MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
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Computational Finance - 3rd Assignment

Deadline: April 27

Exercise 6 (Approximation Formula for F) (3+4 points)

The function

$$f(x) := \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)$$

is the density function of the standard normal distribution. Then

$$F(x) := \int_{-\infty}^{x} f(t)dt$$

is the associated distribution function (compare Exercise 4).
The following relation holds for $0 \leq x < \infty$:

$$F(x) = 1 - f(x)(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5) + \varepsilon(x),$$

where

$$z := \frac{1}{1 + 0.2316419x},$$

$$a_1 = 0.319381530, \quad a_2 = -0.356563782, \quad a_3 = 1.781477937,$$
$$a_4 = -1.821255978, \quad a_5 = 1.330274429$$

and the absolute error ε is bounded by

$$|\varepsilon(x)| < 7.5 \cdot 10^{-8}.$$

Hence we have the approximating formula

$$F(x) \approx 1 - f(x)z((((a_5 z + a_4)z + a_3)z + a_2)z + a_1).$$

a) Use this approximation formula to compute the option prices of Exercise 4 b). Compare these results with those obtained with your computer program.

b) How many subintervals do you need for the computation of $F(x)$ via a trapezoidal sum in order that the error is smaller than $7.5 \cdot 10^{-8}$? Use the error formula for the trapezoidal sum to answer this question.

Exercise 7 (Transforming the Black–Scholes Equation) (5+4+3 points)

Show that the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{BS}$$

for $V(S,t)$ with constant σ and r is equivalent to the equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

for $y(x,\tau)$. For proving this, you may proceed as follows:

a) Use the transformation $S = Ke^x$ and a suitable transformation $t \leftrightarrow \tau$ to show that (BS) is equivalent to

$$-\dot{V} + V'' + \alpha V' + \beta V = 0$$

with $\dot{V} = \frac{\partial V}{\partial \tau}$, $V' = \frac{\partial V}{\partial x}$, $\alpha,\ \beta$ depending on r and σ.

b) Next apply a transformation of the type

$$V = K\exp(\gamma x + \delta\tau)y(x,\tau)$$

for suitable $\gamma,\ \delta$.

c) Transform the terminal condition of the Black–Scholes equation accordingly.

Exercise 8 (On the Binomial Method) (4+5+5 points)

a) For $ud = \gamma$ in the binomial method show

$$u = \beta + \sqrt{\beta^2 - \gamma}, \qquad \text{where}$$
$$\beta := \frac{1}{2}(\gamma e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}).$$

b) For $\gamma = 1$ show

$$u = \exp\left(\sigma\sqrt{\Delta t}\right) + O\left(\sqrt{(\Delta t)^3}\right) \quad \text{and} \quad d = \exp\left(-\sigma\sqrt{\Delta t}\right) + O\left(\sqrt{(\Delta t)^3}\right).$$

c) Show that

$$\Delta_{Binom} := \frac{V^{(u)} - V^{(d)}}{S_0(u-d)},$$

where $V^{(u)} = V(uS_0, t+\Delta t)$ and $V^{(d)} = V(dS_0, t+\Delta t)$, fulfils

$$\Delta_{Binom} = \frac{\partial V}{\partial S} + O(\Delta t).$$

Exercise 9 (Martingale) (2+2 points)

Show that W_t and $W_t^2 - t$ are martingales.

Information:

- The deadline for the programming exercise is April 27. Please turn in a printed version of your code and send it to *numerik_programm@gmx.de*.