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Computational Finance

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Exercise 1 Put-Call Parity

Consider a portfolio consisting of three positions related to the same asset, namely one share (price S), one European put (value V_{P}), plus a short position of one European call (value V_{C}). Put and call have the same expiration date T, and no dividends are paid.

a) Assume a no-arbitrage market without transaction costs. Show that the *put-call parity*

$$S + V_{\mathrm{P}} - V_{\mathrm{C}} = Ke^{-r(T-t)}$$

holds for all t, where K is the strike and r the risk-free interest rate.

b) Use the put-call parity to show

$$V_{\mathrm{C}}(S,t) \geq S - Ke^{-r(T-t)}$$
$$V_{\mathrm{P}}(S,t) \geq Ke^{-r(T-t)} - S\ .$$

Exercise 2 Standard Normal Distribution Function

Establish an algorithm to calculate

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2})\,\mathrm{d}t.$$

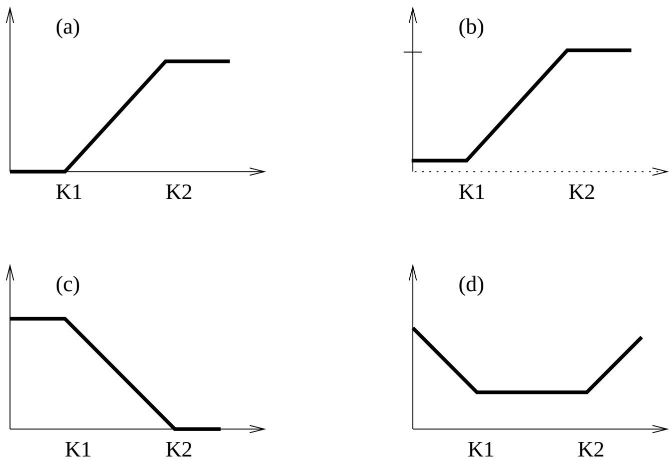
Hint: Construct an algorithm to calculate the *error function*

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-t^2)\,\mathrm{d}t$$

and use $\mathrm{erf}(x)$ to calculate $F(x)$. Use quadrature methods.

Exercise 3 Portfolios

The Figure sketches some payoffs over S. For each of these payoffs, construct portfolios out of vanilla options such that the payoff is met.



Appendix: Black–Scholes Formula

For a European call the analytic solution of the Black–Scholes equation is

$$d_1 := \frac{\log\frac{S}{K} + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 := d_1 - \sigma\sqrt{T-t} = \frac{\log\frac{S}{K} + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$V_{\mathrm{C}}(S,t) = Se^{-\delta(T-t)}F(d_1) - Ke^{-r(T-t)}F(d_2),$$

where F denotes the standard normal cumulative distribution (compare Exercise 3), and δ is a continuous dividend yield. The value $V_{\mathrm{P}}(S,t)$ of a put is obtained by applying the put-call parity

$$V_{\mathrm{P}} = V_{\mathrm{C}} - Se^{-\delta(T-t)} + Ke^{-r(T-t)}$$

from which

$$V_{\mathrm{P}} = -Se^{-\delta(T-t)}F(-d_1) + Ke^{-r(T-t)}F(-d_2)$$

follows.

Exercise 4 Transforming the Black–Scholes Equation

Show that the Black–Scholes equation (BS)

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{(BS)}$$

for $V(S,t)$ with constant σ and r is equivalent to the equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

for $y(x,\tau)$. For proving this, you may proceed as follows:

a) Use the transformation $S = Ke^x$ and a suitable transformation $t \leftrightarrow \tau$ to show that (BS) is equivalent to

$$-\dot{V} + V'' + \alpha V' + \beta V = 0$$

with $\dot{V} = \frac{\partial V}{\partial \tau}$, $V' = \frac{\partial V}{\partial x}$, α, β depending on r and σ.

b) Next apply for suitable γ, δ a transformation of the type

$$V = K\exp(\gamma x + \delta\tau)y(x,\tau)\ .$$

c) Transform the terminal condition of the Black–Scholes equation accordingly.

Exercise 5 On the Binomial Method

a) For $ud = \gamma$ show

$$\beta := \frac{1}{2}(\gamma e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t})$$

$$u = \beta + \sqrt{\beta^2 - \gamma}$$

b) Show for $\gamma = 1$

$$u = \exp\left(\sigma\sqrt{\Delta t}\right) + O\left(\sqrt{(\Delta t)^3}\right)\ .$$

Exercise 6 Anchoring the Binomial Grid at K

The equation $ud = 1$ has established a kind of symmetry for the grid. As an alternative, one may anchor the grid in another way by choosing (for even M)

$$S_0 u^{M/2} d^{M/2} = K \ .$$

a) Give a geometrical interpretation.
b) Derive the relevant formula for u and d.
Hint: Use Exercise 5a.

Exercise 7 Limiting Case of the Binomial Model

Consider a European Call in the binomial model. Suppose the calculated value is $V_0^{(M)}$. In the limit $M \to \infty$ the sequence $V_0^{(M)}$ converges to the value $V_C(S_0, 0)$ of the continuous Black-Scholes model listed on the front page of the second assignment. To prove this, proceed as follows:

a) Let j_K be the smallest index j with $S_{jM} \geq K$. Find an argument why

$$\sum_{j=j_K}^{M} \binom{M}{j} p^j (1-p)^{M-j} (S_0 u^j d^{M-j} - K)$$

is the expectation $\mathsf{E}(V_T)$ of the payoff.

b) The value of the option is obtained by discounting, $V_0^{(M)} = e^{-rT}\mathsf{E}(V_T)$. Show

$$V_0^{(M)} = S_0 B_{M,\tilde{p}}(j_K) - e^{-rT} K B_{M,p}(j_K) \ .$$

Here $B_{M,p}(j)$ is defined by the binomial distribution, and $\tilde{p} := pue^{-r\Delta t}$.

c) For large M the binomial distribution is approximated by the normal distribution with distribution $F(x)$. Show that $V_0^{(M)}$ is approximated by

$$S_0 F\left(\frac{M\tilde{p} - \alpha}{\sqrt{M\tilde{p}(1-\tilde{p})}}\right) - e^{-rT} K F\left(\frac{Mp - \alpha}{\sqrt{Mp(1-p)}}\right) \quad ,$$

where

$$\alpha := -\frac{\log \frac{S_0}{K} + M \log d}{\log u - \log d} \quad .$$

d) Substitute the p, u, d to show

$$\frac{Mp - \alpha}{\sqrt{Mp(1-p)}} \longrightarrow \frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

for $M \to \infty$. Hint: Use Exercise 5b: Up to terms of high order the approximations $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$ hold. (In an analogous way the other argument of F can be analyzed.)

Exercise 8

In the definition of a Wiener process the requirement (a) $W_0 = 0$ is dispensable. Then the requirement in (b) reads

$$\mathsf{E}(W_t - W_0) = 0 \quad , \quad \mathsf{E}((W_t - W_0)^2) = t \quad .$$

Use these relations to deduce for $t > s$

$$\begin{aligned} &\mathsf{E}(W_t - W_s) = 0 \, , \\ &\mathsf{Var}(W_t - W_s) = \mathsf{E}((W_t - W_s)^2) = t - s \, . \end{aligned}$$

Hint: $(W_t - W_s)^2 = (W_t - W_0)^2 + (W_s - W_0)^2 - 2(W_t - W_0)(W_s - W_0)$

Exercise 9

a) Suppose that a random variable X_t satisfies $X_t \sim \mathcal{N}(0, \sigma^2)$. Use

$$\mathsf{E}(X) := \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x$$

to show

$$\mathsf{E}(X_t^4) = 3\sigma^4 \, .$$

b) Apply a) to show the assertion

$$\mathsf{E}\left(\sum_j ((\Delta W_j)^2 - \Delta t_j)\right)^2 = 2 \sum_j (\Delta t_j)^2$$

Exercise 10 Analytical Solution of Special SDEs

Apply Itô's lemma to show

a) $X_t = \exp\left(W_t - \frac{1}{2}t\right)$ solves $\mathrm{d}X_t = X_t \, \mathrm{d}W_t$

b) $X_t = \exp\left(2W_t - t\right)$ solves $\mathrm{d}X_t = X_t \, \mathrm{d}t + 2X_t \, \mathrm{d}W_t$

Hint: Use suitable functions g with $Y_t = g(X_t, t)$. In (a) start with $X_t = W_t$ and $g(x,t) = \exp(x - \frac{1}{2}t)$.

Exercise 11 Moments of the Lognormal Distribution

For the density function $f(S; t - t_0, S_0)$ from

$$f(S; t - t_0, S_0, \mu, \sigma) :=$$

$$\frac{1}{S\sigma\sqrt{2\pi(t - t_0)}} \exp\left\{ - \frac{\left(\log(S/S_0) - \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0)\right)^2}{2\sigma^2(t - t_0)} \right\}$$

show

a) $\int_0^\infty S f(S; t - t_0, S_0) \, \mathrm{d}S = S_0 e^{\mu(t - t_0)}$

b) $\int_0^\infty S^2 f(S; t - t_0, S_0) \, \mathrm{d}S = S_0^2 e^{(\sigma^2 + 2\mu)(t - t_0)}$

Hint: Set $y = \log(S/S_0)$ and transform the argument of the exponential function to a squared term.

Exercise 12 Positive Itô Process

Let X_t be a positive one-dimensional Itô process for $t \geq 0$.
Show that there exist functions α and β such that

$$\mathrm{d}X_t = X_t(\alpha_t\,\mathrm{d}t + \beta_t\,\mathrm{d}W_t)$$

and

$$X_t = X_0 \exp\left\{\int_0^t (\alpha_s - \frac{1}{2}\beta_s^2)\,\mathrm{d}s + \int_0^t \beta_s\,\mathrm{d}W_s\right\}$$

Exercise 13 Black–Scholes Formula

Write a computer program that implements the Black–Scholes formula for given input S, K, r, δ, σ, $T-t$, listed in the first assignment (Appendix). Use Exercise 2 to implement the distribution function $F(x)$. Test your program for a European put with $S=5$, $K=10$, $r=0.06$, $\delta=0$, $\sigma=0.3$, $T-t=1$.

Exercise 14 Heston's model

An **Ornstein–Uhlenbeck process** is defined as solution of the SDE

$$\mathrm{d}X_t = -\alpha X_t\,\mathrm{d}t + \beta\,\mathrm{d}W_t \quad , \quad \alpha > 0$$

Suppose the volatility σ_t is an Ornstein–Uhlenbeck process. Show that the variance $v_t := \sigma_t^2$ follows a Cox–Ingersoll–Ross process, namely

$$\mathrm{d}v_t = \kappa(\Theta - v_t)\,\mathrm{d}t + \sigma_{\mathrm{v}}\sqrt{v_t}\,\mathrm{d}W_t\ .$$

(This is part of Heston's model

$$\begin{aligned}\mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sqrt{v_t}\,S_t\,\mathrm{d}W_t^{(1)}\\ \mathrm{d}v_t &= \kappa(\Theta - v_t)\,\mathrm{d}t + \sigma_{\mathrm{v}}\sqrt{v_t}\,\mathrm{d}W_t^{(2)}\ .\ )\end{aligned}$$

Exercise 15 General Black–Scholes Equation

Assume a portfolio

$$\Pi_t = \alpha_t S_t + \beta_t B_t$$

consisting of a stock S_t and a bond B_t, which obey

$$\begin{aligned}\mathrm{d}S_t &= \mu(S_t,t)\,\mathrm{d}t + \sigma(S_t,t)\,\mathrm{d}W_t\\ \mathrm{d}B_t &= r(t)B_t\,\mathrm{d}t\end{aligned}$$

The functions μ, σ, and r are assumed to be known, and $\sigma > 0$. Further assume the portfolio is *self-financing* in the sense

$$\mathrm{d}\Pi_t = \alpha_t\,\mathrm{d}S_t + \beta_t\,\mathrm{d}B_t\ ,$$

and *replicating* such that Π_T equals the payoff of a European option. (Recall the consequence that Π_t equals the price of the option for all t.)
Derive the Black–Scholes equation for this scenario, assuming $\Pi_t = g(S_t,t)$ with g sufficiently often differentiable.

Hint: coefficient matching of two versions of $\mathrm{d}\Pi_t$

Exercise 16 Implied Volatility

For European options we take the valuation formula of Black and Scholes of the type $V = v(S, \tau, K, r, \sigma)$, where τ denotes the time to maturity, $\tau := T - t$. For the definition of the function v see the first assignment. If actual market data of the price V are known, then one of the parameters considered known so far can be viewed as unknown and fixed via the implicit equation

$$V - v(S, \tau, K, r, \sigma) = 0. \qquad (*)$$

In this *calibration* approach the unknown parameter is calculated iteratively as solution of equation $(*)$. Consider σ to be in the role of the unknown parameter. The volatility σ determined in this way is called *implied volatility* and is zero of $f(\sigma) := V - v(S, \tau, K, r, \sigma)$.

Assignment:

i) Design, implement and test an algorithm to calculate the implied volatility of a call. Use Newton's method to construct a sequence $x_k \to \sigma$. The derivative $f'(x_k)$ can be approximated by the difference quotient

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} .$$

For the resulting *secant iteration* invent a stopping criterion that requires smallness of both $|f(x_k)|$ and $|x_k - x_{k-1}|$.

ii) Calculate the implied volatilities for the data

$$T - t = 0.211 \ , \ S_0 = 5290.36 \ , \ r = 0.0328$$

and the pairs K, V from the Table (for more data see `www.compfin.de`). For each calculated value of σ enter the point (K, σ) into a figure, joining the points with straight lines. (You will notice a convex shape of the curve. This shape has lead to call this phenomenon *volatility smile*.)

Calls on the DAX on 4.Jan 1999

K	6000	6200	6300	6350	6400	6600	6800
V	80.2	47.1	35.9	31.3	27.7	16.6	11.4

Exercise 17 Deficient Random Number Generator

For some time the generator

$$N_i = aN_{i-1} \mod M, \quad \text{with } a = 2^{16} + 3, \ M = 2^{31}$$

was in wide use. Show for the sequence $U_i := N_i / M$

$$U_{i+2} - 6U_{i+1} + 9U_i \ \text{ is integer!}$$

What does this imply for the distribution of the tripels (U_i, U_{i+1}, U_{i+2}) in the unit cube?

Exercise 18 Coarse Approximation of Normal Deviates

Let $U_1, U_2, ...$ be independent random numbers $\sim \mathcal{U}[0, 1]$, and

$$X_k := \sum_{i=k}^{k+11} U_i - 6.$$

Calculate mean and variance of the X_k .

Exercise 19 Inverting the Normal Distribution

Suppose $F(x)$ is the standard normal distribution function. Construct a rough approximation $G(u)$ to $F^{-1}(u)$ for $0.5 \leq u < 1$ as follows:

a) Construct a rational function $G(u)$ with correct asymptotic behavior, point symmetry with respect to $(u, x) = (0.5, 0)$, using only one parameter.
b) Fix the parameter by interpolating a given point $(x_1, F(x_1))$.
c) What is a simple criterion for the error of the approximation?

Exercise 20 Uniform Distribution

For the uniformly distributed random variable (V_1, V_2) on $V_1^2 + V_2^2 < 1$ consider the transformation

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} V_1^2 + V_2^2 \\ \frac{1}{2\pi} \arctan(V_2/V_1) \end{pmatrix}$$

(appropriate branch of arctan). Show that (X_1, X_2) is distributed uniformly.

(Exercise No. 21 does not exist, I miscounted.)

Exercise 22 Continuous Dividend Flow

Assume that a stock pays a dividend D once per year. Calculate a corresponding continuous dividend rate δ under the assumptions

$$\dot{S} = (\mu - \delta)S \ , \ \mu = 0, \quad S(1) = S(0) - D > 0.$$

Generalize the result to general growth rates μ and arbitrary day t_D of dividend payment.

Exercise 23 Stability of the Fully Implicit Method

The backward-difference method is defined via the solution of the equation

$$A_{\text{impl}}\, w^{(\nu)} = w^{(\nu-1)} \quad \text{with} \quad A_{\text{impl}} = \text{tridiag}(-\lambda,\ 2\lambda+1,\ -\lambda).$$

Prove the stability.
Hint: Use $w^{(\nu)} = A_{\text{impl}}^{-1}\, w^{(\nu-1)}$.

Exercise 24 RL-decomposition

Assume a system of linear equations $Ax = b$ with tridiagonal matrix A. Formulate the RL-decomposition as an algorithm.

Exercise 25 Integral Representation

For a European put with time to maturity $\tau := T - t$ prove that $[V(S_t, t) =]$

$$e^{-r\tau} \int_0^\infty (K - S_T)^+ \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp\left\{ -\frac{[\ln(S_T/S_t) - (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau} \right\} \mathrm{d}S_T$$
$$= e^{-r\tau} K F(-d_2) - S_t F(-d_1) \ ,$$

where F, d_1 and d_2 were defined earlier.
Hint: Use $(K - S_T)^+ = 0$ for $S_T > K$, and get two integrals.

Exercise 26 Crank–Nicolson Order

Let the function $y(x, \tau)$ solve the equation

$$y_\tau = y_{xx}$$

and be sufficiently smooth. With the difference quotient

$$\delta_x^2 w_{i\nu} := \frac{w_{i+1,\nu} - 2w_{i\nu} + w_{i-1,\nu}}{\Delta x^2}$$

the local truncation error ϵ of the Crank–Nicolson method is defined

$$\epsilon := \frac{y_{i,\nu+1} - y_{i\nu}}{\delta\tau} - \frac{1}{2}\left(\delta_x^2 y_{i\nu} + \delta_x^2 y_{i,\nu+1}\right).$$

Show

$$\epsilon = O(\Delta\tau^2) + O(\Delta x^2).$$

Exercise 27 Perpetual Put Option

For $T \to \infty$ it is sufficient to analyze the ODE

$$\frac{\sigma^2}{2} S^2 \frac{\mathrm{d}^2 V}{\mathrm{d}S^2} + (r - \delta) S \frac{\mathrm{d}V}{\mathrm{d}S} - rV = 0.$$

Consider an American put with high contact to the payoff $V = (K - S)^+$ at $S = S_\mathrm{f}$.
Show:

a) Upon substituting the boundary condition for $S \to \infty$ one obtains

$$V(S) = c\left(\frac{S}{K}\right)^{\lambda_2},$$

where $\lambda_2 = \frac{1}{2}\left(1 - q_\delta - \sqrt{(q_\delta - 1)^2 + 4q}\right), \quad q = \frac{2r}{\sigma^2}, \quad q_\delta = \frac{2(r-\delta)}{\sigma^2}$
and c is a positive constant.
Hint: Apply the transformation $S = Ke^x$. (The other root λ_1 drops out.)

b) V is convex.

For $S < S_\mathrm{f}$ the option is exercised; then its intrinsic value is $K - S$. For $S > S_\mathrm{f}$ the option is not exercised and has a value $V(S) > K - S$. The holder of the option decides when to exercise. This means, the holder makes a decision on the high contact S_f such that the value of the option becomes maximal.

c) Show: $V'(S_\mathrm{f}) = -1$, if S_f maximizes the value of the option.
Hint: Determine the constant c such that $V(S)$ is continuous in the contact point.

Exercise 28 Semi-Discretization

For a semi-discretization of the Black–Scholes equation (BS) consider the semi-discretized domain

$$0 \le t \le T \quad , \quad S = S_i := i\Delta S \quad , \quad \Delta S := \frac{S_\mathrm{max}}{m} \quad , \quad i = 0, 1, \ldots, m$$

for some value S_max. On this set of parallel lines define for $1 \le i \le m-1$ functions $w_i(t)$ as approximation to $V(S_i, t)$.

a) Using the standard second-order difference schemes, derive the system

$$\dot{w} + Bw = 0\ ,$$

which up to boundary conditions approximates (BS). Here w is the vector $(w_1, \ldots, w_{m-1})^{tr}$.
Show that B is a tridiagonal matrix, and calculate its coefficients.

b) Use a backward-differentiation formula to show that

$$w^{(\nu)} = 4w^{(\nu-1)} - 3w^{(\nu-2)} + 2\Delta t B w^{(\nu-2)}$$

is a valid scheme to integrate $\dot{w} + Bw = 0$.

[Computer persons are encouraged to implement this scheme.]

Exercise 29 Front-Fixing for American Options

Apply the transformation

$$\zeta := \frac{S}{S_\mathrm{f}(t)} \quad , \quad y(\zeta, t) := V(S, t)$$

to the Black–Scholes equation.

a) Show

$$\frac{\partial y}{\partial t} + \frac{\sigma^2}{2}\zeta^2 \frac{\partial^2 y}{\partial \zeta^2} + \left[(r-\delta) - \frac{1}{S_\mathrm{f}}\frac{\mathrm{d}S_\mathrm{f}}{\mathrm{d}t}\right]\zeta \frac{\partial y}{\partial \zeta} - ry = 0$$

b) Set up the domain for (ζ, t) and formulate the boundary conditions for an American call. (Assume $\delta > 0$.)

[Project: Set up a finite-difference scheme to solve the derived boundary-value problem. The curve $S_\mathrm{f}(t)$ is implicitly defined by the above PDE, with final value $S_\mathrm{f}(T) = \max(K, \frac{r}{\delta}K)$.]

Summary of American options, for a put ($r > 0$) or a call ($\delta > 0$), after transformation into (x, τ, y)-variables:

$$q := \frac{2r}{\sigma^2}; \quad q_\delta := \frac{2(r-\delta)}{\sigma^2}$$

put: $g(x,\tau) := \exp\{\frac{1}{4}((q_\delta - 1)^2 + 4q)\tau\} \max\{e^{\frac{1}{2}(q_\delta - 1)x} - e^{\frac{1}{2}(q_\delta + 1)x}, 0\}$

call: $g(x,\tau) := \exp\{\frac{1}{4}((q_\delta - 1)^2 + 4q)\tau\} \max\{e^{\frac{1}{2}(q_\delta + 1)x} - e^{\frac{1}{2}(q_\delta - 1)x}, 0\}$

$$\left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y - g) = 0$$

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0, \quad y - g \geq 0$$

$$y(x, 0) = g(x, 0), \quad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T$$

$$\lim_{x \to \pm\infty} y(x,\tau) = \lim_{x \to \pm\infty} g(x,\tau)$$

Exercise 30 Brennan–Schwartz Algorithm

Let A be a tridiagonal matrix and b and g vectors. The system of equations $Aw = b$ is to be solved such that the side condition $w \geq g$ is obeyed componentwise. Assume for a put $w_i = g_i$ for $1 \leq i \leq i_\mathrm{f}$ and $w_i > g_i$ for $i_\mathrm{f} < i \leq n$, where i_f is unknown.

a) Formulate an algorithm that solves $Aw = b$ in the *backward/forward* approach. In the final forward loop, for each i the calculated candidate w_i is tested for $w_i \geq g_i$: In case $w_i < g_i$ the calculated value w_i is corrected to $w_i = g_i$.

b) Apply the algorithm to the case of a put with relevant A, b, g.

Exercise 31 Programming Assignment

Implement an algorithm for the calculation of American-style options, following the prototype algorithm below. Use Exercises 24 and 30. For this assigment, it is sufficient to implement the case of a put.

Test example: $K = 10$, $r = 0.25$, $\sigma = 0.6$, $T = 1$, $\delta = 0.2$. Calculate approximations to $V(10, 0)$.

Algorithm (prototype algorithm)

Set up the function $g(x, \tau)$ listed in the summary.

Choose θ ($\theta = 1/2$ for Crank–Nicolson).

Fix the discretization by choosing $x_{\text{min}},\ x_{\text{max}},\ m,\ \nu_{\text{max}}$ (for example, $x_{\text{min}} = -5,\ x_{\text{max}} = 5,\ \nu_{\text{max}} = m = 100$).

Calculate $\Delta x := (x_{\text{max}} - x_{\text{min}})/m$,

$\Delta\tau := \frac{1}{2}\sigma^2 T/\nu_{\text{max}}$

$x_i := x_{\text{min}} + i\Delta x$ for $i = 0, \ldots, m$

$\lambda := \Delta\tau/\Delta x^2$ and $\alpha := \lambda\theta$.

Initialize the iteration vector w with

$g^{(0)} = (g(x_1, 0), \ldots, g(x_{m-1}, 0))$.

Arrange for the matrix A

τ-loop: for $\nu = 0, 1, \ldots, \nu_{\text{max}} - 1$:

$\tau_\nu := \nu\Delta\tau$

initialize the vector b with

$b_i := w_i + \lambda(1-\theta)(w_{i+1} - 2w_i + w_{i-1})$ for $2 \le i \le m-2$

$b_1 := w_1 + \lambda(1-\theta)(w_2 - 2w_1 + g_{0\nu}) + \alpha g_{0,\nu+1}$

$b_{m-1} := w_{m-1} + \lambda(1-\theta)(g_{m\nu} - 2w_{m-1} + w_{m-2}) + \alpha g_{m,\nu+1}$

subroutine for the LCP solution w, directly as in Exercises 24, 30

$w^{(\nu+1)} = w$

Exercise 32 Integration by Parts for Itô Integrals

a) Show

$$\int_{t_0}^{t} s\, \mathrm{d}W_s = tW_t - t_0 W_{t_0} - \int_{t_0}^{t} W_s\, \mathrm{d}s$$

Hint: Start with the Wiener process $X_t = W_t$ and apply the Itô Lemma with the transformation $y = g(x, t) := tx$.

b) Denote $\Delta Y := \int_{t_0}^{t}\int_{t_0}^{s} \mathrm{d}W_z \mathrm{d}s$. Show by using a) that

$$\int_{t_0}^{t}\int_{t_0}^{s} \mathrm{d}z\, \mathrm{d}W_s = \Delta W \Delta t - \Delta Y.$$

Exercise 33 Bias of Numerical Integration

Assume the SDE of the Black-Scholes GBM $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$ is integrated over the interval $0 \le t \le T$ with an Euler scheme with m equidistant steps of size Δt. The expectation is $x := \mathsf{E}(S_T) = S_0 \mathrm{e}^{\mu T}$. Analyze the bias of an Euler approximation $\hat{x} := S_m$ of the value S_T.

Exercise 34 Monte Carlo for European Options

Implement a Monte Carlo method for single-asset European options, based on the Black–Scholes model. Perform experiments with various values of N and a random number generator of your choice. Compare results obtained by using the analytic solution formula for S_t with results obtained by using Euler's discretization. For c) B is the barrier such that the option expires worthless when $S_t \geq B$ for some t.

input: S_0, number of simulations (trajectories) N, payoff function $\Lambda(S)$, risk-neutral interest rate r, volatility σ, time to maturity T, strike K.

payoffs:

a) vanilla put, with $\Lambda(S) = (K - S)^+$, $S_0 = 5$, $K = 10$, $r = 0.06$, $\sigma = 0.3$, $T = 1$.

b) binary call, with $\Lambda(S) = \mathbf{1}_{S>K}$, $S_0 = K = \sigma = T = 0.5$, $r = 0.1$

c) up-and-out barrier: call with $S_0 = 5$, $K = 6$, $r = 0.05$, $\sigma = 0.3$, $T = 1$, $B = 8$.

Hint: Correct values are: a) 4.43046 b) 0.46220 [Quecke] c) 0.0983 [Hig04]