## Hints on Some of the Exercises

of the book

R. Seydel: Tools for Computational Finance. Springer, 2002/2004/2006/2009/2012.

**Preparatory Remarks:**

Some of the hints suggest ideas that may simplify solving the exercises significantly. Hence one should first try to solve the exercise without a hint. In any case one should argue why the hint may be reasonable.

The exercises differ strongly in their degree of difficulty. Several of the exercises appear so simple that no hint is given. For some exercises, the hint on the solution is incorporated into the assignment.

**Hints on Exercise 1.1:**

The assumed scenario implies for the value $\Pi(t)$ of the portfolio

$$\Pi = S + V_{\mathrm{P}} - V_{\mathrm{C}}.$$

In this way $\Pi(T)$ is determined, which must be discounted.

**Hints on Exercise 1.2:**

You find the final transformation in Section 4.1. This exercise suggests a constructive way to derive the transformation. Actually the transformation consists of several transformations. The terms of the type

$$S^{\nu} \frac{\partial^{\nu} V}{\partial S^{\nu}}$$

suggest a transformation to constant coefficients. One possibility is to introduce a new independent variable $x$ by $S = e^x$. But one should prefer $S = Ke^x$ for better scaling. The backward running time $t$ transforms by $\tilde{t} = -t$ to a forward running time $\tilde{t}$. This is just a basic version of the time transformation. Modifications and suitable scalings may lead to artificial time variables $\tilde{t}$ (or $\tau$) with different properties. The scaling $\tilde{t} = -\frac{1}{2}\sigma^2 t$ leads to coefficients $\alpha$ and $\beta$ that only depend on the dimensionless parameter $2r/\sigma^2$. We suggest to introduce an artificial time variable $\tau$ via

$$t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \ .$$

The involved additional translation by $T$ maps the terminal condition for $t = T$ to an initial condition for $\tau = 0$. The transformation of the assignment in b) allows to eliminate the terms $\alpha V'$ and $\beta V$ by choosing suitable $\gamma$ and $\delta$.

**Hints on Exercise 1.3:**

Make use of $F(0) = \frac{1}{2}$ to show first

$$F(x) = \frac{1}{2}\left(1 + \text{ erf}\left(\frac{x}{\sqrt{2}}\right)\right).$$

It remains to calculate $\text{erf}\left(\frac{x}{\sqrt{2}}\right)$. The integral $\text{erf}(z)$ for $z = \frac{x}{\sqrt{2}}$ is approximated numerically by quadrature methods. This is easily done using trapezoidal sums. Extrapolation should be applied to obtain high accuracy. Both are introduced in Appendix A4 (C1 in third edition).

**Hints on Exercise 1.4:**

The derivation of the alternative formula for $s_M^2$ is simple calculus. The recursion follows from a suitable definition of the $\alpha_i$ and $\beta_i$ as partial sums. For the rounding error analysis use the overbar to designate values subjected to rounding errors. For example, the formula

$$\zeta_i := \frac{x_i - \alpha_{i-1}}{i}$$

leads to

$$\bar{\zeta}_i = (x_i - \alpha_{i-1})(1 + \sigma_i)(1 + \delta_i)/i,$$

where $\sigma_i$ is the relative error of the subtraction, and $\delta_i$ that of the division. Both errors $\sigma_i$ and $\delta_i$ are bounded by the relative machine precision eps, $|\sigma_i| <$ eps , $|\delta_i| <$ eps . In the computer, a quadratic term $\sigma_i\delta_i$ is not visible compared to eps, and hence can be neglected. So, up to first order we have

$$\bar{\zeta}_i \doteq \zeta_i\ (1 + \sigma_i + \delta_i),$$

where the symbol $\doteq$ stands for "first order." In summary, the rounding error-subjected result satisfies

$$\bar{\zeta}_i \doteq \zeta_i\ (1 + \gamma_i),$$

with $|\gamma_i| = |\delta_i + \sigma_i| \leq 2$ eps . We conclude that the relative rounding error $\gamma_i$ of the formula for $\zeta_i$ is small as long as the numbers $x_i$ and $\alpha_{i-1}$ are regarded free of error. But since $x_i$ and $\alpha_{i-1}$ import an initial error into the $i$-th loop, the relative error in $\bar{\zeta}_i$ can become large because of *cancellation.* The error bound then involves a term with $|x_i - \alpha_{i-1}|$ in the denominator. (Why?) The further propagation of the error in $\bar{\zeta}_i$ can become disturbing when $\alpha_i$ is calculated and $\alpha_i \to 0$, even in case $\zeta_i \to 0$. When the $\beta_i$ are calculated, the algorithm benefits from the fact that only non-negative numbers are added. The possible cancellation in the calculation of $x_i - \alpha_{i-1}$ has little effect on the final result $s_M^2$. (Try to find a proper argument for this claim.)

**Hints on Exercise 1.11:**

Start from $X_t = W_t$ and find the coefficients $a$ and $b$ by comparison with equation (1.31) /Definition 1.10. Apply Itô's lemma to derive

$$dy = \exp\left(x - \frac{t}{2}\right)\, dW.$$

For the assignment b) start from the result of a) and use $g(x,t) = x^2$.

**Hints on Exercise 1.12:**

Since both assignments a) and b) are proved in an analogous way, it may suffice to prove one of the two assignments, say b): A first substitution is $y = \log(S/S_0)$. To save scratch paper, use the abbreviation $\Delta t := t - t_0$. With suitable manipulations bring the integrand to the form

$$\exp\left\{-u(y) + \Delta t(\sigma^2 + 2\mu)\right\};$$

the function $u(y)$ is obtained in this way. For such an integrand, the next substitution is standard. The final step is to make use of

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}.$$

**Hints on Exercise 1.13:**

Assignment d): Use the information of equation (1.34).

**Hints on Exercise 1.16:**

In Figure 1.24 (c), the horizontal middle part has the value 0.

**Hints on Exercise 2.1:**

Set up a list of possible pairs of integers $(z_0, z_1)$, for which $z_0 + 2z_1 = 0 \bmod 11$ holds. For each pair investigate the number of parallel straight lines in the $(U_{i-1}, U_i)$-plane that touch the unit square and satisfy

$$z_0 U_{i-1} + z_1 U_i = c \in \mathbb{Z}.$$

**Hints on Exercise 2.2:**

Express $N_{i+1}$ in dependence on $N_i$. Then do the same with $N_{i+2}$ and substitute into

$$N_{i+2} - 6N_{i+1} + 9N_i\ .$$

**Hints on Exercise 2.3:**

For motivation test the assertions by means of the example of Exercise 2.1. The assignment b) uses the result of a). As a simplification show b) first for the two-dimensional case ($m = 2$).

**Hints on Exercise 2.4:**

Recall the properties of a random variable $U$ that is uniformly distributed on the interval $a \leq u \leq b$: Its expectation is $\frac{1}{2}(a+b)$, and its variance is $\frac{1}{12}(b-a)^2$.

**Hints on Exercise 2.6:**

Realize that a zero of $G$ is zero of the numerator of the rational function, and a pole of $G$ (vertical tangent) is zero of the denominator. If a rational function satisfies a point symmetry and involves only one parameter, then its formula has a simple form. The general *ansatz* is

$$G(u) = \frac{\tilde{u} \sum_{l=0}^{n} a_l \tilde{u}^{2l}}{\sum_{l=0}^{m} b_l \tilde{u}^{2l}}, \qquad \text{with } \tilde{u} := u - \frac{1}{2}$$

**Hints on Exercise 2.7:**

The transformation of Exercise 2.7 maps the unit disc to the unit square. (Correction: first line, replace: on $[-1,1]^2$ by: on the unit disk) In order to satisfy $0 \leq X_2 \leq 1$, several branches of "arctan" are required.

In order to obtain a uniform distribution on the unit disc $\mathcal{D}$, all outside pionts $(U_1, U_2)$ are discarded, where $U_1$ and $U_2$ are uniformly distributed on $[-1, 1]$. The density of the remaining points in $\mathcal{D}$ is the inverse of the area,

$$f(V_1, V_2) = \begin{cases} \frac{1}{\pi} & \text{for } (V_1, V_2) \in \mathcal{D} \\ 0 & \text{elsewhere .} \end{cases}$$

The rest follows from the transformation (Theorem 2.11). Hence the inverse function must be determined first. The density on the unit square again equals the inverse of the area.

**Hints on Exercise 2.9:**

Following Algorithm 2.14, the required transformation is $AZ$, with $\Sigma = AA^{tr}$. For the frequently used special case $n = 2$ it is worthwhile to derive an explicit solution formula. To this end set

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

and calculate $a, b$ and $c$ from $\rho$, $\sigma_1$ and $\sigma_2$.

**Hints on Exercise 2.12:**

In the definition of the discrepancy the supremum (viz the smallest upper bound) is taken of expressions of the form $d := |\alpha - \beta|$, for $0 \leq \alpha, \beta \leq 1$. This reasoning leads to a). For b) show the left-hand inequality $D_N^* \leq D_N$ first. Motivate the right-hand inequality by menas of the case $m = 1$. For general $m$ apply the fact that each rectangle that is parallel to the axes, can be represented as a combination of $2^m$ rectangles $Q^*$. For $m = 2$ we suggest a graphical illustration. For c) analyze the saw-tooth function $d$ and the height of its jumps.

**Hints on Exercise 3.2:**

It is recommendable to use the abbreviation $\Delta W_j := W_{t_{j+1}} - W_{t_j}$. Whereas proving a) only requires basic calculus, the assertion in b) needs stochastic analysis. For $n \to \infty$ the left-hand side of the assertion a) converges in the square mean to the stochastic integral

$$\int_0^t W_s \, dW_s$$

of equation (3.9). (This is not part of the assignment.) The remaining step in the proof of b) amounts to show that the final sum in a) converges to $t$ in the mean. After carrying out the multiplications of $(\Sigma_j \Delta W_j^2 - t)^2$, split off the sum $\Sigma_j \Delta W_j^4$. Incorporate the expectation and use the formula $\mathsf{E}(\Delta W_j^4) = 3\Delta t_j^2$ for the fourth moments of a normal deviate. Finally you reach

$$\mathsf{E}\left(\left(\sum_{j=1}^{n} \Delta W_j^2 - t\right)^2\right) = 2\sum_{j=1}^{n} \Delta t_j^2,$$

which converges to 0 for $n \to \infty$.

**Hints on Exercise 3.3:**

$\Delta t := t - t_0$

**Hints on Exercise 3.12:**

The parabola is defined by the intersection of its left branch with the $S$-axis.

**Hints on Exercise 4.1:**

a) The model $\dot{S} = -\delta S$, $S(1) = S(0) - D > 0$ defines a continuous dividend flow with rate $\delta$. (Try to motivate this model assumption.) By means of the solution of this differential equation one obtains a simple formula for $\delta$ depending on the ratio $D/S(0)$.
b) $S = 0$ is not needed and meaningless.

**Hints on Exercise 4.3:**

To show

$$\epsilon := \frac{y_{i,\nu+1} - y_{i\nu}}{\Delta\tau} - \frac{1}{2}\left(\delta_x^2 y_{i\nu} + \delta_x^2 y_{i,\nu+1}\right) = O(\Delta\tau^2) + O(\Delta x^2),$$

perform a Taylor expansion about $(x_i, \tau_\nu)$. We suggest to abbreviate fourth-order ($\Delta x^4$ and $\Delta t^4$) and higher-order terms by $O(\Delta x^4)$ and $O(\Delta t^4)$, and to summarize them appropriately. Make use of the fact that $y$ solves the differential equation.

**Hints on Exercise 4.4:**

Transform the boundary conditions of the $(S, t)$-world into the $(x, \tau)$-world, and obtain

$$\exp\left(\frac{x}{2}(q_\delta + 1) + \frac{\tau}{4}(q_\delta + 1)^2\right) - \exp\left(\frac{x}{2}(q_\delta - 1) + \frac{\tau}{4}(q_\delta - 1)^2\right)$$

(case of the call).

**Hints on Exercise 4.5:**

The functions $g$ for a put and a call are listed in Section 4.6. Do the derivation only for the put, because the call is analogous.

**Hints on Exercise 4.8:**

(In earlier editions, typo in the text before part c): the intrinsic value of a put is $K - S$, and not $S - K$.) Replace in line 3 "with high contact to" by: contacting. (Because the "high" is shown by the analysis of this exercise.)

**Hints on Exercise 4.9:**

For $S > S_\text{f}$ use the Lemma of Itô. The drift term is abbreviated by $O(dt)$. Apply arbitrage arguments to answer the final question.

**Hints on Exercise 5.1:**

The main work is the construction of the specific spline of assignment a). Instead of using in each subinterval a general ansatz such as $\alpha+\beta x+\gamma x^2+\delta x^3$, it is recommendable to start "on the right" (for $x > 2$) with the trivial polynomial 0 and continue at each node $\bar{x}_4 = 2$, $\bar{x}_3 = 1$, $\bar{x}_2 = 0, \ldots$ by adding a polynomial of the type $a_k(\bar{x}_k - x)^3$. (Why?) For example, the polynomial 0 for $x > 2$ is extended at $\bar{x}_4 = 2$ by the polynomial $(2 - x^3)$ in $\mathcal{C}^2$-smooth way, with $a_4 = 1$. When $\bar{x}_0 = -2$ is reached, the parameters must be determined such that the polynomial 0 for $x < -2$ is matched in a $\mathcal{C}^2$-smooth way.

**Hints on Exercise 5.4:**

The normal distribution is characterized by the density in (2.10). So apply $Y_1 := \log S_1$, $Y_2 := \log S_2$ and continue in the $(Y_1, Y_2)$-plane. In the $(S_1, S_2)$-plane, the $Y$-ellipse is distorted.

**Hints on Exercise 5.5:**

In a), $D$ and $b$ are defined in equation (5.28). In c), $D$ and $b$ are different from (5.28).

**Hints on Exercise 5.7:**

Recall $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^{tr}$. Use Cramer's rule for the inversion.

**Hints on Exercise 5.8:**

Follow the convention to number the vertices of each triangle in the counterclockwise fashion. It does not matter at which vertex you start with the numbering. In b) you better implement your algorithm to make sure that it works correctly. By testing the algorithm you can determine the bandwidth.

**Hints on Exercise 5.10 (5.4 in earlier editions):**

First verify for $\delta := u - u_I$ on an arbitrary subinterval the existence of a $\zeta$ with $\delta'(\zeta) = 0$. Let $x_j$ be the node that is the closest to $\zeta$. Expand $\delta$ about $\zeta$ to obtain $\delta(x_j)$. For the assignment (b) of the lemma the relation

$$\delta'(x) = \int_\zeta^x \delta''(\xi)\, d\xi$$

is helpful.

**Hints on Exercise Exercise 6.7:**

in c) disregard the hint $C \exp(\ldots) = B$ .
Replace the numbers at the end by: $\delta_1 = 0.05$, $\delta_2 = 0.3$. For $M = 2000$ an approximation of the American-style option is 0.1303, and for the European-style 0.1200.

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