MATHEMATISCHES INSTITUT
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Computational Finance - 9th Assignment

Deadline: June 8

Exercise 27 (Moments of Itô Integrals) (3+7+3 points)

a) Use the Itô isometry

$$\mathrm{E}\left(\left[\int_a^b f(t,\omega)\,dW_t\right]^2\right) = \int_a^b \mathrm{E}\left(f^2(t,\omega)\right)\,dt$$

to show its generalization

$$\mathrm{E}(I(f)I(g)) = \int_a^b \mathrm{E}(fg)\,dt, \quad \text{where} \quad I(f) = \int_a^b f(t,\omega)\,dW_t.$$

Hint: $4fg = (f+g)^2 - (f-g)^2$

b) Show for ΔY, ΔW and Δt defined in Exercise 24 by using a) and $\mathrm{E}\left(\int_a^b f(t,\omega)\,dW_t\right) = 0$ the following assertions for the moments:

$$\mathrm{E}(\Delta Y) = 0, \quad \mathrm{E}(\Delta Y^2) = \frac{\Delta t^3}{3}, \quad \mathrm{E}(\Delta Y \Delta W) = \frac{\Delta t^2}{2}, \quad \mathrm{E}(\Delta Y \Delta W^2) = 0.$$

c) By transformation of two independent standard normally distributed random variables $Z_i \sim \mathcal{N}(0,1), i = 1,2$, two new random variables are obtained by

$$\Delta\widehat{W} := Z_1\sqrt{\Delta t}, \quad \Delta\widehat{Y} := \frac{1}{2}(\Delta t)^{3/2}(Z_1 + \frac{1}{\sqrt{3}}Z_2).$$

Show that $\Delta\widehat{W}$, $\Delta\widehat{Y}$ and their corresponding products have the same moments like ΔW and ΔY in b).

Exercise 28 (Second-order Weakly Convergent Method) (3+2 points)

In addition to the moments of Exercise 27 b) further moments of the random variables ΔW and ΔY of Exercises 24 and 27 are

$$\mathrm{E}(\Delta W) = \mathrm{E}(\Delta W^3) = \mathrm{E}(\Delta W^5) = 0, \quad \mathrm{E}(\Delta W^2) = \Delta t, \quad \mathrm{E}(\Delta W^4) = 3\Delta t^2.$$

Assume a new random variable $\Delta\widetilde{W}$ satisfying

$$\mathrm{P}\left(\Delta\widetilde{W} = \pm\sqrt{3\Delta t}\right) = \frac{1}{6}, \quad \mathrm{P}\left(\Delta\widetilde{W} = 0\right) = \frac{2}{3}$$

and the additional random variable

$$\Delta\widetilde{Y} := \frac{1}{2}\Delta\widetilde{W}\Delta t.$$

a) Show that the random variables $\Delta\widetilde{W}$ and $\Delta\widetilde{Y}$ have up to terms of order $\mathcal{O}(\Delta t^3)$ the same moments as ΔW and ΔY.

b) Deduce that the method

$$\begin{aligned} y_{j+1} = y_j + a\Delta t + b\Delta\widetilde{W} + \frac{1}{2}bb'\left((\Delta\widetilde{W})^2 - \Delta t\right) \\ + \frac{1}{2}\left(a'b + ab' + \frac{1}{2}b^2b''\right)\Delta\widetilde{W}\Delta t + \frac{1}{2}\left(aa' + \frac{1}{2}b^2a''\right)\Delta t^2 \end{aligned}$$

is second-order weakly convergent.

Exercise 29 (Mean Square Error) (1+2+1 points)

For the mean square error the relation

$$\mathrm{MSE}(\hat{x}) = (\mathrm{bias}(\hat{x}))^2 + \mathrm{Var}(\hat{x})$$

is valid. Let $\hat{x} := y_T^h$ be the result of a weakly convergent discretization scheme with order β and $g =$ identity, where h denotes the step length. Then, the bias is of the order β,

$$\mathrm{bias}(\hat{x}) = \alpha_1 h^\beta, \quad \alpha_1 \text{ a constant.}$$

Since the variance of Monte Carlo is of the order $\frac{1}{N}$ (N being the number of samples), we have

$$\zeta(h, N) := \mathrm{MSE}(\hat{x}) = \alpha_1^2 h^{2\beta} + \frac{\alpha_2}{N}, \quad \alpha_2 \text{ another constant.}$$

a) Argue why for some constant α_3

$$C(h, N) := \alpha_3 \frac{N}{h}$$

is a reasonable model for the costs of the MC simulation.

b) Minimize $\zeta(h, N)$ with respect to h, N subject to the side condition

$$\alpha_3 \frac{N}{h} = C$$

for given budget C.

c) Show that for the optimal h, N

$$\sqrt{\mathrm{MSE}(\hat{x})} = \alpha_4 C^{-\frac{\beta}{1+2\beta}}$$

for some constant α_4.