MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
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Computational Finance - 8th Assignment

Deadline: June 1

Exercise 24 (Integration by Parts for Itô Integrals) (2+3 points)

a) Show

$$\int_{t_0}^{t} s\, dW_s = tW_t - t_0 W_{t_0} - \int_{t_0}^{t} W_s\, ds.$$

Hint: Start with the Wiener process $X_t = W_t$ and apply the Itô Lemma with the transformation $y = g(x,t) := tx$.

b) Denote $\Delta Y := \int_{t_0}^{t}\int_{t_0}^{s} dW_z ds$, $\Delta W := W_t - W_{t_0}$ and $\Delta t := t - t_0$. Show by using a) that

$$\int_{t_0}^{t}\int_{t_0}^{s} dz\, dW_s = \Delta W \Delta t - \Delta Y.$$

Exercise 25 (Integral Representation) (8 points)

For an European put with time to maturity $\tau := T - t$ prove that

$$\begin{aligned}[V(S_t,t) =]e^{-r\tau} &\int_0^\infty (K - S_T)^+ \frac{1}{S_T\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{[\ln(S_T/S_t) - (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau}\right\} dS_T \\ &= e^{-r\tau} K F(-d_2) - S_t F(-d_1),\end{aligned}$$

where F, d_1 and d_2 were defined in Exercise 4.
Hint: Use $(K - S_T)^+ = 0$ for $S_T > K$, and get two integrals.

Exercise 26 (Random Number Generators and Monte Carlo) (25P points)

a) Implement the *linear congruential generator* given by

$$N_i = (aN_{i-1} + b) \mod M \quad \text{with} \;\; a = 1366,\; b = 150889,\; M = 714025.$$

The seed N_0 should be the input value.
Use your program to compute 10000 pairs (U_{i-1}, U_i) in the unit square and plot them.

b) Implement the *Fibonacci generator* given by

$$\begin{aligned} U_i &:= U_{i-17} - U_{i-5}, \\ U_i &:= U_i + 1 \quad \text{if} \quad U_i < 0. \end{aligned}$$

Calculate $U_1, \dots, U_{17}$ with the linear congruential generator of a).
Use your program to compute 10000 pairs (U_{i-1}, U_i) in the unit square and plot them.

c) Implement the *polar method of Marsaglia*. Calculate the initial values with the Fibonacci generator of b).
Use your program to compute 10000 standard normally distributed numbers and plot them in two dimensions by separating them vertically with distance 10^{-4}. Furthermore, divide the x-axis into subintervals having the same length and count the computed numbers in each subinterval. Then set up the corresponding histogram.

d) Implement a Monte Carlo method for single-asset European options, based on the Black-Scholes model. Perform experiments with various values of N (see below) and a random number generator of your choice. To obtain values for S_T, use the analytic solution formula for S_t and also alternatively Milstein's discretization. (Compare the different results.)

Input values: S_0, number of simulations (trajectories) N, payoff function $\Lambda(S)$, risk-neutral interest rate r, volatility σ, time to maturity T, strike K.
Output value: approximated value of the option $V_0^{(N)}$.

Compute approximations $V_0^{(N)}$ for $N = 1, 10, 100, 1000, 10000$ for the following option prices at time $t = 0$:

i) European put with $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 5$, $\delta = 0$,

ii) European call with the same parameters,

iii) binary call with the same parameters.

For i) and ii), compare your results with the values obtained via the Black Scholes formula.

<u>Information</u>:

- The deadline for the programming exercise is June 8. Please turn in a printed version of your code and send it to *numerik_programm@gmx.de*.
- The first session of the Cologne Computational Finance Laboratory takes place after exercises on June 6 and on June 8.