MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
Prof. Dr. Rüdiger Seydel
Dipl.-Math. Christian Jonen

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Computational Finance - 10th Assignment

Deadline: June 22

Exercise 30 (*LU* Decomposition of a Special Tridiagonal Matrix) (4 points)

A matrix of the form

$$A = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \dots & \dots & \dots & \\ & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & b_n & a_n \end{pmatrix} =: \text{tridiag}(b_\mu, a_\mu, c_\mu)$$

with

$$\begin{aligned} |a_1| &> |c_1| > 0, \\ |a_\mu| &\geq |b_\mu| + |c_\mu|,\ b_\mu \neq 0, c_\mu \neq 0,\ 2 \leq \mu \leq n-1, \\ |a_n| &\geq |b_n| > 0 \end{aligned}$$

is called an *irreducible diagonally dominant* tridiagonal matrix.
Show that an irreducible diagonally dominant tridiagonal matrix $A = \text{tridiag}(b_\mu, a_\mu, c_\mu)$ has an LU decomposition $A = LU$ with $L = \text{tridiag}(b_\mu, \alpha_\mu, 0)$ and $U = \text{tridiag}(0, 1, \gamma_\mu)$, where $\alpha_1 := a_1$, $\gamma_1 := c_1\alpha_1^{-1}$ and

$$\begin{aligned} \alpha_\mu &:= a_\mu - b_\mu\gamma_{\mu-1}, \quad 2 \leq \mu \leq n, \\ \gamma_\mu &:= c_\mu\alpha_\mu^{-1}, \quad 2 \leq \mu \leq n-1. \end{aligned}$$

Exercise 31 (Crank-Nicolson Order) (8 points)

Let the function $y(x, \tau)$ solve the equation

$$y_\tau = y_{xx}$$

and be sufficiently smooth. With the difference quotient

$$\delta_x^2 w_{i\nu} := \frac{w_{i+1,\nu} - 2w_{i\nu} + w_{i-1,\nu}}{\Delta x^2}$$

the local truncation error ϵ of the Crank-Nicolson method is defined as

$$\epsilon := \frac{y_{i,\nu+1} - y_{i\nu}}{\delta\tau} - \frac{1}{2}\left(\delta_x^2 y_{i\nu} + \delta_x^2 y_{i,\nu+1}\right).$$

Show

$$\epsilon = O(\Delta\tau^2) + O(\Delta x^2).$$

Exercise 32 (Transformation of the Boundary Conditions of BS) (3 points)

In Exercise 7 the Black-Scholes equation is transformed into the equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2} \, .$$

Transform the boundary conditions of the Black-Scholes equation accordingly.