MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
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Computational Finance - 6th Assignment

Deadline: May 18

Exercise 16 (Analytical Solution of SDE) (4 points)

Solve the stochastic differential equation

$$dY_t = Y_t^2[(4Y_t - 1)\,dt - 2\,dW_t].$$

Hint: Start with $dX_t = a\,dt + b\,dW_t$ und use a suitable function g with $Y_t = g(X_t, t)$.

Exercise 17 (Integration by Parts) (3 points)

The *multi-dimensional Itô lemma* is as follows:
Let X_t be a n-dimensional Itô process, i.e.

$$dX_t = a(X_t, t)\,dt + b(X_t, t)\,dW_t$$

where

$$X_t = \begin{pmatrix} X_t^{(1)} \\ \vdots \\ X_t^{(n)} \end{pmatrix}, \quad W_t = \begin{pmatrix} W_t^{(1)} \\ \vdots \\ W_t^{(m)} \end{pmatrix}, \quad a(X_t, t) = \begin{pmatrix} a_1(X_t^{(1)}, \ldots, X_t^{(n)}, t) \\ \vdots \\ a_n(X_t^{(1)}, \ldots, X_t^{(n)}, t) \end{pmatrix}$$

and

$$b(X_t, t) = ((b_{ik}(X_t, t)))_{i=1,\ldots,n}^{k=1,\ldots,m}.$$

Further let $g : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^p$ be a C^2 map.
Then the process

$$Y_t = g(X_t, t)$$

is again an Itô process, whose component number $k \in \{1, \ldots, p\}$, $Y_t^{(k)}$, is given by

$$dY_t^{(k)} = \frac{\partial g_k}{\partial t}(X_t, t)\,dt + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(X_t, t)\,dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(X_t, t)\,dX_t^{(i)} dX_t^{(j)}$$

where $dW_t^{(i)} dW_t^{(j)} = \delta_{ij}\,dt,\ dt\,dt = dW_t^{(i)} dt = dt\,dW_t^{(i)} = 0$.

Use this multi-dimensional Itô lemma for the following assignment:
Let X_t, Y_t be Itô processes in $\mathbb{R}$. Prove that

$$d(X_t Y_t) = X_t\,dY_t + Y_t\,dX_t + dX_t \cdot dY_t\,.$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s\,dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s\,dX_s - \int_0^t dX_s \cdot dY_s\,.$$

Exercise 18 (General Black–Scholes Equation) (5 points)

Assume a portfolio

$$\Pi_t = \alpha_t S_t + \beta_t B_t$$

consisting of a stock S_t and a bond B_t, which obey

$$\begin{aligned} dS_t &= \mu(S_t, t)\, dt + \sigma(S_t, t)\, dW_t \\ dB_t &= r(t) B_t\, dt\,. \end{aligned}$$

The functions μ, σ and r are assumed to be known; $\sigma > 0$. Furthermore, assume the portfolio is *self-financing* in the sense

$$d\Pi_t = \alpha_t\, dS_t + \beta_t\, dB_t\,,$$

and *replicating* such that Π_T equals the payoff of a European option. (Recall the consequence that Π_t equals the price of the option for all t.)
Derive the Black–Scholes equation for this scenario, assuming $\Pi_t = g(S_t, t)$ with g sufficiently often differentiable.

Hint: coefficient matching of two versions of $d\Pi_t$

Exercise 19 (Implied Volatility) (20P points)

For European options we take the valuation formula of Black and Scholes of the type $V = v(S, \tau, K, r, \sigma)$, where τ denotes the time to maturity, $\tau := T - t$. For the definition of the function v see exercise 3. If actual market data of the price V are known, then one of the parameters considered known so far can be viewed as unknown and fixed via the implicit equation

$$V - v(S, \tau, K, r, \sigma) = 0. \tag{$*$}$$

In this *calibration* approach the unknown parameter is calculated iteratively as solution of equation $(*)$. Consider σ to be in the role of the unknown parameter. The volatility σ determined in this way is called *implied volatility* and is a zero of $f(\sigma) := V - v(S, \tau, K, r, \sigma)$.

Assignment:

a) Design, implement and test an algorithm to calculate the implied volatility of a call. Use Newton's method to construct a sequence $x_k \to \sigma$. The derivative $f'(x_k)$ can be approximated by the difference quotient

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\,.$$

For the resulting *secant iteration* invent a stopping criterion that requires smallness of both $|f(x_k)|$ and $|x_k - x_{k-1}|$.

b) Consider the following market data for call options on the same underlying:

$$T - t = 0.32787,\ S_0 = 7133.06,\ r = 0.0487,$$

K	6400	6700	7000	7300	7600	7900	8200	8500	8800
V	934.0	690.0	469.0	283.0	145.0	62.0	22.0	7.5	2.1

Calculate the implied volatilities for these data. For each calculated value of σ enter the point (K, σ) into a figure and join the points with straight lines. (You will notice a convex shape of the curve. This shape has led to call this phenomenon *volatility smile*.)

c) In order to calculate implied volatilities for call options on the DAX with real market data, visit the homepage `www.optionsscheine.onvista.de`. The underlying has the security identification number 846900. Consider an issuer and determine a set of data by choosing options with different strikes for the same maturity. Proceed as in b) with your downloaded data.

Information:

- The deadline for the programming exercise is May 25. Please turn in a printed version of your code and send it to *numerik_programm@gmx.de*.