MATHEMATISCHES INSTITUT
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Summer 2011
May 4

Computational Finance - 5th Assignment

Deadline: May 11

Exercise 13 (Limiting Case of the Binomial Model) (3+4+3+6 points)

Consider a European call option in the binomial model. Suppose the calculated value is $V_0^{(M)}$. In the limit $M \to \infty$ the sequence $V_0^{(M)}$ converges to the value $V_{\mathrm{C}}(S_0, 0)$ of the continuous Black-Scholes model listed in Exercise 4. To prove this, proceed as follows:

a) Let j_K be the smallest index j with $S_{jM} \geq K$. Find an argument why

$$\sum_{j=j_K}^{M} \binom{M}{j} p^j (1-p)^{M-j} (S_0 u^j d^{M-j} - K)$$

is the expectation $\mathrm{E}(V_T)$ of the payoff.

b) The approximated value of the option is obtained by discounting: $V_0^{(M)} = e^{-rT}\mathrm{E}(V_T)$. Show

$$V_0^{(M)} = S_0 B_{M,\tilde{p}}(j_K) - e^{-rT} K B_{M,p}(j_K).$$

Here $B_{M,p}(j)$ is defined by the binomial distribution, and $\tilde{p} := pue^{-r\Delta t}$.

c) For large M the binomial distribution is approximated by the normal distribution with distribution $F(x)$. Show that $V_0^{(M)}$ is approximated by

$$S_0 F\left(\frac{M\tilde{p} - \alpha}{\sqrt{M\tilde{p}(1-\tilde{p})}}\right) - e^{-rT} K F\left(\frac{Mp - \alpha}{\sqrt{Mp(1-p)}}\right),$$

where

$$\alpha := -\frac{\log \frac{S_0}{K} + M \log d}{\log u - \log d}.$$

d) Substitute p, u, d to show

$$\frac{Mp - \alpha}{\sqrt{Mp(1-p)}} \longrightarrow \frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

for $M \to \infty$.
Hint: Use Exercise 8b): Up to terms of high order the approximations $u \approx e^{\sigma\sqrt{\Delta t}}$, $d \approx e^{-\sigma\sqrt{\Delta t}}$ hold.
(The other argument of F can be analyzed in an analogous way.)

Exercise 14 (Proof of the Lemma of Paragraph 1.4) (4+3+1 points)

a) Suppose that a random variable X_t satisfies $X_t \sim \mathcal{N}(0, \sigma^2)$. Use

$$\mathrm{E}(X) := \int_{-\infty}^{\infty} x f(x) dx$$

to show

$$\mathrm{E}(X_t^4) = 3\sigma^4.$$

b) Apply a) to show the assertion

$$\mathrm{E}\left(\left[\sum_j ((\Delta W_j)^2 - \Delta t_j) \right]^2 \right) = 2 \sum_j (\Delta t_j)^2.$$

c) Deduce the assertion of the lemma of paragraph 1.4 from b).

Exercise 15 (Moments of the Lognormal Distribution) (4+4 points)

For the density function $f(S; t - t_0, S_0, \mu, \sigma)$ defined by

$$f(S; t - t_0, S_0, \mu, \sigma) := \frac{1}{S\sigma\sqrt{2\pi(t - t_0)}} \exp\left\{ -\frac{\left(\log(S/S_0) - \left(\mu - \frac{\sigma^2}{2}\right)(t - t_0)\right)^2}{2\sigma^2(t - t_0)} \right\}$$

show

a) $\displaystyle\int_0^{\infty} S f(S; t - t_0, S_0, \mu, \sigma) dS = S_0 e^{\mu(t - t_0)}$,

b) $\displaystyle\int_0^{\infty} S^2 f(S; t - t_0, S_0, \mu, \sigma) dS = S_0^2 e^{(\sigma^2 + 2\mu)(t - t_0)}$.

Hint: Set $y = \log(S/S_0)$ and transform the argument of the exponential function to a squared term.