Modular-invariance in rational conformal field theory: past, present and future: Lecture 1

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Overview

Lecture 1.

- vertex algebras - definition
- locality, quantum fields
- existence theorem
- vertex operator algebras - definition
- partition functions
- example: Virasoro VOAs
- example: Heisenberg VOA
- example: Moonshine module VOA
- example: lattice VOAs

Lecture 2.

- representations of VOAs
- rational VOAs
- examples of rational VOAs
- the modular-invariance conjecture
- C_2-cofinite and regular VOAs
- the associated vector-valued modular form
- holomorphic VOAs
- approach using MTCs
- approach using vector-valued modular forms
- unbounded denominators and ASD conjecture

Vertex algebras

Definition A *vertex algebra* V is a $\mathbb{C}$-linear space equipped with a countable infinity of $\mathbb{C}$-bilinear products

$$V \otimes V \to V, \ (u, v) \mapsto u(n)v \quad (n \in \mathbb{Z})$$

and a distinguished *vacuum element* $\mathbf{1}$ satisfying some axioms as follows:

$\forall\ u, v, w \in V,\ \forall r, s, t \in \mathbb{Z}$:

- $u(n)v = 0$ for $n \geq n_0(u, v)$
- $u(-1)\mathbf{1} = u, u(n)\mathbf{1} = 0$ for $n \geq 0$
- $\displaystyle\sum_{i \geq 0} \binom{r}{i} (u(t+i)v)(r+s-i)w =$

$$\sum_{i \geq 0} (-1)^i \binom{t}{i} \{u(r+t-i)v(s+i)w - (-1)^t v(s+t-i)u(r+i)w\}$$

The first identity ensures that these sums are *finite*.

These products are generally neither commutative or associative.

Commutative rings are VAs

Let A be a commutative, associative $\mathbb{C}$-algebra with identity 1. For $a, b \in A$, let

$$a(-1)b := ab, \qquad a(n)b := 0 \quad (n \neq -1).$$

Then A is a vertex algebra with vacuum element 1.

Vertex algebras are the objects of a category **Valg** in which a morphism $U \to V$ is a $\mathbb{C}$-linear map preserving vacuum vectors and all products. The last example gives us an inclusion of categories

$$\mathbf{Alg} \hookrightarrow \mathbf{Valg}$$

Valg is in some ways a natural extension of the category **Alg** of $\mathbb{C}$-algebras.

Locality and quantum fields

We will recouch the basic identity (the Jacobi identity, or JI) in terms of *vertex operators*, or *quantum fields*.

For fixed $u \in V$ and $n \in \mathbb{Z}$ we consider

$$u(n) : V \to V, \quad v \mapsto u(n)v \qquad (v \in V)$$

as a $\mathbb{C}$*-linear operator acting on the left of* V.

The *vertex operator* defined by u is the *formal generating function*

$$Y(u, z) := \sum_{n \in \mathbb{Z}} u(n) z^{-n-1} \in End(V)[[z, z^{-1}]]$$

We write

$$Y(u, z)v := \sum_{n \in \mathbb{Z}} u(n) v z^{-n-1} \in V[[z]][z, z^{-1}]$$

State-field correspondence

The space of *fields* on V is

$$\mathfrak{F}(V) := \left\{ \sum_n a_n z^{-n-1} \in End(V)[[z, z^{-1}]] \mid a_n(v) = 0 \ \forall n \geq n_0(v) \right\}$$

Y becomes the *state-field correspondence*

$$\begin{aligned} Y : V &\longrightarrow \mathfrak{F}(V) \\ u &\longmapsto Y(u, z) \end{aligned}$$

Creative fields

Y is an injection because

$$\begin{aligned} Y(u,z)\mathbf{1} &= u(-1)\mathbf{1} + u(-2)\mathbf{1}z + \ldots \\ &= u + (\text{higher powers of } z) \end{aligned}$$

We say that $Y(u,z)$ is *creative* and *creates the state u from the vacuum*.

Locality

Let $t \geq 0$ be large enough so that $u(t+i)v = 0$ for all $i \geq 0$. JI says

$$\sum_{i \geq 0} (-1)^i \binom{t}{i} \{u(r+t-i)v(s+i)w - (-1)^t v(s+t-i)u(r+i)w\} = 0$$

Now notice that

$$\begin{aligned}
& (z_1 - z_2)^t Y(u, z_1) Y(v, z_2) \\
= \ & \sum_{i \geq 0} (-1)^i \binom{t}{i} z_1^{t-i} z_2^i \sum_{m,n} u(m)v(n) z_1^{-m-1} z_2^{-n-1} \\
= \ & \sum_{r,s} \left\{ \sum_{i \geq 0} (-1)^i \binom{t}{i} u(r+t-i)v(s+i) \right\} z_1^{-r-1} z_2^{-s-1}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& (z_1 - z_2)^t Y(v, z_2) Y(u, z_1) \\
= \; & (-1)^t \sum_{r,s} \left\{ \sum_{i \geq 0} (-1)^i \binom{t}{i} v(s+t-i) u(r+i) \right\} z_2^{-s-1} z_1^{-r-1}
\end{aligned}$$

We obtain

$$\boxed{(z_1 - z_2)^t \, [Y(u, z_1), Y(v, z_2)] = 0}$$

This remarkable identity is called *locality*.

We write this property as

$$Y(u,z) \sim Y(v,z) \quad \text{or} \quad Y(u,z) \sim_t Y(v,z)$$

and say that $Y(u,z)$, $Y(v,z)$ are *mutually local* of order t.

Examples.

$Y(u,z) \sim_0 Y(v,z) \Leftrightarrow [u(r), v(s)] = 0$

$Y(u,z) \sim_1 Y(v,z) \Leftrightarrow [u(r+1), v(s)] - [u(r), v(s+1)] = 0$

$Y(u,z) \sim_2 Y(v,z) \Leftrightarrow$
$[u(r+2), v(s)] - 2[u(r+1), v(s+1)] + [u(r), v(s+2)] = 0$

Translation-covariance

One more easy consequence of JI. Define a special endomorphism

$$D : V \to V, \qquad u \mapsto u(-2)\mathbf{1}$$

Then

$$\begin{aligned} [D, Y(u,z)] &= \partial_z Y(u,z) \\ \text{i.e.,} \quad [D, u(n)] &= -nu(n-1) \end{aligned}$$

This is called *translation-covariance*.

Existence Theorem

We have proved half of

Theorem A vertex algebra gives us a quadruple $(V, Y, \mathbf{1}, D)$: a linear space V, a vacuum vector $\mathbf{1}$, a special endomorphism D, and a state-field correspondence $Y : V \to \mathfrak{F}(V)$ whose image consists of *creative, translation-covariant, mutually local fields*.

Conversely, given such a quadruple, the products $u(n)v$ defined by

$$Y(u, z)v = \sum_n u(n)vz^{-n-1}$$

satisfy the JI hence define a vertex algebra structure on V.

Theorem. Let $(V, \mathbf{1}, D)$ consist of a $\mathbb{C}$-linear space, $\mathbf{1} \in V$, and $D \in End(V)$. Suppose given a subset $U \subseteq V$ and a map

$$U \longrightarrow \mathfrak{F}(V), \;\; u \longmapsto Y(u,z) = \sum_n u(n) z^{-n-1}$$

such that the set $\{Y(u,z) \mid u \in U\}$ consists of mutually local, translation-covariant, creative fields. Assume

$$V = \mathrm{span}\langle u_1(n_1) ... u_n(n_k) \mathbf{1} \mid u_i \in U, n_i \in \mathbb{Z} \rangle$$

Then there is a *unique extension* of Y to a state-field correspondence $Y : V \to \mathfrak{F}(V)$ such that $(V, Y, \mathbf{1}, D)$ is a vertex algebra.

We say that U *generates* V.

Vertex operator algebras

A VOA is a vertex algebra V with a second distinguished element ω satisfying several special properties.

The vertex operator for $\omega \in V$ is

$$Y(\omega, z) := \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$$
$$L(n) = \omega(n+1)$$

Virasoro algebra

The $L(n)$ close on the *Virasoro algebra of central charge* c:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m,-n} c Id_V,$$

a central extension of the *Witt-Zassenhaus* Lie algebra in which the central element acts as on V as multiplication by c.

Translation-covariance

Note that

$$
\begin{aligned}
[L(-1), Y(\omega, z)] &= \sum_n [L(-1), L(n)] z^{-n-2} \\
&= (-1-n) L(-1+n) z^{-n-2} \\
&= \partial_z Y(\omega, z)
\end{aligned}
$$

We require this for *all* fields: $L(-1)$ *is* the endomorphism D.

Spectral decomposition

$L(0) \in End(V)$ satisfies

- semisimple
- eigenvalues in $\mathbb{Z}$
- finite-dimensional eigenspaces
- eigenvalues bounded below

This can be summarized in the decomposition of V into $L(0)$-eigenspaces

$$\boxed{V = \bigoplus_{n \geq n_0} V_n}$$

$$V_n := \{v \in V | L(0)v = nv\}, \quad \dim V_n < \infty$$

These conditions arise from the exigencies of CFT:

ω is the stress-energy tensor, $L(0)$ the Hamiltonian, and we create bosonic particles of integral energy(*) n from the vacuum, a finite number for each n.

The VA axioms capture the idea of locality, but one gets a rich theory *only* for VOAs.

(Added (*): In the original lecture I used the term 'spin n' - which is not unknown in this context - but some in the MPI audience were not happy with this.)

VOA - summary

- $(V, Y, \mathbf{1}, \omega)$
- $Y : V \longrightarrow \mathfrak{F}(V)$
  - $Y(u, z) \sim Y(v, z)$
  - $Y(u, z)\mathbf{1} = u + ...$
  - $[L(-1), Y(u, z)] = \partial_z Y(u, z)$
- $Y(\omega, z) = \sum_n L(n) z^{-n-2}$
- $[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m,-n} c Id_V$
- $V = \oplus_{n \geq n_0} V_n$

Partition function

There are a number of *formal trace functions* associated to a VOA. These hold the keys to the connections with elliptic modular and other kinds of automorphic forms. Recall that we have

$$V = \oplus_{n \geq n_0} V_n$$

The *partition function* of V is

$$Z_V(q) := q^{-c/24} \sum_{n \geq n_0} \dim V_n q^n$$

Zero modes

If $u \in V_k$ then $u(m)$ *permutes* the V_n:

$$u(m) : V_n \longrightarrow V_{n+m-k-1}$$

The *zero mode* of v is

$$o(u) := u(k-1) : V_n \longrightarrow V_n$$

We set

$$Z_V(u,q) := q^{-c/24} \sum_n Tr(o(u)|V_n)q^n$$

If $u = \mathbf{1}$ this reduces to the partition function Z_V.

Zero mode trace map

Z_V defines a linear map

$$Z_V : V \longrightarrow q^{-c/24}\mathbb{C}[[q]]$$

$$u \mapsto Z_V(u, q) = q^{-c/24} \sum_n Tr(o(u)|V_n)q^n$$

A basic problem is to describe the image of this map for a given VOA V. Only partial results are known.

For ‘good’ VOAs, the image should consist of elliptic modular and other kinds of automorphic objects. We discuss this later.

Example 1. Virasoro VOAs

Vir is the abstract Virasoro Lie algebra

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m,-n}K$$

By definition, a VOA V necessarily has a field $Y(\omega, z) = \sum_n L(n)z^{-n-2}$ whose modes close on *Vir*. So V is, in particular, a *Vir*-module for which K acts as a scalar c, and we may look for VOAs in the category of such *Vir*-modules. Verma modules will furnish us with some examples.

$\mathbb{C}\mathbf{1}$ is a 1-dimensional linear space, $c \in \mathbb{C}$.

- $Vir^+ := \oplus_{n \geq 0} \mathbb{C} L(n) + \mathbb{C} K$
- $L(n).\mathbf{1} := 0 \quad (n \geq 0)$
- $K.\mathbf{1} := c\mathbf{1}$

  This makes $\mathbb{C}\mathbf{1}$ into a Vir^+-module
- $Ver_c := Ind_{Vir^+}^{Vir} \mathbb{C}\mathbf{1}$

Ver_c is a Verma module of central charge c

We define $Y(\omega, z) := \sum_n L(n) z^{-n-2}$, where now $L(n)$ means the action of the $L(n)$ generator of Vir on Ver_c. Then

$$
\begin{aligned}
&Y(\omega, z) \in \mathfrak{F}(Ver_c) \\
&Y(\omega, z) \sim_4 Y(\omega, z) \\
&Z_{Ver_c} = q^{-c/24} \prod_{n \geq 1} (1 - q^n)^{-1}
\end{aligned}
$$

and we know translation covariance holds automatically because it is satisfied by Vir.

The only thing that fails is creativity, because

$$Y(\omega, z)\mathbf{1} = \sum_n L(n)\mathbf{1}z^{-n-2} = L(-1)\mathbf{1}z^{-1} + \ldots$$

This is resolved by *modding out* the *Vir* ideal generated by $L(-1)\mathbf{1}$. Using the Existence Theorems we obtain

Theorem. There is a *Virasoro VOA* V_c of central charge c generated by a single Virasoro field $Y(\omega, z)$. We have $V_c = Ver_c / VirL(-1)\mathbf{1}$ and

$$Z_{V_c}(q) = \frac{1}{q^{c/24}\prod_{n\geq 2}(1-q^n)}$$

Example 2. Heisenberg VOA

The Heisenberg Lie algebra has basis $h(n)$ $(n \in \mathbb{Z}), K$ with

$$[h(m), h(n)] = m\delta_{m,-n}K$$

It is easier to deal with than *Vir*. Proceed just as in the last case. The corresponding Verma module itself - rather than a quotient - is a VOA. One difference is that we must take central charge $c = 1$, i.e., K acts on the Verma module as Id_V.

Theorem There is a *Heisenberg VOA* $M(1)$ of central charge 1 generated by a single field $Y(h,z) = \sum_n h(n)z^{-n-1}$.

$$Z_{M(1)}(q) = \frac{1}{q^{1/24}\prod_{n\geq 1}(1-q^n)} = \eta(q)^{-1}$$

The Virasoro vector is $\omega := \frac{1}{2}h(-1)^2\mathbf{1}$.

Theorem The image of the map

$$Z_{M(1)} : M(1) \longrightarrow q^{-1/24}\mathbb{C}[[q]]$$

consists of *all functions* $\frac{f(q)}{\eta(q)}$ where $f(q)$ is a *quasimodular form*, i.e.,

$$f \in \mathbb{C}[E_2, E_4, E_6]$$

Example 3. Moonshine module

Theorem There is a VOA $V^\natural$ of central charge $c = 24$ whose automorphism group is the Monster sporadic simple group M, and which has partition function

$$Z_{V^\natural}(q) = J(q) = q^{-1} + 196884q + \ldots$$

equal to the absolute modular invariant (constant term 0).
The image of the map

$$Z_{V^\natural} : V^\natural \longrightarrow q^{-1}\mathbb{C}[[q]]$$

consists of *all* modular forms $f(q)$ of level 1 (i.e., on $PSL_2(\mathbb{Z})$) that satisfy

$$f(q) \text{ is holomorphic in the upper half-plane } \mathcal{H}$$
$$f(q) = aq^{-1} + bq + \ldots \quad (a, b \in \mathbb{C})$$

Example 4. Lattice theories

Let L be an *even lattice*, i.e., a free abelian group of finite rank ℓ equipped with a positive-definite symmetric bilinear form $(\ ,\ )$ such that $(\alpha, \alpha) \in 2\mathbb{Z} \quad (\alpha \in L)$.

Theorem There is a VOA V_L of central charge $c = \ell$ which has partition function

$$Z_{V_L} = \frac{\theta_L(q)}{\eta(q)^\ell},$$

This is a modular function of weight 0 on a congruence subgroup of $PSL_2(\mathbb{Z})$.