A New Risk Index

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Key words risk analysis, bifurcation
MSC (2000) 91B30, 70K50, 37N, 68J10, 37M20

This paper contributes to the novel approach of a deterministic risk analysis. We introduce a risk index, which is designed such that large values of the index indicate a deterministic risk. The approach is illustrated by two examples. The new risk index allows to give the “risk area” a quantitative meaning.

1 Introduction

Risk theory has been entirely based on probabilistic approaches. The framework provided by probability theory and statistics is perfectly suited for the traditional tasks of insurance, which go along with large numbers of claims or premiums, and for which large amounts of data are available. In such standard scenarios the assumption of randomness is natural.

But when a single risk is investigated, the amount of available data often is insufficient. This holds in particular for rare events where often enormous fatalities are expected to occur with almost vanishing probability [1], [5]. For single risks it is important to ask for the causes of risks. Which effect is prompted by what kind of cause? Typically, little is known that might provide relevant insight. As long as nothing happens, it is convenient to hide our ignorance under the assumption of randomness. Only after a failure we begin to understand what might have been the cause of the failure. Then it often turns out that the assumption of randomness is deceptive and the causes of the failure were fully deterministic.

There is a class of risks for which a deterministic analysis was introduced [7]. This class of risks is characterized by the potential of modeling key state variables by deterministic equations. Mostly these equations are differential equations. The class of risks for which model equations exist, is quite large. It includes most of the technological apparatus studied by mechanics and engineering. The risky events we have in mind are jumps in the state, loss of stability, and other structural changes. Obviously, such phenomena are the causes of events that are often regarded as failures. The underlying mechanism is bifurcation. Therefore [7] has suggested bifurcation as integral for a deterministic risk analysis. The mentioned paper has outlined in a qualitative way the direction to pursue, suggesting the “distance” to the next bifurcation as a measure for risk. So far it was left open how such a strategy might work quantitatively, and how the “distance” might relate to a risk measure.

The present paper fills the gap. We introduce a new risk index that is designed to give the “distance” to a deterministic risk (viz bifurcation) a quantitative meaning. Our risk index is scaling invariant and should apply to a wide range of problems. By means of

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this risk index we are able to assign a definition to what a “feasible parameter” range may be.

The outline of the paper is as follows. Section 2 illustrates the basic principle for the simple case of a scalar parameter; a risk measure is introduced. In Section 3 the procedure is illustrated by an example. The two-parameter case is more complicated. A related example is presented in Section 4, and an algorithm is outlined and tested in Section 5.

2 The one-parameter situation

We denote the state vector by $y$, and assume a model equation is known that defines the law of the behavior of $y$. The model can be represented, for example, by a system of ordinary differential equations,

$$\dot{y} = f(t, y, \lambda).$$

In this model equation, $t$ is the independent variable, and $\lambda$ denotes the parameter. We begin with the simple situation of a scalar parameter. The dot in (1) indicates differentiation with respect to $t$. For what follows it is not essential what kind of model equation is underlying. We only assume that all bifurcation parameter values in the parameter interval of practical interest are calculated. Admittedly, this assumption is stringent. If relevant bifurcations are missing then the risk analysis may give false clues in estimating the distance to the next bifurcation. We denote an arbitrary operation point by $\lambda_1$, and let the closest bifurcation value of the underlying model be $\lambda_0$. For texts on bifurcation phenomena we refer, for example, to [4], [6]. For reviews on how to calculate bifurcation we in addition mention [2] or the tutorials [9] in the domain www.bifurcation.de.

Figure 1 illustrates the basic idea. If $d$ denotes the distance to the next bifurcation, $d = |\lambda_1 - \lambda_0|$, then the prototype of our risk index $R(\lambda_1)$ will be $R(\lambda_1) := d^{-1}$. Then a large value of $R$ indicates closeness of a risk. To be of practical relevance, the risk index must be properly scaled. Also uncertainties in the modeling (that is, in $\lambda_0$) and in the measurement of $\lambda_1$ must be compensated. To this end, let $\Delta\lambda_1 > 0$ be the absolute error in $\lambda_1$, and $\Delta\lambda_0 > 0$ be the absolute error in $\lambda_0$ as compared to the (somewhat vague notion of the) bifurcation of the real problem. Typically, $\Delta\lambda_1$ can be neglected, $\Delta\lambda_1 = 0$. More involved is the role of $\Delta\lambda_0$, which is the result of several sources of error:

(i) inadequacy of the law represented by model (1)
(ii) errors in the coefficients of (1)
(iii) computational error in approximating $\lambda_0$

Compared to the modeling errors due to (i) or (ii), the computational error (iii) usually is much smaller and often can be neglected. In many cases, it remains disclosed how the model errors (i) and (ii) affect $\Delta \lambda_0$. Then $\Delta \lambda_0$ can be set to an arbitrary symbolic number, such as $10^{-2}$, or may be neglected again. Note that $\Delta \lambda_0$ compensates for all remaining sources of randomness in the formally deterministic model (1). Finally the data are scaled by $\lambda_1$ or $\lambda_0$. Since the scaling by the test point $\lambda_1$ has advantages in the two-dimensional situation (Section 5), we scale by $\lambda_1$ for the sake of analogy.

Summarizing, the prototype risk index $R(\lambda_1) = d^{-1}$ is replaced by the definition

$$R(\lambda_1) := \frac{\lambda_1}{\max\{\epsilon, |\lambda_1 - \lambda_0| - \Delta \lambda_1 - \Delta \lambda_0\}}.$$  \hspace{1cm} (2)

This function is to be evaluated for suitable test points $\lambda_1$. The number $\epsilon > 0$ in (2) is chosen close to zero, such as $10^{-10}$, to prevent dividing by 0. Large values of $R$ indicate closeness of risk. The definition of the risk index $R$ allows to define a feasible range of parameters,

$$\mathcal{F}_c := \{\lambda \mid R(\lambda) < c\}.$$ \hspace{1cm} (3)

The risk area of level $c$ is the complement of $\mathcal{F}_c$. This is an attempt to quantify deterministic risk. Of course the risk depends on $c$ and on the quality of the modeling, but the applied scaling suggests that there may be a choice of $c$ that is applicable for a wide range of examples.

**Fig. 2** Simulation $u(t)$ of (4), for $0 \leq t \leq 200$, with the parameter $\lambda$ linearly increasing from 0.1 to 0.3.

### 3 Example: CSTR

To illustrate ideas we discuss an example. The time-dependent behavior of a specific continuous stirred tank reactor (the CSTR from [10]) is described by two ordinary differential equations

$$\dot{u} = -u + Da(1 - u) \exp(v)$$
$$\dot{v} = -v + 16.2 \cdot Da(1 - u) \exp(v) - 3v.$$ \hspace{1cm} (4)
Solutions $u(t)$ (material balance) and $v(t)$ (energy balance) are sought in dependence on the time $t$, and on the parameter $\lambda = Da$, which is the Damköhler number. The two components $u(t), v(t)$ set up the vector $y(t)$. We do not go into chemical or mathematical details. The risk and bifurcation scenario is explained by Figures 2 and 3. A simulation of (4) shows for $t \approx 30$ a jump from a stationary state into a periodic regime with large amplitude. The jump must be regarded as risk. The bifurcation diagram of Figure 3 (same scaling of axes as in Figure 2) explains the risk, which is caused by a passing of a Hopf bifurcation point with hard loss of stability. The Figure 4 shows how our risk index works. To this end imagine Figure 4 copied on Figure 2. The related risk index “warns” in time. Reaching a value of, say, $R = 10$ would suggest to freeze the parameter value.

4 Example: catalytic reaction

Scaling becomes a more serious issue in multi-parameter situations. For illustration we choose a two-parameter model that exemplifies the difficulties. Heat and mass transfer within a porous catalyst of a flat particle with first-order reaction is described by the
boundary-value problem \[3\]
\[
\frac{d^2 y}{dx^2} = \vartheta^2 y \exp \left[ \frac{\gamma 0.4(1 - y)}{1 + 0.4(1 - y)} \right], \quad \frac{dy(0)}{dx} = 0, \quad y(1) = 1. \quad (5)
\]

\(y(x)\) is the dimensionless concentration, \(x\) is a dimensionless coordinate, \(\vartheta\) is the Thiele modulus, and \(\gamma\) is the dimensionless energy of activation. This is a two-parameter problem, where we take as a bifurcation parameter the modified Thiele modulus \(\lambda := \vartheta^2\); \(\gamma\) is the second parameter. The \((\lambda, \gamma)\)-parameter chart is shown in Figure 5. The dynamical behavior is dominated by a cusp, which is bounded by two bifurcation curves of turning-point type. (The curves have been calculated by the author’s branching system \[6\].) Note that in the range of interest the two parameters differ by a factor of 100. Hence a scaling is badly needed to be able to define a distance in the parameter plane.

![Fig. 5 Parameter chart of (5).](image)

![Fig. 6 Distance defined via discretization.](image)

5 Algorithm for the two-parameter case

The algorithm for the two-dimensional situation is sketched below. The procedure is illustrated by Figures 6 and 7. Scaling is done by the test point \((\lambda_1, \gamma_1)\) because \((\lambda_0, \gamma_0)\) is no point but in general forms a curve \((\lambda_0(s), \gamma_0(s))\) for some curve parameter \(s\).
Algorithm (2D)

(a) scale the data by \((\lambda_1, \gamma_1)\):

\[
(\lambda_1 \pm \Delta \lambda, \gamma_1 \pm \Delta \gamma) \rightarrow \left(1 \pm \frac{\Delta \lambda}{\lambda_1}, 1 \pm \frac{\Delta \gamma}{\gamma_1}\right)
\]

\[
(\lambda_0(s), \gamma_0(s)) \rightarrow \left(\frac{\lambda_0(s)}{\lambda_1}, \frac{\gamma_0(s)}{\gamma_1}\right)
\]

(b) choose the closest straight-line segment on the discretized bifurcation curve such that \((\lambda_1, \gamma_1)\) is “perpendicular” (see Figure 6).

If not possible, take the Euclidian distance.

(c) calculate the distance \(d\)

(d) if necessary, correct \(d\) by relevant error margins (see Figure 7)

(e) risk index \(R(\lambda_1, \gamma_1) := d^{-1}\)

Fig. 7 On the definition of the risk index \(R = d^{-1}\).

Fig. 8 \((\lambda, \gamma)\)-parameter plane of (5), with a bifurcation curve and the values of \(R(\lambda, \gamma)\) for selected test points \((\lambda_1, \gamma_1)\).

The distance in step (c) is calculated by applying elementary calculus. We apply the algorithm to the example (5). The scaling invariance is seen in Figures 8 and 9. The level curves in Figure 9 are calculated by gnuplot; they are not smooth.
6 Outlook

In this paper, we have introduced a risk index based on a distance in the parameter space. The distance was taken to a threshold defined by bifurcation — that is, attractors were studied. Analogously, one may investigate a distance in state space, namely the distance to the closest boundary of the domain of attraction. In this way, one may attempt to characterize a risk of transient phases. In principle, a related approach would work analogously. But since trajectories can cross the thresholds formed by the basin boundaries, such a procedure appears less recommendable. In addition, the size of the domains of attraction varies depending on the distance to bifurcations. So, our approach may be general enough.

This paper may also be understood as a call for applications. We wonder how the risk index of this paper applies to other examples. Apart from the two examples of this paper, the approach was successfully tested on an electrical power generator [8]. We offer the interested reader our FORTRAN program for the 2D-situation, and are interested to see how the concept of a deterministic risk may apply to other fields of modeling, in particular in mechanics.

Fig. 9 Parameter plane as in Figure 8, with level curves $R(\lambda, \gamma) = c$ with $c = 5$ (outer curve), $c = 10$, $c = 20$ (inner curve).

References


