Radial eigenfunctions for the game-theoretic p-Laplacian on a ball

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Abstract

The normalized or game-theoretic p-Laplacian operator given by $-\Delta_p^N u := -\frac{1}{p}|\nabla u|^{2-p}\Delta_p(u)$ for $p \in (1,\infty)$ with $\Delta_p u = \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})$ has no apparent variational structure. Showing the existence of a first (positive) eigenvalue of this fully nonlinear operator requires heavy machinery as in [6]. Restricted to the class of radial functions, however, the normalized p-Laplacian transforms into a linear Sturm-Liouville operator. We investigate radial eigenfunctions to this operator under homogeneous Dirichlet boundary conditions and come up with an explicit complete orthonormal system of Bessel functions in a suitably weighted L^2-space. This allows us to give a Fourier-series representation for radial solutions to the corresponding evolution equation $u_t - \Delta_p^N u = 0$.

1 Introduction

In [9] (and [20], [4], [16]) radial solutions to the eigenvalue problem

$$\Delta_p u + \lambda |u|^{p-2}u = 0 \text{ in } B_R(0), \quad u = 0 \text{ on } \partial B_R(0), \tag{1.1}$$

were investigated, and it was shown that there exists an increasing sequence λ_k of eigenvalues, each of them simple, and that the k-th radial eigenfunction $v_k(r) = u_k(|x|)$ has exactly $(k-1)$ simple zeroes in $(0,R)$. There are also nonradial eigenfunctions, a fact that is well known for the linear case $p = 2$, but much harder to prove for the nonlinear case, see [3]. Notice that (1.1) is quasilinear and homogeneous of degree $(p-1)$, since for any $t \in \mathbb{R}^+$ we have $\Delta_p tu = t^{p-1}\Delta_p u$.

In [7] similar results were shown for eigenvalues and radial eigenfunctions of the Pucci operator, which associates to a symmetric matrix M and two positive real numbers $a < A$ the operator $\mathcal{M}_{a,A}(M) = A\, trace(M^+) - a\, trace(M^-)$. This time the eigenvalue problem

$$\mathcal{M}_{a,A}(D^2u) + \lambda u = 0 \text{ in } B_R(0), \quad u = 0 \text{ on } \partial B_R(0), \tag{1.2}$$

is fully nonlinear and homogeneous of degree 1.

More recently there have been investigations of closely related problems with different degrees of homogeneity. To be precise, for any $\alpha > -1$, the operators $|Du|^{\alpha+2-p}\Delta_p u$ and $|Du|^\alpha \mathcal{M}_{a,A}(D^2u)$ were studied in [6, 10] and again the existence of countably many (radial) eigenvalues tending to ∞ was shown. For $\alpha = 0$ these equations are homogeneous of degree 1. In [17] the authors study an evolution equation in the limit case of $\frac{1}{p}|Du|^{\alpha+2-p}\Delta_p u$ as $p \to \infty$ for $\alpha = h - 1 > 0$, while [11, 2] look at the evolution equation for finite p and $\alpha = 0$. It is the purpose of this note to point out that in the case $\alpha = 0$ we can explicitly calculate the radial eigenvalues and eigenfunctions of the 1-homogeneous version of (1.1), i.e. of (2.2) given below. As a byproduct we obtain an explicit Fourier-series representation for radial solutions to a corresponding initial boundary value problem for $u_t - \Delta_p^N u = 0$.

One should also expect many nonradial eigenfunctions to exist, but for $p \neq 2$ we are not aware of any results in this direction, not even in two dimensions. The normalized p-Laplacian Δ_p^N is not of divergence type. That rules out variational characterizations of eigenfunctions as in the case of the usual p-Laplace operator Δ_p. On the other hand Δ_p^N is in general strongly nonlinear, but then spectral theory for linear operators which are not in divergence form cannot be applied either.

2 Result

As shown in [14], in intrinsic coordinates the normalized p-Laplacian is a convex combination of the normalized ∞-Laplacian and the normalized 1-Laplacian. If $\nu = -\frac{\nabla u}{|\nabla u|}$ denotes the unit vector pointing out of a level set $\Omega_c := \{x \in \Omega \ ; \ u(x) \geq c\}$, then

$$\Delta_p^N u = \frac{p-1}{p}\Delta_\infty^N u + \frac{1}{p}\Delta_1^N u = \frac{p-1}{p}u_{\nu\nu} + \frac{1}{p}(n-1)Hu_\nu, \tag{2.1}$$

with H denoting the mean curvature of $\partial\Omega_c$. Parabolic versions, i.e. equations of type $u_t - \Delta_p^N u = 0$ have recently been investigated in [11, 2], and for $p = \infty$ in [13, 1]. The asymptotic decay of solutions depends on the first eigenvalue.

For radial functions the eigenvalue problem

$$\Delta_p^N u + \lambda u = 0 \text{ in } B_R(0), \quad u = 0 \text{ on } \partial B_R(0), \tag{2.2}$$

using the decomposition (2.1) and the Ansatz $u(x) = v(|x|)$, our eigenvalue problem (2.2) transforms into

$$v''(r) + \frac{n-1}{p-1}\frac{1}{r}v'(r) + \frac{p}{p-1}\lambda v(r) = 0 \text{ in } (0, R) \tag{2.3}$$

with boundary conditions

$$v'(0) = 0 = v(R). \tag{2.4}$$

In this context, weighted spaces appear naturally, and we will use the following notation for them throughout the article: Given a domain $\Omega \subset \mathbb{R}^n$ and a weight function $w : \Omega \to [0, \infty)$ we write $L^2_{w(x)}(\Omega)$ for the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|^2_{L^2_{w(x)}} := \int_\Omega w(x)^2 u(x)^2 \, dx < \infty.$$

If w is positive a.e. in Ω, this defines a norm, and $L^2_{w(x)}(\Omega)$, equipped with the associated scalar product, is a Hilbert space.

Remark 2.1. *For $p = \infty$, (2.3),(2.4) is the equation for a vibrating string, and as noted in* [13]*, Section 4, the first eigenvalue is $\lambda_1 = (\frac{\pi}{2R})^2$ and the first eigenfunction is just $\cos(\sqrt{\lambda}r)$. This and higher eigenfunctions for $p = \infty$ are obviously given by*

$$v_k(r) = \sqrt{\frac{2}{R}}\cos\left(\sqrt{\lambda_k}r\right) \ \textit{with} \ \lambda_k = \left(\frac{(2k-1)\pi}{2R}\right)^2 \ \textit{for} \ k \in \mathbb{N}.$$

They form a complete orthonormal system normalized in $L^2(0,R)$, that spans $\{v \in W^{1,2}(0,R)\,;\, v(R)=0\}$.

Correspondingly, the sequence of functions $u_k(x) = v_k(|x|)$, $k \in \mathbb{N}$, forms an orthonormal basis in the subspace W of the weighted space $L^2_{|x|^{\frac{1-n}{2}}}(B_R(0))$, where W consists of all radial functions $u : B_R(0) \to \mathbb{R}$ satisfying $u(R) = 0$. We should also point out that the case $n = 1$ is well understood, and that for $p = 1$ the Dirichlet-eigenvalue problem (2.2) has no nontrivial radial solution.

From now on we study the case $n \geq 2$, $p \in (1,\infty)$ and note that for $p = n$ the differential equation (2.3) is known as Bessel's equation. The corresponding Bessel functions, evaluated at $|x|$, are then radial eigenfunctions of (2.2) on a ball in $\mathbb{R}^n$. If we choose

$$m = \frac{p+n-2}{p-1}, \tag{2.5}$$

then (2.3) reads

$$v'' + \tfrac{m-1}{r}v' + \tilde{\lambda} v = 0$$

(with $\tilde{\lambda} = p\lambda$) and can be interpreted as the usual Bessel equation for balls in $\mathbb{R}^m$. Notice that $m > 1$ is in general not a natural number. Nevertheless we can state and prove the following result.

Theorem 2.2. *Suppose that $\alpha = \frac{p-n}{2(p-1)}$, $\beta = \frac{n-1}{p-1}$, $\gamma = \beta - (n-1)$ and $J_{-\alpha}$ is the Bessel function, cf.* (4.9). *Then for $R = 1$ the eigenfunctions of* (2.3), (2.4) *are given by*

$$v_k(r) = c_k r^\alpha J_{-\alpha}\left(\mu_k^{(-\alpha)} r\right) \quad \textit{with } c_k = \left(\int_0^1 r^\beta \left(r^\alpha J_{-\alpha}\left(\mu_k^{(-\alpha)} r\right)\right)^2 dr\right)^{-1/2},$$

and they form a complete orthonormal system in the weighted space $L^2_{r^{\beta/2}}((0,1))$. Here $\mu_k^{-\alpha}$ are the positive zeroes of the Besselfunction $J_{-\alpha}$.

Correspondingly, the radially symmetric eigenfunctions of (2.2) are given by $u_k(x) = v_k(|x|/R)$ and the associated eigenvalues are

$$\lambda_k = \frac{p-1}{p}\left(\frac{\mu_k^{(-\alpha)}}{R}\right)^2.$$

These eigenfunctions form a complete orthonormal system in the subspace $L^2_{rad,|x|^{\gamma/2}}(B_R(0))$ of radial functions in $L^2_{|x|^{\gamma/2}}(B_R(0))$.

The proof is the content of the final two sections.

Remark 2.3. *The weight exponent $\gamma = (2-p)(n-1)/(p-1)$ is positive for $p \in (1,2)$, negative for $p \in (2,\infty)$ and vanishes only for $p = 2$.*

Remark 2.4. *The radial eigenfunctions of* (2.2) *also form a complete orthogonal system in the weighted Sobolev space*

$$W^{1,2}_{0,\mathrm{rad},|x|^{\gamma/2}}(B_R(0)) := \left\{ u \in W^{1,2}_{|x|^{\gamma/2}}(B_R(0)) \;\middle|\; \begin{array}{r} u \textit{ is radially symmetric} \\ \textit{and } u|_{\partial B_R(0)} = 0 \end{array} \right\}.$$

To see this, observe that for all sufficiently smooth, compactly supported, radially symmetric functions in $W^{1,2}_{|x|^{\gamma/2}}$ (a dense subspace containing our eigenfunctions), an integration by parts in the scalar product of $W^{1,2}_{|x|^{\gamma/2}}$ gives

$$\langle u_1, u_2\rangle_{W^{1,2}_{|x|^{\gamma/2}}} = \int_{B_R(0)} (-\Delta_p^N u_1 + u_1)u_2\ |x|^\gamma\ dx.$$

($W^{1,2}_{0,\mathrm{rad},|x|^{\gamma/2}}$ and its scalar product correspond to the space D^0 and the scalar product $\langle\cdot,\cdot\rangle_+$ in Section 3 below.) As a consequence, the eigenfunctions of Δ_p^N, which are pairwise orthogonal in $L^2_{|x|^{\gamma/2}}$, inherit this property with respect to the scalar product in $W^{1,2}_{|x|^{\gamma/2}}$, and the constant zero is the only function in $W^{1,2}_{0,\mathrm{rad},|x|^{\gamma/2}}$ orthogonal to all of them, because any such function also has to be orthogonal to all eigenfunctions with respect to the scalar product in $L^2_{|x|^{\gamma/2}}$.

Our result has an immediate application to the corresponding initial-boundary value problem IBVP

$$\begin{aligned} u_t(x,t) - \Delta_p^N u(x,t) &= 0 && \text{in } B_R(0)\times(0,\infty),\\ u(x,0) &= u_0(x) && \text{in } B_R(0),\\ u(x,t) &= 0 && \text{on } \partial B_R(0)\times(0,\infty). \end{aligned}$$

Corollary 2.5. *With the notation of Theorem 2.2 and Remark 2.4 suppose that $u_0 \in L^2_{rad,|x|^{\gamma/2}}(B_R(0))$. Then the solution of IBVP is given by*

$$u(x,t) = \sum_{k=1}^{\infty} d_k\, e^{-\lambda_k t}\, u_k(x) \quad \textit{where} \quad d_k = \langle u_0, u_k\rangle_{L^2_{|x|^{\gamma/2}}(B_R(0))}. \tag{2.6}$$

3 Radial eigenfunctions as a complete orthogonal system

Following a standard approach in functional analysis, we will show that solutions of (2.3) can be interpreted as eigenfunctions of an unbounded self-adjoint linear operator in $H := L^2_{r^{\beta/2}}((0,1))$ with compact inverse. By and large, our arguments are certainly known in the context of Sturm-Liouville eigenvalue problems, but we are not aware of any references fully covering the singular problem at hand, and for this reason, proofs are given below.

We recall that the scalar product in H is given by

$$\langle v_1, v_2\rangle_H := \int_0^1 r^\beta v_1(r)v_2(r)\,dr, \quad \text{with } \beta = \frac{n-1}{p-1}.$$

Multiplying (2.3) by (-1) and adding v on either side (which ensures invertibility of the self-adjoint extension later on), we are led to study the operator $B : D(B) \to H$ defined by

$$B(v)(r) := -v''(r) - \frac{n-1}{p-1}\frac{v'(r)}{r} + v(r) = -v''(r) - \beta\frac{v'(r)}{r} + v(r),$$

for

$$v \in D(B) := \{v \in C^\infty([0,1]) \mid v'(0) = v(1) = 0\} \subset H.$$

An integration by parts shows that B is symmetric, and

$$\langle Bv, v\rangle_H = \int_0^1 [(v')^2 + v^2] r^\beta\,dr \geq \|v\|_H^2 \quad \text{for every } v \in D(B),$$

whence B is positive definite and, in particular, semi-bounded. Since $D(B)$ is dense in H, this implies that B has a self-adjoint extension B_F (Friedrichs' extension,

for instance see [12], Section XII.5, or [19], Section 17). Its domain of definition is given by
$$D(B_F) = \{v \in H \mid v \in D(B^*) \cap D^0\},$$
where B^* denotes the adjoint of B, and $D^0 \subset H$ is the closure of $D(B)$ with respect to the norm $\|\cdot\|_+$ generated by the scalar product
$$\langle v, w\rangle_+ := \langle Bv, w\rangle_H .$$
Moreover, B_F is semibounded with the same bound as B, i.e.,
$$\langle B_F v, v\rangle_H \geq \|v\|_H^2 \quad \text{for every } v \in D(B_F).$$
Note that in our case, D^0 can be explicitly characterized as $D^0 = W^{1,2}_{r^{\beta/2}}((0,1)) \cap \{v(1) = 0\}$, the space of weakly differentiable functions v in H with weak derivative v' in H and zero trace on the right boundary point $r = 1$.

The compactness of the inverse of B_F rests on the following compact embedding. For the reader's convenience, we provide a short proof based on well known standard embeddings without weights.

Lemma 3.1. *$W^{1,2}_{r^{\beta/2}}((0,1))$ is compactly embedded in $H = L^2_{r^{\beta/2}}((0,1))$.*

Proof. Since $W^{1,2}$ is compactly embedded in L^2 (on any bounded interval, without the weights), the only possible problem is the behavior of functions near $r = 0$, which can be controlled as follows: For every $0 < \varepsilon < 1$, using that $\beta \geq 0$,
$$\begin{aligned}
\int_0^\varepsilon u^2(r) r^\beta \, dr &\leq \frac{\varepsilon^{\beta+1}}{\beta+1} u^2(1) + \int_0^\varepsilon \int_r^1 2\,|u(s)u'(s)| \, ds\, r^\beta \, dr \\
&\leq \frac{\varepsilon^{\beta+1}}{\beta+1} u^2(1) + \varepsilon \int_0^1 [u^2(s) + u'^2(s)] s^\beta \, ds \\
&\leq (C+1) \max\left\{\varepsilon, \frac{\varepsilon^{\beta+1}}{\beta+1}\right\} \|u\|^2_{W^{1,2}_{r^{\beta/2}}((0,1))} .
\end{aligned}$$
Here, $C > 0$ is a constant coming from the continous embedding of $W^{1,2}$ into C^0 on $(\frac{1}{2}, 1)$ which we employed to estimate $|u(1)|$. Consequently, the norm of u in $L^2_{r^{\beta/2}}$ on $(0, \varepsilon)$ becomes small for small ε, uniformly for all u in a bounded subset of $W^{1,2}_{r^{\beta/2}}((0,1))$. Together with the compact embedding of $W^{1,2}_{r^{\beta/2}}((\varepsilon,1))$ into $L^2_{r^{\beta/2}}((\varepsilon,1))$ for fixed ε, this implies the assertion. $\square$

Proposition 3.2. *$B_F : D(B_F) \to H$ is invertible, and as a linear operator from H into H, the inverse B_F^{-1} is bounded and compact.*

Proof. Being positive definite, B_F is obviously one-to-one. Let R denote the range of B_F. Recall that D^0 is a closed subspace of $W^{1,2}_{r^{\beta/2}}((0,1))$ and that the scalar product on D^0, given by $\langle \cdot, \cdot\rangle_+ = \langle B_F \cdot, \cdot\rangle_H$, coincides with that of $W^{1,2}_{r^{\beta/2}}((0,1))$ restricted to D^0. Consequently, for every $z \in R$, we have that
$$\left\|B_F^{-1} z\right\|_+^2 = \left\langle B_F B_F^{-1} z, B_F^{-1} z\right\rangle_H \leq \|z\|_H \left\|B_F^{-1} z\right\|_H \leq \|z\|_H \left\|B_F^{-1} z\right\|_+ ,$$
whence $B_F^{-1} : R \to D^0$ is bounded, with respect to the norm of $H = L^2_{r^{\beta/2}}((0,1))$ in R and the norm of $W^{1,2}_{r^{\beta/2}}((0,1))$ in D^0. Since D^0 is compactly embedded in H, we see that $B_F^{-1} : R \to H$ is compact.

We proceed to show that R is closed in H. Suppose that $B_F w_k \to z$ in H with some $z \in H$ and a sequence $(w_k) \subset D(B_F)$. As a convergent sequence, $(B_F w_k)$ is bounded in H, whence $(w_k) = (B_F^{-1} B_F w_k)$ is relatively compact in H. Selecting a convergent subsequence of (w_k) (not relabeled), we get that $w_k \to w$ in H for some $w \in H$. Due to the fact that B_f is a self-adjoint and hence closed operator, we infer that $w \in D(B_F)$ and $B_F w = z$. In particular, $z \in R$.

It remains to show that $R = H$. Since B_F is positive definite, the kernel of $B_F^* = B_F$ is $\{0\}$. Since R is closed, the orthogonal complement of R in H is $\{0\}$ by the closed range theorem, and as a consequence, $R = H$. □

In view of the spectral theorem for compact self-adjoint operators (Theorem VI.16 in [18], e.g.), applied to B_F^{-1} and combined with the fact that each eigenfunction of B_F^{-1} is also an eigenfunction of B_F associated to the reciprocal eigenvalue, we conclude:

Theorem 3.3. *There exist countably many eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ of B_F with $\lambda_k \to \infty$ as $k \to \infty$, and the associated eigenfunctions $u_k \in D(B_F)$ form a complete orthogonal system in $H = L^2_{r^{\beta/2}}((0,1))$.*

Finally, we observe that each u_k is a classical solution of (2.3), which allows us to explicitly calculate the eigenfunctions and eigenvalues in the last section.

Proposition 3.4. *For each of the eigenfunctions u_k obtained in Theorem 3.3, we have that $u_k \in C^\infty((0,1])$,*

$$\lim_{r \to 0^+} u_k'(r) r^\beta = 0, \qquad u_k(1) = 0, \tag{3.7}$$

and u_k solves (2.3) in the classical sense, for $\lambda = \frac{p-1}{p}(1 - \lambda_k)$.

Proof. Clearly, $\langle B_F u_k - \lambda_k u_k, v \rangle_H = \langle u_k, Bv - \lambda_k v \rangle_H = 0$ for every $v \in D(B)$. Using that $u_k \in D^0 = W^{1,2}_{r^{\beta/2}}((0,1)) \cap \{u(1) = 0\} \subset D(B_F)$, an integration by parts yields that

$$\int_0^1 [u_k' v' + (1 - \lambda_k) u_k v] r^\beta \, dr = 0 \tag{3.8}$$

for every $v \in D(B)$ and thus, by density, even every $v \in D^0$. In particular, u_k is a weak solution of (2.3) with the appropriate choice of λ, and by standard elliptic theory, $u_k \in C^\infty([\varepsilon, 1])$ for every $\varepsilon > 0$. As a consequence, $u_k \in C^\infty((0,1])$ solves (2.3) in the classical sense. Finally, (3.7) is the natural boundary condition for u_k at $r = 0$ implied by (3.8) (using that for arbitrary $v \in D^0$, the value of $v(0)$ is unrestricted). □

4 Classical radial eigenfunctions

In this section we will finally derive the classical solutions of (2.3), (2.4). Below J_ν denotes the Bessel function of order ν,

$$J_\nu(t) = \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{t}{2}\right)^{\nu+2j}}{j! \Gamma(\nu + j + 1)} \tag{4.9}$$

(see §84 in [8]), and as defined in Theorem 2.2 we have $\alpha = \frac{p-n}{2(p-1)}$ and $\beta = \frac{n-1}{p-1}$. Notice that $2\alpha - 1 = -\beta$. For any $\delta > 0$ such that $J_\nu \neq 0$ on $(0, \delta)$, we have the representation

$$u(r) = c_1 J_\nu(r) + c_2 J_\nu(r) \int_r^\delta \tau^{-1} J_\nu^{-2}(\tau) d\tau$$

of the general solution of Bessel's equation

$$r^2u''(r) + ru'(r) + (r^2 - \nu^2)u(r) = 0$$

(see §102 in [8]). Following the derivation of transformed Bessel's equations in §104 in [8], we obtain

$$c_1 r^\theta J_\nu(\eta r) + c_2 r^\theta J_\nu(\eta r) \int_{\eta r}^{\varepsilon} \tau^{-1} J_\nu^{-2}(\tau) d\tau,$$

where $\varepsilon > 0$ can be any sufficiently small number, as general solution of the transformed Bessel equation

$$u''(r) - \frac{2\theta - 1}{r} u'(r) + \left(\frac{\theta^2 - \nu^2}{r^2} + \eta^2 \right) u(r) = 0. \tag{4.10}$$

Comparing the coefficients of equation (4.10) to the coefficients of equation (2.3), we infer that general solution of (2.3) is given by

$$v(r) = c_1 r^\alpha J_{-\alpha}(\eta r) + c_2 r^\alpha J_{-\alpha}(\eta r) \int_{\eta r}^{\varepsilon} \tau^{-1} J_{-\alpha}^{-2}(\tau) d\tau =: c_1 z_1(r) + c_2 z_2(r). \tag{4.11}$$

Here, $\eta = \sqrt{\lambda \frac{p}{p-1}}$. Note that ν and η appear only quadratically in (4.10). We can get another representation of the general solution if we replace $J_{-\alpha}$ by J_α in (4.11) or η by $-\eta$. But, since choosing $-\eta$ only yields a reflection in 0 and the term $r^\alpha J_{-\alpha}$ is easier to handle with respect to the following calculations, we stick to (4.11).

To single out the solutions satisfying the boundary conditions, in particular (3.7), we now discuss the asymptotic behavior of the terms $z_1(r)$ and $z_2(r)$ in (4.11) near $r = 0$. As this is not affected by the value of $\eta > 0$, we choose $\eta = 1$ for simplicity.

We start with $z_2(r)$. In the following, for each i, $\mathfrak{B}_i(r)$ denotes a power series with positive radius of convergence and $\mathfrak{B}_i(0) \neq 0$. Using this notation, the explicit representation of Bessel function (4.9) gives $r^\alpha J_{-\alpha}(r) = \mathfrak{B}_1(r^2)$. Note that $J_{-\alpha}^2$ has no roots close to the origin. That is why for all small r,

$$\begin{aligned}
\int_r^\varepsilon \tau^{-1} J_{-\alpha}^{-2}(\tau) d\tau &= \int_r^\varepsilon \tau^{-1+2\alpha} \mathfrak{B}_2(\tau^2) d\tau \\
&= \int_r^\varepsilon b_0 \tau^{-1+2\alpha} + b_1 \tau^{1+2\alpha} + b_2 \tau^{3+2\alpha} + \dots d\tau \\
&= c_\alpha + r^{2\alpha} \mathfrak{B}_3(r^2) + b_{-\alpha} \log(r)
\end{aligned}$$

with a constant $c_\alpha \in \mathbb{R}$, and $b_{-\alpha} := 0$ if $-\alpha$ is not a non-negative integer. Hence

$$\begin{aligned}
z_2(r) &= r^\alpha J_{-\alpha}(r) \int_r^\varepsilon \tau^{-1} J_{-\alpha}^{-2}(\tau) d\tau \\
&= \mathfrak{B}_1(r^2) \left(c_\alpha + r^{2\alpha} \mathfrak{B}_3(r^2) + b_{-\alpha} \log(r) \right).
\end{aligned}$$

The derivative is

$$\begin{aligned}
z_2'(r) =& 2r \mathfrak{B}_1'(r^2) \left(c_\alpha + r^{2\alpha} \mathfrak{B}_2(r^2) + b_{-\alpha} \log(r) \right) \\
&+ \mathfrak{B}_1(r^2) \left(2\alpha r^{2\alpha-1} \mathfrak{B}_3(r^2) + 2r^{2\alpha+1} \mathfrak{B}_3'(r^2) + b_{-\alpha} r^{-1} \right)
\end{aligned}$$

If we take into account that $b_{-\alpha} = 0$ for $\alpha \notin \mathbb{Z}$, $b_0 \neq 0$, $2\alpha - 1 < 0$ and $\mathfrak{B}_i(0) \neq 0$, the leading term in $z_2'(r)$ as $r \to 0^+$ is either

$$\mathfrak{B}_1(0) b_{-\alpha} r^{-1} \quad (\text{if } \alpha = 0) \quad \text{or} \quad \mathfrak{B}_3(0) 2\alpha r^{2\alpha-1} \mathfrak{B}_2(0) \quad (\text{if } \alpha \neq 0).$$

In both cases,

$$\liminf_{r\to 0^+} \left| r^\beta z_2'(r) \right| = \liminf_{r\to 0^+} \left| r^{1-2\alpha} z_2'(r) \right| > 0,$$

i.e, z_2' violates the natural boundary condition (3.7) and a fortiori (2.4) at the origin.

Analyzing the first term $z_1(r) = r^\alpha J_{-\alpha}(\eta r)$ of the general solution (4.11), the explicit representation of the Bessel function yields

$$z_1(r) = \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{\eta}{2}\right)^{-\alpha+2j} r^{2j}}{j!\Gamma(-\alpha + j + 1)}$$

and the derivative is

$$z_1'(r) = \sum_{j=1}^{\infty} (-1)^j \frac{\left(\frac{\eta}{2}\right)^{-\alpha+2j} 2j}{j!\Gamma(-\alpha + j + 1)} r^{2j-1}.$$

This shows that z_1 satisfies the boundary condition $z_1'(0) = 0$ from (2.4) and a fortiori (3.7). Consequently, for classical solutions of (2.3), (3.7) is equivalent to $v'(0) = 0$, and subject to this condition, the general solution of (2.3) is given by

$$v(r) = c r^\alpha J_{-\alpha}(\eta r), \qquad c \in \mathbb{R}.$$

The other boundary condition $v(R) = 0$ holds true if and only if $\eta = \frac{\mu_k^{(-\alpha)}}{R}$ for some k, where $\{\mu_k^{(-\alpha)}\}_{k\in\mathbb{N}}$ are the positive zeros of $J_{-\alpha}$ as before. Recalling that $\eta = \sqrt{\lambda \frac{p}{p-1}}$, we get

$$v_k(r) = c_k r^\alpha J_{-\alpha}\left(\frac{\mu_k^{(-\alpha)}}{R} r \right) \tag{4.12}$$

as the classical solution of (2.3), for

$$\lambda = \lambda_k := \left(\frac{\mu_k^{(-\alpha)}}{R} \right)^2 \frac{p-1}{p}.$$

Since all results of Section 3 also hold in the interval $(0, R)$, we can deduce from Theorem 3.3 and Proposition 3.4 that the solutions in (4.12) form a complete orthonormal system in $H = L^2_{r^{\beta/2}}((0, R))$, which concludes the proof of Theorem 2.2.

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