Mock theta functions and representation theory of affine Lie superalgebras and superconformal algebras

Abstract: One of the beautiful properties in representation theory of affine Lie algebras is the $SL_2(\mathbb{Z})$-invariance of the space of characters of integrable modules discovered by Kac-Peterson in the early 1980's.

However, for affine Lie superalgebras, modular invariance had long been quite unclear except for only a few cases. Recently a remarkable breakthrough was brought by Zwegers, who constructed a modular function from the supercharacter of the affine sl(2|1)-module of level 1 by adding non-holomorphic correction term, which is called the "modification" procedure.

In my lecture, I will explain some recent progress on mock modular forms and representation theory of affine Lie superalgebras, which has been obtained by joint works with Victor Kac. The main subjects are concerned to the followings, on some of which I will give detail explanations in my talk.

After quick review on the Zwegers method which was the starting point of our work, we introduce the "basic" mock theta functions and explain their modification to non-holomorphic modular forms. These basic mock theta functions play basically important roles in our modification theory in general case.

Then, just corresponding to classical theta functions, we consider mock theta functions in genaral setting, namely mock theta functions of higher level and higher rank and higher atypicality. I will explain how to construct modification of these functions and how to compute their modular transformation formulas explicitely.

Then we turn to integrable representations of affine Lie superalgebras with maximal atypicality and compute their characters and supercharacters by using the Weyl-Kac type character formula. These are mock theta functions and the above method works to give $SL_2(\mathbb{Z})$ -invariant family of modified supercharacters for any affine Lie superalgebra. In this way, we observe that modular properties hold also for all affine Lie superalgebras. Furthermore, by considering admissible representations just like the case of affine Lie algebras, we obtain $SL_2(\mathbb{Z})$-invariant family of modified (super)characters of admissible modules for affine Lie superalgebras.

Then we go to the quantum Hamiltonian reduction, namely representations of W-algebras associated to affine Lie superalgebras. The theory of W-algebra is very important, since many of superconformal algebras are constructed as W-algebras. The modification of characters of affine Lie superalgebras naturally gives rise to the modification of characters of W-algebras. So we get modular invariant family of representations also for superconformal algebras.

As a consequence, vast numbers of $SL_2(\mathbb{Z})$-invariant families of modified mock theta functions are in our hands obtained from representations of affine Lie superalgebras and superconformal algebras. It will be natural to expect that there should exist "new" conformal field theory corresponding to these new "mock modular series" representations of superconformal algebras.

References:

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