Rogers-Ramanujan and Moonshine: Lecture 1

Ken Ono (Emory University)

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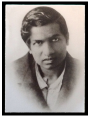
1. Framework of Rogers-Ramanujan identities



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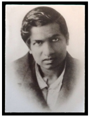

2. Monstrous Moonshine

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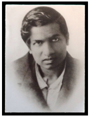

2. Monstrous Moonshine

3. Umbral Moonshine

A special continued fraction

Famous Fact

The **golden ratio** *is the* **algebraic integral unit**

$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+\cdots}}}.$$

A special continued fraction

Famous Fact

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$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+\cdots}}}.$$

Question

Is there a theory of **special values** *for*

$$R(q) := \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1+\cdots}}}}?$$

Ramanujan's first letter to Hardy

Ramanujan's unit:

$$e^{-2\pi/5} \cdot R(e^{-2\pi}) = -\phi + \sqrt{2+\phi}.$$

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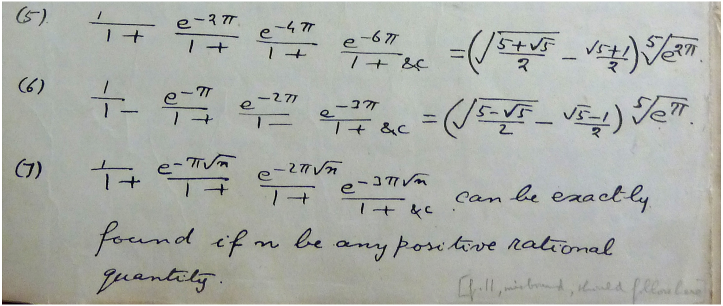
...in his own handwriting....


(5) $\frac{1}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-6\pi}}{1+}\&c = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$

(6) $\frac{1}{1-}\frac{e^{-\pi}}{1+}\frac{e^{-2\pi}}{1-}\frac{e^{-3\pi}}{1+}\&c = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}.$

(7) $\frac{1}{1+}\frac{e^{-\pi\sqrt{n}}}{1+}\frac{e^{-2\pi\sqrt{n}}}{1+}\frac{e^{-3\pi\sqrt{n}}}{1+}\&c$ can be exactly found if n be any positive rational quantity.

Hardy's reaction

"[These formulas] defeated me completely. . . . they could only be written down by a mathematician of the highest class. They must be true because no one would have the imagination to invent them."

G. H. Hardy 

Rogers-Ramanujan

Theorem (Rogers-Ramanujan)

$$R(q) = \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cdots}}} = \prod_{n=0}^{\infty} \frac{(1 - q^{5n+1})(1 - q^{5n+4})}{(1 - q^{5n+2})(1 - q^{5n+3})}.$$

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Theorem (Rogers, Ramanujan)

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Ubiquity of the RR Identities

- Number theory
- Conformal field theory
- K-theory
- Kac-Moody Lie algebras
- Knot theory
- Probability theory
- Statistical mechanics
- $\ldots$

Ramanujan's Claim

Conjecture (Folklore)

If τ is a CM point, then

$$e^{2\pi i\tau/5} \cdot R(e^{2\pi i\tau})$$

is an **algebraic integral unit.**

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Theorem (Berndt-Chan-Zhang (1996), Cais-Conrad (2006))

The Folklore Conjecture is true.

Important remarks

1. RR identities are of the form

$$\text{“Summatory } q\text{-series”} \quad = \quad \text{“Infinite product } \textbf{modular function}\text{”}.$$

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3. ....further ad hoc identities of Bailey, Dyson, Slater.
4. RR and AG identities $\Longrightarrow$ Lepowsky-Wilson program ...$\Longrightarrow$ **vertex operator theory** $\Longrightarrow$ **Moonshine**.

Fundamental Problems

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Is there a larger (and conceptual) framework of identities:

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Fundamental Problem 2

If so, do **natural ratios** *generalizing* $R(q)$ *give* **integral units***?*

Key observation

Remark

If we let $\lambda = (\lambda_1, \lambda_2, \dots)$ *denote* **partitions** *of integers, then*

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Idea

Seek a framework of identities where the "sum sides" are

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} \mathrm{Stuff}$$

.

Fundamental Problem 1

"Theorem" (Griffin-O-Warnaar)

Let $P_\lambda(x_1, \ldots; q)$ *be (extended)* **Hall-Littlewood polynomials**.

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Let $P_\lambda(x_1, \ldots; q)$ *be (extended)* **Hall-Littlewood polynomials**.
There are four triples (a, b, c) *such that* **for all** $m, n \geq 1$ *we have*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{bn+c}) = \text{"}\textbf{Explicit infinite product modular function}\text{"}.$$

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Remark

RR identities when $m=n=1$ and $(a,b,c)=(1,2,-1),(2,2,-1)$.

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Remark

Part (2) with $m = n = 1$ *is the original Folklore Conjecture.*

Representation theoretic interpretation

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- Given an affine Kac-Moody algebra, one has the “principal specialization homomorphism.”
- Weyl-Kac “product formula” gives infinite products.

Notation

Definition (q-Pochammer symbols)

We let

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and

$$\theta(a;q) := (a;q)_\infty(q/a;q)_\infty.$$

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$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \ldots ; q^{2n-1})$$

$$= \frac{(q^\kappa; q^\kappa)_\infty^n}{(q)_\infty^n} \cdot \prod_{i=1}^{n} \theta(q^{i+m}; q^\kappa) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa)$$

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Remarks

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2. If $n = 1$, then we obtain the Andrews-Gordon identities.

Example (Theorem 1 with $m = n = 2$)

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^3) = \prod_{n=1}^{\infty} \frac{(1 - q^{9n})}{(1 - q^n)}$$

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$$= \prod_{n=1}^{\infty} \frac{(1 - q^{9n})(1 - q^{9n-1})(1 - q^{9n-8})}{(1 - q^n)(1 - q^{9n-4})(1 - q^{9n-5})}.$$

Integrality properties

Theorem 2 (Griffin-O-Warnaar)

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2. *The singular value $\Phi_*(m, n; \tau)$ is a unit over $\mathbb{Z}[1/\kappa]$.*
3. *The ratio $\Phi_{1a}(m, n; \tau)/\Phi_{1b}(m, n; \tau)$ is an* **integral unit**.

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- They **are not** algebraic integers, but are roots of:

$$3x^2 - 1$$

$$3^9x^{18} - 3^7 \cdot 37x^{12} - 2 \cdot 3^9x^9 + 2^3 \cdot 3^4 \cdot 17x^6 - 2 \cdot 3^5x^3 - 1.$$

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- By Theorem 2 (2), **both** $\sqrt{3}\Phi_{1*}(2,2;i/3)$ are integral units.

Example when $m = n = 2$ continued.

- Which gives Theorem 2 (3) that

$$\Phi_{1a}(2, 2; i/3)/\Phi_{1b}(2, 2; i/3) = 4.60627\ldots$$

is an **integral unit**.

Example when $m = n = 2$ continued.

- Which gives Theorem 2 (3) that

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is an **integral unit**.

- Indeed, $\Phi_{1a}(2,2;i/3)/\Phi_{1b}(2,2;i/3)$ is a root of

$$x^{18} - 102x^{15} + 420x^{12} - 304x^9 - 93x^6 + 6x^3 + 1.$$

The Monster

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Conjecture (Fischer and Griess (1973))

There is a huge simple group (containing a double cover of Fischer's B) with order

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

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Theorem (Griess (1982))

The Monster group $\mathbb{M}$ *exists.*

Classification of Finite Simple Groups

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Finite simple groups live in natural infinite families, apart from the **sporadic groups**.

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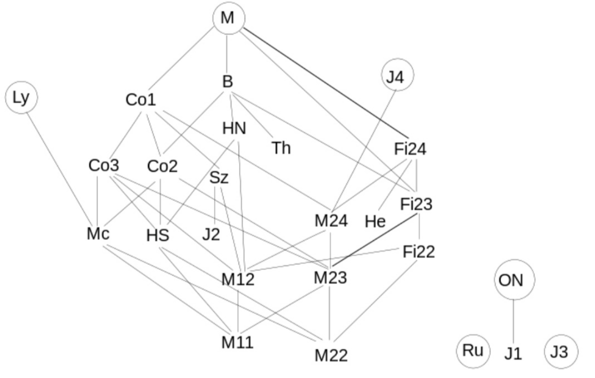

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i.e. $p \in Ogg_{ss} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$.

Ogg's Jack Daniels Problem

Remarque 1. – Dans sa leçon d'ouverture au Collège de France, le 14 janvier 1975, J. TITS mentionna le groupe de Fischer, "le monstre", qui, s'il existe, est un groupe simple "sporadique" d'ordre

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i. e. divisible exactement par les quinze nombres premiers de la liste du corollaire. Une bouteille de Jack Daniels est offerte à celui qui expliquera cette coïncidence.

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Remark

This is the first hint of **Moonshine**.

Second hint of moonshine

John McKay observed that

$$196884 = 1 + 196883$$

John Thompson's generalizations

Thompson further observed:

$$\underbrace{\begin{aligned} 196884 \\ 21493760 \\ 864299970 \end{aligned}}_{\text{Coefficients of } j(\tau)} \begin{aligned} &= \\ &= \\ &= \end{aligned} \underbrace{\begin{aligned} &1 + 196883 \\ &1 + 196883 + 21296876 \\ &1 + 1 + 196883 + 196883 + 21296876 + 842609326 \end{aligned}}_{\text{Dimensions of irreducible representations of the Monster } \mathbb{M}}$$

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Definition

Klein's j-function

$$j(\tau) - 744 = \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = \sum_{n=-1}^{\infty} c(n)q^n$$
$$= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots .$$

The Monster characters

The character table for $\mathbb{M}$ (ordered by size) gives dimensions:

$$\begin{aligned}
\chi_1(e) &= 1\\
\chi_2(e) &= 196883\\
\chi_3(e) &= 21296876\\
\chi_4(e) &= 842609326\\
&\vdots\\
\chi_{194}(e) &= 258823477531055064045234375.
\end{aligned}$$

Monster module

Conjecture (Thompson)

There is an infinite-dimensional graded module

$$V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$$

with

$$\dim(V_n^\natural) = c(n).$$

The McKay-Thompson Series

Definition (Thompson)

Assuming the conjecture, if $g \in \mathbb{M}$, then define the **McKay–Thompson series**

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|V_n^{\natural})q^n.$$

Conway and Norton

Conjecture (Monstrous Moonshine)

For each $g \in \mathbb{M}$ there is an explicit genus 0 discrete subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$

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Conjecture (Monstrous Moonshine)

For each $g \in \mathbb{M}$ there is an explicit genus 0 discrete subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the unique modular function with

$$T_g(\tau) = q^{-1} + O(q).$$

Borcherds' work

Theorem (Frenkel–Lepowsky–Meurman)

The moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *is a vertex operator algebra of central charge* 24 *whose graded dimension is given by* $j(\tau) - 744$*, and whose automorphism group is* $\mathbb{M}$.

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Theorem (Borcherds)

The Monstrous Moonshine Conjecture is true.

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Question A

Do order p elements in $\mathbb{M}$ know the $\overline{\mathbb{F}}_p$ supersingular j-invariants?

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If $p \notin \mathrm{Ogg}_{ss}$, then why do we expect $p \nmid \#\mathbb{M}$?

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Question A

Do order p elements in $\mathbb{M}$ know the $\overline{\mathbb{F}}_p$ supersingular j-invariants?

Question B

If $p \notin Ogg_{ss}$, then why do we expect $p \nmid \#\mathbb{M}$?

Question C

If $p \in Ogg_{ss}$, then why do we expect (a priori) that $p \mid \#\mathbb{M}$?

Witten's Conjecture (2007)

Conjecture (Witten, Li-Song-Strominger)

The vertex operator algebra $V^\natural$ is dual to a 3d quantum gravity theory. Thus, there are 194 "black hole states".

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Question (Witten)

How are these different kinds of black hole states distributed?

Distribution of Monstrous Moonshine

Open Problem

Question

Consider the moonshine expressions

$$
\begin{aligned}
196884 &= 1 + 196883 \\
21493760 &= 1 + 196883 + 21296876 \\
864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \\
&\vdots \\
c(n) &= \sum_{i=1}^{194} \mathbf{m}_i(n)\chi_i(e)
\end{aligned}
$$

How many '1's, '196883's, etc. show up in these equations?

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
-1	1	0	$\cdots$	0
1	1/2	1/2	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$
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100	$4.427\ldots \times 10^{-18}$	$1.077\ldots \times 10^{-16}$	$\cdots$	$0.04428\ldots$
120	$1.377\ldots \times 10^{-21}$	$5.501\ldots \times 10^{-20}$	$\cdots$	$0.04428\ldots$
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160	$2.621\ldots \times 10^{-27}$	$3.443\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
180	$1.877\ldots \times 10^{-28}$	$3.371\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
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Distribution of Moonshine

Theorem 3 (Duncan, Griffin, O)

If $1 \leq i \leq 194$*, then as* $n \to +\infty$ *we have*

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|n|}}$$

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Remark

We have exact formula for the $\mathbf{m}_i(n)$ analogous to Rademacher's infinite series expansion for $p(n)$.

Corollary

Corollary (Duncan, Griffin, O)

We have that

$$\delta\left(\mathbf{m}_i\right) := \lim_{n\to+\infty} \frac{\mathbf{m}_i(n)}{\sum_{i=1}^{194} \mathbf{m}_i(n)}$$

is well defined

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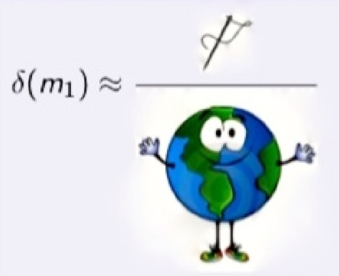
$$\delta(\mathbf{m}_i) := \lim_{n \to +\infty} \frac{\mathbf{m}_i(n)}{\sum_{i=1}^{194} \mathbf{m}_i(n)}$$

is well defined, and

$$\delta(\mathbf{m}_i) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$

How many '1's?

Corollary (Duncan, Griffin, O)



Umbral (shadow) Moonshine

Present day moonshine

Observation (Eguchi, Ooguri, Tachikawa (2010))

Using the K3 surface elliptic genus, there is a **mock modular form**

$$H(\tau) = q^{-\frac{1}{8}}\left(-2 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \ldots\right).$$

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The degrees of the irreducible repn's of the Mathieu group M_{24} *are:*

$$1, 23, 45, 231, 252, 253, 483, 770, 990, 1035,$$
$$1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.$$

Mathieu Moonshine

Theorem (Gannon (2013))

There is an infinite dimensional graded M_{24}-module whose McKay-Thompson series are specific mock modular forms.

What are mock modular forms?

Notation. Throughout, let

$$\tau = x + iy \in \mathbb{H} \quad \text{with} \quad x, y \in \mathbb{R}.$$

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Harmonic Maass forms

Definition

A *harmonic Maass form of weight k on a subgroup* $\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

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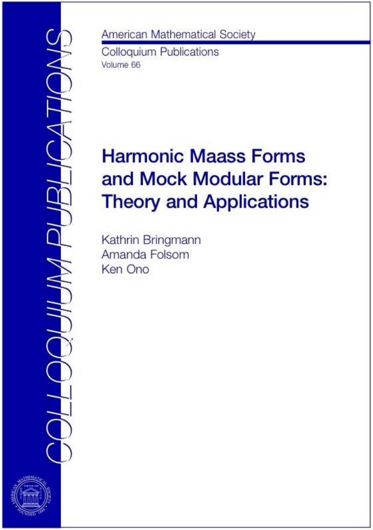
2. We have that $\Delta_k M = 0$.

Coming in 2016...



Fourier expansions

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Fundamental Lemma

If $M \in H_{2-k}$ and $\Gamma(a,x)$ is the incomplete Γ-function, then

$$M(\tau) = \sum_{n \gg -\infty} c^{+}(n) q^{n} + \sum_{n<0} c^{-}(n) \Gamma(k-1, 4\pi|n|y) q^{n}.$$

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Holomorphic part M^{+} **Nonholomorphic part** M^{-}

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Remark

- *We call M^{+} a* **mock modular form**.

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Remark

- *We call M^+ a* **mock modular form**.
- *If $\xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial \overline{\tau}}}$, then the* **shadow of** M *is $\xi_{2-k}(M^-)$.*

Shadows are modular forms

Fundamental Lemma

The operator $\xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial\overline{\tau}}}$ *defines a* **surjective** *map*

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Remark

In M_{24} *Moonshine, the McKay-Thompson series are mock modular forms with* **classical Jacobi theta series** *shadows!*

Larger Framework of Moonshine?

Remark

There are well known connections with even unimodular positive definite rank 24 lattices:

$$\mathbb{M} \quad \longleftrightarrow \quad \textbf{Leech lattice}$$

$$M_{24} \quad \longleftrightarrow \quad A_1^{24} \text{ lattice.}$$

Umbral Moonshine Conjecture

Conjecture (Cheng, Duncan, Harvey (2013))

Let L^X (up to isomorphism) be an even unimodular positive-definite rank 24 lattice, and let :

- *X be the corresponding ADE-type root system.*

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- *For each $g \in G^X$ let $H_g^X(\tau)$ be a specific automorphic form with "minimal principal parts".*

Then there is an infinite dimensional graded G^X module K^X for which $H_g^X(\tau)$ is the McKay-Thompson series for g.

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- For $X = A_2^{12}$ we have $G^X = M_{24}$ and Gannon's Theorem.
- There are 22 other isomorphism classes of X, the $H_g^X(\tau)$ constructed from X and its Coxeter number $m(X)$.

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Remark

Apart from the Leech case, the $H_g^X(\tau)$ are always weight 1/2 mock modular forms.

Preview of our results....

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Theorem 4 (Duncan, Griffin, O)

The Umbral Moonshine Conjecture is true.

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Example

For M_{12} the MT series include Ramanujan's mock thetas:

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2},$$

$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4) \cdots (1+q^{2n})},$$

$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4) \cdots (1-q^n+q^{2n})}$$

Framework of Rogers-Ramanujan Identities

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Theorem 1 (Griffin,O, Warnaar)

Framework of Rogers-Ramanujan identities in terms of Hall-Littlewood q-series.

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Theorem 2 (Griffin, O, Warnaar)

Folklore Conjecture on algebraic CM values.

Ogg's Jack Daniels Problem

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Do order p elements in $\mathbb{M}$ know the $\overline{\mathbb{F}}_p$ supersingular j-invariants?

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Question C

If $p \in Ogg_{ss}$, then why do we expect (a priori) that $p \mid \#\mathbb{M}$?

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Umbral Moonshine

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