algorithms for valuing high-dimensional options with an early exercise feature. By driving paths in regions which are more important for variance, Jonen's change of drift technique promises fast convergence. Further, acceleration techniques including quasi Monte Carlo and some simple but yet powerful approaches are introduced.

The method of Andersen and Broadie (2004) for calculating lower and upper bounds for the true option value may be combined with Jonen's approaches. Compared with their state-of-the-art dual method — which is based on the Least Squares Monte Carlo method — Jonen's ultimate algorithm shows a remarkable performance; speed-up factors of up to over sixty are quite possible.

References:

L. Andersen and M. Broadie: *A primal-dual simulation algorithm for pricing multi-dimensional American options*, Management Science, 50 (2004), pp. 1222-1234.

C. Jonen: *Efficient pricing of high-dimensional American-style derivatives: a robust regression Monte Carlo method*, PhD dissertation, Universität Köln (2011), http://kups.ub.uni-koeln.de/4442.

F. Longstaff and E. Schwartz: *Valuing American options by simulation: a simple least-squares approach*, Review of Financial Studies, 14 (2001), pp. 113-147.