MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
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Winter 2009/10
November 19

Computational Finance 2 - 6th Assignment

Deadline: November 26

Exercise 12 (Finite Element Approach - Heston Model) (3+4+6 Points)

Heston's model is an extension of the Black Scholes model; it is a stochastic volatility model and is given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt \end{aligned}$$

where dW_t^1 and dW_t^2 are Wiener processes with correlation ρ; μ is the rate of return of the asset; θ is the long vol; κ is the rate at which v_t reverts to θ; and ξ is the volatility of the volatility. Then, the value function U(t,v,S) of a vanilla put option satisfies

$$U_t + 0.5\xi^2 vU_{vv} + \rho\xi vSU_{vS} + 0.5vS^2U_{SS} + (\kappa(\theta - v) - \lambda v)U_v + rSU_S - rU = 0$$

$$U(T, v, S) = (K - S)^+$$

$$U_t + rSU_S + \kappa\theta U_v - rU = 0,\ v \to 0$$

$$U_t + 0.5\xi^2 vU_{vv} + (\kappa(\theta - v) - \lambda v)U_v - rU = 0,\ S \to 0$$

$$U(t, v, \infty) = 0$$

$$U_t + 0.5vS^2U_{SS} + rSU_S - rU = 0,\ v \to \infty$$

where λ is the market price of volatility risk. Apply a finite element approach to solve Heston's PDE numerically as follows:

a) Apply the change of variables $S = Ke^x$.

b) Following some algebraic manipulations, the PDE resulting from a) can be put into the following form:

$$u_t - V \cdot \nabla u = -\nabla \cdot D \cdot \nabla u + ru$$

Determine V and D.

c) Formulate the variational problem.

d) Approximate the variational problem by finite elements. To this end, proceed as in Exercise 11 b).

Exercise 14 (Inline Options) (6 Points)

What is an inline option? Sketch the payoff. Does it make sense to buy an inline option on the DAX with maturity in February 2010?

Bonus Exercise 15 (Forward Equation - Black Scholes Model) (6 Points)

The Forward equation (or Fokker Planck equation) for the geometric Brownian motion is

$$\frac{\partial p}{\partial T} + p(\mu - \sigma^2) - 0.5\sigma^2 S_T^2 \frac{\partial^2 p}{\partial S_T^2} + (\mu - 2\sigma^2) S_T \frac{\partial p}{\partial S_T} = 0$$

$$p(S_T, T; S_0, t_0) = \delta(S_T - S_0)$$

$$p(0, T; S_0, t_0) = 0$$

$$\lim_{S_T \to \infty} p(S_T, T; S_0, t_0) = 0$$

Show that

$$p(S_T, T; S_0, t_0) = \frac{1}{\sqrt{2\pi(\sigma S_T)^2(T - t_0)}} \exp\left(-\frac{(\ln(S_T/S_0) - (\mu - \sigma^2/2)(T - t_0))^2}{2\sigma^2(T - t_0)}\right).$$

In so doing, find an appropriate change of variables and use Green's function.