Euclidean and Hyperbolic Planes

A minimalistic introduction with metric approach

Anton Petrunin

Contents

Introduction **7**
Prerequisite. Overview.

1 Preliminaries **11**
Metric spaces. Examples. Shortcut for distance. Isometries and motions. Lines. Congruent triangles. Continuous maps. Angles. Reals modulo $2\cdot\pi$. Geometric constructions.

Euclidean geometry

2 The Axioms **21**
Models and axioms. The Axioms. Angle and angle measure. Lines and half-lines. Zero angle. Straight angle. Vertical angles.

3 Half-planes **27**
Sign of angle. Intermediate value theorem. Same sign lemmas. Half-planes. Triangle with the given sides.

4 Congruent triangles **33**
Side-angle-side condition. Angle-side-angle condition. Isosceles triangles. Side-side-side condition.

5 Perpendicular lines **37**
Right, acute and obtuse angles. Perpendicular bisector. Uniqueness of perpendicular. Reflection. Angle bisectors. Circles.

6 Parallel lines and similar triangles **45**
Parallel lines. Similar triangles. Pythagorean theorem. Angles of triangle. Parallelograms.

7 Triangle geometry **53**
Circumcircle and circumcenter. Altitudes and orthocenter. Medians and centroid. Bisector of triangle. Incenter.

Inversive geometry

8 Inscribed angles **61**
Angle between a tangent line and a chord. Inscribed angle. Inscribed quadrilateral. Arcs.

9 Inversion **67**
Cross-ratio. Inversive plane and clines. Ptolemy's identity. Perpendicular circles. Angles after inversion.

Non-Euclidean geometry

10 Absolute plane **79**
Two angles of triangle. Three angles of triangle. How to prove that something can not be proved? Curvature.

11 Hyperbolic plane **87**
Poincaré disk model. Description of the model. What has to be proved? Auxiliary statements. The proofs.

12 Geometry of h-plane **97**
Angle of parallelism. Inradius of triangle. Circles, horocycles and equidistants. Hyperbolic triangles. Conformal interpretation.

Additional topics

13 Spherical geometry **109**
Space and spheres. Pythagorean theorem. Inversion of the space. Stereographic projection. Central projection.

14 Klein model **115**
Special bijection of h-plane to itself. Klein model. Hyperbolic Pythagorean theorem. Bolyai's construction.

15 Complex coordinates **123**

Complex numbers. Complex coordinates. Conjugation and absolute value. Euler's formula. Argument and polar coordinates. Möbius transformations. Elementary transformations. Complex cross-ratio.

Index **131**

Used resources **135**

Introduction

This is an introduction to Euclidean and Hyperbolic plane geometries and their development from postulate systems.

The lectures are meant to be rigorous, conservative, elementary and minimalistic. At the same time it includes about the maximum what students can absorb in one semester.

Approximately half of the material used to be covered in high school, not any more.

The lectures are oriented to sophomore and senior university students. These students already had a calculus course. In particular they are familiar with the real numbers and continuity. It makes possible to cover the material faster and in a more rigorous way than it could be done in high school.

Prerequisite

The students has to be familiar with the following topics.

- ◇ Elementary set theory: $\in$, $\cup$, $\cap$, $\setminus$, $\subset$, $\times$.
- ◇ Real numbers: intervals, inequalities, algebraic identities.
- ◇ Limits, continuous functions and Intermediate value theorem.
- ◇ Standard functions: absolute value, natural logarithm, exponent. Occasionally, basic trigonometric functions are used, but these parts can be ignored.
- ◇ Chapter 13 use in addition elementary properties of *scalar product*, also called *dot product*.
- ◇ To read Chapter 15, it is better to have some previous experience with complex numbers.

Overview

We use so called *metric approach* introduced by Birkhoff. It means that we define Euclidean plane as a *metric space* which satisfies a list of properties. This way we minimize the tedious parts which are unavoidable in the more classical Hilbert's approach. At the same time the students have chance to learn basic geometry of metric spaces.

Euclidean geometry is discussed in the the chapters 1–7. In the Chapter 1 we give all definitions necessary to formulate the axioms; it includes metric space, lines, angle measure, continuous maps and congruent triangles. In the Chapter 2, we formulate the axioms and prove immediate corollaries. In the chapters 3–6 we develop Euclidean geometry to a dissent level. In Chapter 7 we give the most classical theorem of triangle geometry; this chapter included mainly as an illustration.

In the chapters 8–9 we discuss geometry of circles on the Euclidean plane. These two chapters will be used in the construction of the model of hyperbolic plane.

In the chapters 10–12 we discuss non-Euclidean geometry. In Chapter 10, we introduce the axioms of absolute geometry. In Chapter 11 we describe so called Poincaré disc model (discovered by Beltrami). This is a construction of hyperbolic plane, an example of absolute plane which is not Euclidean. In the Chapter 12 we discuss some geometry of hyperbolic plane.

The last few chapters contain additional topics: Spherical geometry, Klein model and Complex coordinates. The proofs in these chapters are not completely rigorous.

When teaching the course, I used to give additional exercises in compass-and-ruler constructions[1]. These exercises work perfectly as an introduction to the proofs. I used extensively java applets created by C.a.R. which are impossible to include in the printed version.

Disclaimer

I am not doing history. It is impossible to find the original reference to most of the theorems discussed here, so I do not even try. (Most of the proofs discussed in the lecture appeared already in the Euclid's Elements and the Elements are not the original source anyway.)

Recommended books

- ◇ Kiselev's textbook [11] — a classical book for school students. Should help if you have trouble to follow the lectures.
- ◇ Moise's book, [8] — should be good for further study.

[1] see `www.math.psu.edu/petrunin/fxd/car.html`

- ◇ Greenberg's book [4] — a historical tour through the axiomatic systems of various geometries.
- ◇ Methodologically these lecture notes are very close to Sharygin's textbook [10]. Which I recommend to anyone who can read Russian.

Chapter 1

Preliminaries

Metric spaces

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set and d be a function which returns a real number $d(A,B)$ for any pair $A,B \in \mathcal{X}$. Then d is called* metric *on $\mathcal{X}$ if for any $A,B,C \in \mathcal{X}$, the following conditions are satisfied.*

(a) Positiveness:

$$d(A,B) \geqslant 0.$$

(b) $A = B$ if and only if

$$d(A,B) = 0.$$

(c) Symmetry:

$$d(A,B) = d(B,A).$$

(d) Triangle inequality:

$$d(A,C) \leqslant d(A,B) + d(B,C).$$

A metric space *is a set with a metric on it. More formally, a metric space is a pair $(\mathcal{X}, d)$ where $\mathcal{X}$ is a set and d is a metric on $\mathcal{X}$.*

Elements of $\mathcal{X}$ are called points *of the metric space. Given two points $A,B \in \mathcal{X}$ the value $d(A,B)$ is called* distance *from A to B.*

Examples

◇ *Discrete metric.* Let $\mathcal{X}$ be an arbitrary set. For any $A,B \in \mathcal{X}$, set $d(A,B) = 0$ if $A = B$ and $d(A,B) = 1$ otherwise. The metric d is called *discrete metric* on $\mathcal{X}$.

◇ *Real line.* Set of all real numbers ($\mathbb{R}$) with metric defined as

$$d(A,B) \stackrel{\text{def}}{=\!=} |A-B|.$$

◇ *Metrics on the plane.* Let us denote by $\mathbb{R}^2$ the set of all pairs (x,y) of real numbers. Assume $A=(x_A,y_A)$ and $B=(x_B,y_B)$ are arbitrary points in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics.

- *Euclidean metric,* denoted as d_2 and defined as

$$d_2(A,B) = \sqrt{(x_A-x_B)^2+(y_A-y_B)^2}.$$

- *Manhattan metric,* denoted as d_1 and defined as

$$d_1(A,B) = |x_A-x_B| + |y_A-y_B|.$$

- *Maximum metric,* denoted as d_∞ and defined as

$$d_\infty(A,B) = \max\{|x_A-x_B|, |y_A-y_B|\}.$$

1.2. Exercise. *Prove that d_1, d_2 and d_∞ are metrics on $\mathbb{R}^2$.*

Shortcut for distance

Most of the time we study only one metric on the space. For example $\mathbb{R}$ will always refer to the real line. Thus we will not need to name the metric function each time.

Given a metric space $\mathcal{X}$, the distance between points A and B will be further denoted as

$$AB \quad \text{or} \quad d_{\mathcal{X}}(A,B);$$

the later is used only if we need to emphasize that A and B are points of the metric space $\mathcal{X}$.

For example, the triangle inequality can be written as

$$AB + BC \geqslant AC.$$

For the multiplication we will always use "$\cdot$", so AB should not be confused with $A{\cdot}B$.

Isometries and motions

Recall that a map $f\colon \mathcal{X}\to\mathcal{Y}$ is a *bijection* if it gives an exact pairing of the elements of two sets. Equivalently, $f\colon \mathcal{X}\to\mathcal{Y}$ is a bijection if it has an *inverse*; i.e., a map $g\colon \mathcal{Y}\to\mathcal{X}$ such that $g(f(A))=A$ for any $A\in\mathcal{X}$ and $f(g(B))=B$ for any $B\in\mathcal{Y}$.

1.3. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$ be their metrics. A map*

$$f\colon \mathcal{X}\to\mathcal{Y}$$

is called distance-preserving *if*

$$d_{\mathcal{Y}}(f(A),f(B))=d_{\mathcal{X}}(A,B)$$

for any $A,B\in\mathcal{X}$.

A bijective distance-preserving map is called an isometry*.*

Two spaces are isometric *if there exists an isometry from one to the other.*

The isometry from space to itself is also called motion *of the space.*

1.4. Exercise. *Show that any distance preserving map is injective; i.e., if $f\colon \mathcal{X}\to\mathcal{Y}$ is a distance preserving map then $f(A)\neq f(B)$ for any pair of distinct points $A,B\in\mathcal{X}$*

1.5. Exercise. *Show that if $f\colon\mathbb{R}\to\mathbb{R}$ is a motion of the real line then either*

$$f(X)=f(0)+X \quad \textit{for any} \quad X\in\mathbb{R}$$

or

$$f(X)=f(0)-X \quad \textit{for any} \quad X\in\mathbb{R}.$$

1.6. Exercise. *Prove that $(\mathbb{R}^2,d_1)$ is isometric to $(\mathbb{R}^2,d_\infty)$.*

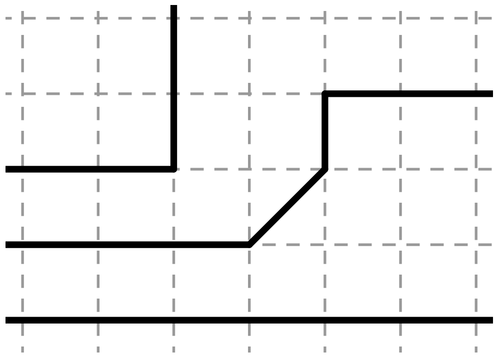
1.7. Exercise. *Describe all the motions of the Manhattan plane.*

Lines

If $\mathcal{X}$ is a metric space and $\mathcal{Y}$ is a subset of $\mathcal{X}$, then a metric on $\mathcal{Y}$ can be obtained by restricting the metric from $\mathcal{X}$. In other words, the distance between points of $\mathcal{Y}$ is defined to be the distance between the same points in $\mathcal{X}$. Thus any subset of a metric space can be also considered as a metric space.

1.8. Definition. *A subset ℓ of metric space is called* line *if it is isometric to the real line.*

Note that a space with discrete metric has no lines. The following picture shows examples of lines on the Manhattan plane, i.e. on $(\mathbb{R}, d_1)$.



Half-lines and segments. Assume there is a line ℓ passing through two distinct points P and Q. In this case we might denote ℓ as (PQ). There might be more than one line through P and Q, but if we write (PQ) we assume that we made a choice of such line.

Let us denote by $[PQ)$ the half-line which starts at P and contains Q. Formally speaking, $[PQ)$ is a subset of (PQ) which corresponds to $[0, \infty)$ under an isometry $f\colon (PQ) \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

The subset of line (PQ) between P and Q is called segment between P and Q and denoted as $[PQ]$. Formally, segment can defined as the intersection of two half-lines: $[PQ] = [PQ) \cap [QP)$.

An ordered pair of half-lines which start at the same point is called *angle*. An angle formed by two half-lines $[PQ)$ and $[PR)$ will be denoted as $\angle QPR$. In this case the point P is called *vertex* of the angle.

1.9. Exercise. *Show that if $X \in [PQ]$ then $PQ = PX + QX$.*

1.10. Exercise. *Consider graph $y = |x|$ in $\mathbb{R}^2$. In which of the following spaces (a) $(\mathbb{R}^2, d_1)$, (b) $(\mathbb{R}^2, d_2)$ (c) $(\mathbb{R}^2, d_\infty)$ it forms a line? Why?*

1.11. Exercise. *How many points M on the line (AB) for which we have*

1. $AM = MB$?
2. $AM = 2{\cdot}MB$?

Congruent triangles

An *ordered* triple of distinct points in a metric space, say A, B, C is called *triangle* and denoted as $\triangle ABC$. So the triangles $\triangle ABC$ and $\triangle ACB$ are considered as different.

Two triangles $\triangle A'B'C'$ and $\triangle ABC$ are called *congruent* (briefly $\triangle A'B'C' \cong \triangle ABC$) if there is a motion $f\colon \mathcal{X} \to \mathcal{X}$ such that $A' = f(A)$, $B' = f(B)$ and $C' = f(C)$.

Let $\mathcal{X}$ be a metric space and $f, g\colon \mathcal{X} \to \mathcal{X}$ be two motions. Note that the inverse $f^{-1} : \mathcal{X} \to \mathcal{X}$, as well as the composition $f \circ g : \mathcal{X} \to \mathcal{X}$ are also motions.

It follows that "$\cong$" is an equivalence relation; i.e., the following two conditions hold.

◇ If $\triangle A'B'C' \cong \triangle ABC$ then $\triangle ABC \cong \triangle A'B'C'$.

◇ If $\triangle A''B''C'' \cong \triangle A'B'C'$ and $\triangle A'B'C' \cong \triangle ABC$ then

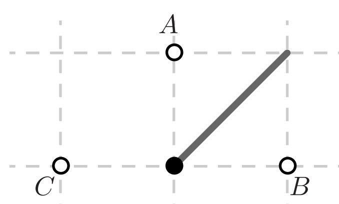
$$\triangle A''B''C' \cong \triangle ABC.$$

Note that if $\triangle A'B'C' \cong \triangle ABC$ then $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$.

For discrete metric, as well some other metric spaces the converse also holds. The following example shows that it does not hold in the Manhattan plane.

Example. Consider three points $A = (0, 1)$, $B = (1, 0)$ and $C = = (-1, 0)$ on the Manhattan plane $(\mathbb{R}^2, d_1)$. Note that

$$d_1(A, B) = d_1(A, C) = d_1(B, C) = 2.$$



On one hand

$$\triangle ABC \cong \triangle ACB.$$

Indeed, it is easy to see that the map $(x, y) \mapsto (-x, y)$ is an isometry of $(\mathbb{R}^2, d_1)$ which sends $A \mapsto A$, $B \mapsto C$ and $C \mapsto B$.

On the other hand

$$\triangle ABC \ncong \triangle BCA.$$

Indeed, assume there is a motion f of $(\mathbb{R}^2, d_1)$ which sends $A \mapsto B$ and $B \mapsto C$. Note that a point M is a midpoint[1] of A and B if and only if $f(M)$ is a midpoint of B and C. The set of midpoints for A

[1] M is a midpoint of A and B if $d_1(A, M) = d_1(B, M) = \frac{1}{2} \cdot d_1(A, B)$.

and B is infinite, it contains all points (t,t) for $t \in [0,1]$ (it is the dark gray segment on the picture). On the other hand the midpoint for B and C is unique (it is the black point on the picture). Thus the map f can not be bijective, a contradiction.

Continuous maps

Here we define continuous maps between metric spaces. This definition is a straightforward generalization of the standard definition for the real-to-real functions.

Further $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$ be their metrics.

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous at point $A \in \mathcal{X}$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that if $d_{\mathcal{X}}(A, A') < \delta$ then

$$d_{\mathcal{Y}}(f(A), f(A')) < \varepsilon.$$

The same way one may define a continuous map of several variables. Say, assume $f(A,B,C)$ is a function which returns a point in the space $\mathcal{Y}$ for a triple of points (A,B,C) in the space $\mathcal{X}$. The map f might be defined only for some triples in $\mathcal{X}$.

Assume $f(A,B,C)$ is defined. Then we say that f continuous at the triple (A,B,C) if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_{\mathcal{Y}}(f(A,B,C), f(A',B',C')) < \varepsilon.$$

if $d_{\mathcal{X}}(A,A') < \delta$, $d_{\mathcal{X}}(B,B') < \delta$ and $d_{\mathcal{X}}(C,C') < \delta$ and $f(A',B',C')$ is defined.

1.12. Exercise. *Let $\mathcal{X}$ be a metric space.*

(a) *Let $A \in \mathcal{X}$ be a fixed point. Show that the function*

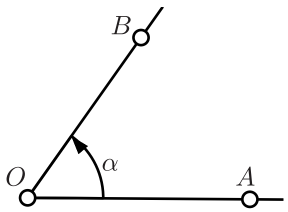
$$f(B) \stackrel{\text{def}}{=\!=} d_{\mathcal{X}}(A,B)$$

is continuous at any point B.

(b) *Show that $d_{\mathcal{X}}(A,B)$ is a continuous at any pair $A, B \in \mathcal{X}$.*

1.13. Exercise. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be a metric spaces. Assume that the functions $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are continuous at any point and $h = g \circ f$ is its composition; i.e., $h(x) = g(f(A))$ for any $A \in \mathcal{X}$. Show that $h\colon \mathcal{X} \to \mathcal{Z}$ is continuous.*

Angles



Before formulating the axioms, we need to develop a language which makes possible rigorously talk about angle measure.

Intuitively, the angle measure of an angle is how much one has to rotate the first half-line counterclockwise so it gets the position of the second half-line of the angle.

Note that the angle measure is defined up to full rotation which is $2{\cdot}\pi$ if measure in radians; so the angles $\dots, \alpha - 2{\cdot}\pi$, α, $\alpha + 2{\cdot}\pi$,$\alpha + 4{\cdot}\pi, \dots$ should be regarded to be the same.

Reals modulo $2{\cdot}\pi$

Let us introduce a new notation; we will write

$$\alpha \equiv \beta \quad \text{or} \quad \alpha \equiv \beta \pmod{2{\cdot}\pi}$$

if $\alpha = \beta + 2{\cdot}\pi{\cdot}n$ for some integer n. In this case we say

"α is equal to β modulo $2{\cdot}\pi$".

For example

$$-\pi \equiv \pi \equiv 3{\cdot}\pi \quad \text{and} \quad \tfrac{1}{2}{\cdot}\pi \equiv -\tfrac{3}{2}{\cdot}\pi.$$

The introduced relation "$\equiv$" behaves roughly as equality. We can do addition subtraction and multiplication by integer number without getting into trouble. For example, if

$$\alpha \equiv \beta \quad \text{and} \quad \alpha' \equiv \beta'$$

then

$$\alpha + \alpha' \equiv \beta + \beta', \qquad \alpha - \alpha' \equiv \beta - \beta' \quad \text{and} \quad n{\cdot}\alpha \equiv n{\cdot}\beta$$

for any integer n. But "$\equiv$" does not in general respect multiplication by non-integer numbers; for example

$$\pi \equiv -\pi \quad \text{but} \quad \tfrac{1}{2}{\cdot}\pi \not\equiv -\tfrac{1}{2}{\cdot}\pi.$$

1.14. Exercise. *Show that $2{\cdot}\alpha \equiv 0$ if and only if $\alpha \equiv 0$ or $\alpha \equiv \pi$.*

Geometric constructions

In the next chapter we will define plane as a metric space which satisfies certain properties. The *geometric constructions* in the plane is the construction of points, lines, and circles using only an idealized ruler and compass; they provide a valuable source of exercises in geometry.

The idealized ruler can be used only to draw a line through given two points. The idealized compass can be used only to draw a *circle* with given center and radius. I.e., given three points A, B and O we can draw the set of all points on distant AB from O; the value AB is called *radius* and O is called *center* of the circle. We may also mark new points in the plane as well as on the constructed lines, circles and their intersections (assuming that such points exist).

We can also look at the different set of instruments. For example, we may only use the ruler or we may invent a new instrument, say an instrument which produce midpoint for given two points.

The geometric constructions provide a sourse of exercises which we will use further in the lectures.

Euclidean geometry

Chapter 2

The Axioms

Models and axioms

The metric space $(\mathbb{R}^2, d_2)$ described on page 12, may be taken as a definition of Euclidean plane. It can be called *numerical model* of Euclidean plane; it builds the Euclidean plane from the real numbers while the later is assumed to be known.

In the axiomatic approach, one describes Euclidean plane as anything which satisfy a list of properties called *axioms*. Axiomatic system for the theory is like rules for the game. Once the axiom system is fixed, a statement considered to be true if it follows from the axioms and nothing else is considered to be true.

The formulations of the first axioms were not rigorous at all. For example, Euclid described a *line* as *breadthless length* and *straight line* as a line which *lies evenly with the points on itself*. On the other hand, these formulations were clear enough so that one mathematician could understand the other.

The best way to understand an axiomatic system is to make one by yourself. Look around and choose a physical model of the Euclidean plane, say imagine an infinite and perfect surface of chalk board. Now try to collect the key observations about this model. Let us assume that we have intuitive understanding of such notions as *line* and *point*.

- ◇ We can measure distances between points.
- ◇ We can draw unique line which pass though two given points.
- ◇ We can measure angles.
- ◇ If we rotate or shift we will not see the difference.
- ◇ If we change scale we will not see the difference.

These observations are good enough to start with. In the next section we use the language developed in the previous chapter to formulate them rigorously.

The observations above are intuitively obvious. On the other hand, it is not intuitively obvious that Euclidean plane is isometric to $(\mathbb{R}^2, d_2)$. This gives the first advantage of the axiomatic approach.

An other advantage lies in the fact that the axiomatic approach is easily adjustable. For example we may remove one axiom from the list, or exchange it to an other axiom. We will do such modifications in Chapter 10 and further.

The Axioms

In this section we set an axiomatic system of the Euclidean plane. This set of axioms is very close to the one given by Birkhoff in [3].

2.1. Definition. *The* Euclidean plane *is a metric space with at least two points which satisfies the following axioms:*

I. There is one and only one line, that contains any two given distinct points P and Q.

II. Any angle $\angle AOB$ defines a real number in the interval $(-\pi, \pi]$. This number is called angle measure of $\angle AOB$ *and denoted by $\measuredangle AOB$. It satisfies the following conditions:*

(a) Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$ there is unique half-line $[OB)$ such that $\measuredangle AOB = \alpha$

(b) For any points A, B and C distinct from O we have

$$\measuredangle AOB + \measuredangle BOC \equiv \measuredangle AOC.$$

(c) The function

$$\measuredangle\colon (A, O, B) \mapsto \measuredangle AOB$$

is continuous at any triple of points (A, O, B) such that $O \neq A$ and $O \neq B$ and $\measuredangle AOB \neq \pi$.

III. $\triangle ABC \cong \triangle A'B'C'$ if and only if

$$A'B' = AB, \quad A'C' = AC, \quad \text{and} \quad \measuredangle C'A'B' = \pm\measuredangle CAB.$$

IV. If for two triangles $\triangle ABC$, $\triangle AB'C'$ and $k > 0$ we have

$$\begin{aligned} B' &\in [AB), & C' &\in [AC) \\ AB' &= k\cdot AB, & AC' &= k\cdot AC \end{aligned}$$

then

$$B'C' = k\cdot BC, \quad \measuredangle ABC = \measuredangle AB'C' \quad \text{and} \quad \measuredangle ACB = \measuredangle AC'B'.$$

From now on, we can use no information about Euclidean plane which does not follow from the Definition 2.1.

Angle and angle measure

The notations $\angle AOB$ and $\measuredangle AOB$ look similar, they also have close but different meaning, which better not to be confused. The angle $\angle AOB$ is a pair of half-lines $[OA)$ and $[OB)$ while $\measuredangle AOB$ is a number in the interval $(-\pi, \pi]$.

The equality

$$\angle AOB = \angle A'O'B'$$

means that $[OA) = [O'A')$ and $[OB) = [O'B')$, in particular $O = O'$. On the other hand the equality

$$\measuredangle AOB = \measuredangle A'O'B'$$

means only equality of two real numbers; in this case O may be distinct from O'.

Lines and half-lines

2.2. Proposition. *Any two distinct lines intersect at most at one point.*

Proof. Assume two lines ℓ and m intersect at two distinct points P and Q. Applying Axiom I, we get $\ell = m$. □

2.3. Exercise. *Suppose $A' \in [OA)$ and $A' \neq O$ show that $[OA) = [OA')$.*

2.4. Proposition. *Given $r \geqslant 0$ and a half-line $[OA)$ there is unique $A' \in [OA)$ such that $OA = r$.*

Proof. According to definition of half-line, there is an isometry

$$f\colon [OA) \to [0, \infty),$$

such that $f(O) = 0$. By the definition of isometry, $OA' = f(A')$ for any $A' \in [OA)$. Thus, $OA' = r$ if and only if $f(A') = r$.

Since isometry has to be bijective, the statement follows. □

Zero angle

2.5. Proposition. *$\measuredangle AOA = 0$ for any $A \ne O$.*

Proof. According to Axiom IIb,

$$\measuredangle AOA + \measuredangle AOA \equiv \measuredangle AOA$$

Subtract $\measuredangle AOA$ from both sides, we get $\measuredangle AOA \equiv 0$. Hence $\measuredangle AOA = 0$. □

2.6. Exercise. *Assume $\measuredangle AOB = 0$. Show that $[OA) = [OB)$.*

2.7. Proposition. *For any A and B distinct from O, we have*

$$\measuredangle AOB \equiv -\measuredangle BOA.$$

Proof. According to Axiom IIb,

$$\measuredangle AOB + \measuredangle BOA \equiv \measuredangle AOA$$

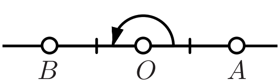
By Proposition 2.5 $\measuredangle AOA = 0$. Hence the result follows. □

Straight angle

If $\measuredangle AOB = \pi$, we say that $\angle AOB$ is a *straight angle*. Note that by Proposition 2.7, if $\angle AOB$ is a straight angle then so is $\angle BOA$.

We say that point O *lies between* points A and B if $O \ne A$, $O \ne B$ and $O \in [AB]$.

2.8. Theorem. *The angle $\angle AOB$ is straight if and only if O* lies *between A and B.*



Proof. By Proposition 2.4, we may assume that $OA = OB = 1$.

($\Leftarrow$). Assume O lies between A and B. Let $\alpha = \measuredangle AOB$.

Applying Axiom IIa, we get a half-line $[OA')$ such that $\alpha = \measuredangle BOA'$. We can assume that $OA' = 1$. According to Axiom III, $\triangle AOB \cong \triangle BOA'$; denote by h the corresponding motion of the plane.

Then $(A'B) = h(AB) \ni h(O) = O$. Therefore both lines (AB) and $(A'B)$, contain B and O. By Axiom I, $(AB) = (A'B)$.

By the definition of the line, (AB) contains exactly two points A and B on distance 1 from O. Since $OA' = 1$ and $A' \neq B$, we get $A = A'$.

By Axiom IIb and Proposition 2.5, we get

$$\begin{aligned} 2\cdot\alpha &\equiv \measuredangle AOB + \measuredangle BOA' \equiv \\ &\equiv \measuredangle AOB + \measuredangle BOA \equiv \\ &\equiv \measuredangle AOA \equiv \\ &\equiv 0 \end{aligned}$$

Since $[OA) \neq [OB)$, Axiom IIa implies $\alpha \neq 0$. Hence $\alpha = \pi$ (see Exercise 1.14).

$(\Rightarrow)$. Suppose that $\measuredangle AOB \equiv \pi$. Consider line (OA) and choose point B' on (OA) so that O lies between A and B'.

From above, we have $\measuredangle AOB' = \pi$. Applying Axiom IIa, we get $[OB) = [OB')$. In particular, O lies between A and B. □

A triangle $\triangle ABC$ is called *degenerate* if A, B and C lie on one line. The following corollary is just a reformulation of Theorem 2.8.

2.9. Corollary. *A triangle is degenerate if and only if one of its angles is equal to π or 0.*

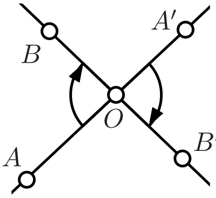
2.10. Exercise. *Show that three distinct points A, O and B lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 0.$$

2.11. Exercise. *Let A, B and C be three points distinct from O. Show that B, O and C lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 2\cdot\measuredangle AOC.$$

Vertical angles



A pair of angles $\angle AOB$ and $\angle A'OB'$ is called *vertical* if O leis between A and A' and at the same time O lies between B and B'.

2.12. Proposition. *The vertical angles have equal measures.*

Proof. Assume that the angles $\angle AOB$ and $\angle A'OB'$ are vertical.

Note that the angles $\angle AOA'$ and $\angle BOB'$ are straight. Therefore $\measuredangle AOA' = \measuredangle BOB' = \pi$. It follows that

$$\begin{aligned} 0 &= \measuredangle AOA' - \measuredangle BOB' \equiv \\ &\equiv \measuredangle AOB + \measuredangle BOA' - \measuredangle BOA' - \measuredangle A'OB' \equiv \\ &\equiv \measuredangle AOB - \measuredangle A'OB'. \end{aligned}$$

Hence the result follows. □

2.13. Exercise. *Assume O is the midpoint for both segments $[AB]$ and $[CD]$. Prove that $AC = BD$.*

Chapter 3

Half-planes

This chapter contains long proofs of self-evident statements. It is okay to skip it, but make sure you know definitions of positive/negative angles and that your intuition agrees with 3.8, 3.10, 3.11 and 3.16.

Sign of angle

- ⋄ The angle $\angle AOB$ is called *positive* if $0 < \measuredangle AOB < \pi$;
- ⋄ The angle $\angle AOB$ is called *negative* if $\measuredangle AOB < 0$.

Note that according to the above definitions the straight angle as well as zero angle are neither positive nor negative.

3.1. Exercise. *Show that $\angle AOB$ is positive if and only if $\angle BOA$ is negative.*

3.2. Exercise. *Let $\angle AOB$ is a straight angle. Show that $\angle AOX$ is positive if and only if $\angle BOX$ is negative.*

3.3. Exercise. *Assume that the angles $\angle AOB$ and $\angle BOC$ are positive. Show that*

$$\measuredangle AOB + \measuredangle BOC + \measuredangle COB = 2\cdot\pi.$$

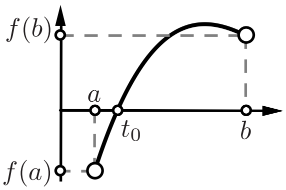
if $\angle COB$ is positive and

$$\measuredangle AOB + \measuredangle BOC + \measuredangle COB = 0.$$

if $\angle COB$ is negative.

Intermediate value theorem

3.4. Intermediate value theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Assume $f(a)$ and $f(b)$ have the opposite signs then $f(t_0) = 0$ for some $t_0 \in [a,b]$.*



The Intermediate value theorem should be covered in any calculus course. We will use the following corollary.

3.5. Corollary. *Assume that for any $t \in [0,1]$ we have three points in the plane O_t, A_t and B_t such that*

(a) Each function $t \mapsto O_t$, $t \mapsto A_t$ and $t \mapsto B_t$ is continuous.

(b) For for any $t \in [0,1]$, the points O_t, A_t and B_t do not lie on one line.

Then the angles $\angle A_0O_0B_0$ and $\angle A_1O_1B_1$ have the same sign.

Proof. Consider the function $f(t) = \measuredangle A_tO_tB_t$.

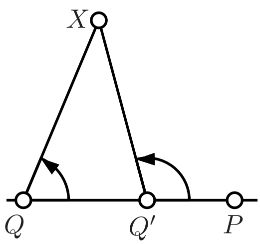
Since the points O_t, A_t and B_t do not lie on one line, Theorem 2.8 implies that $f(t) = \measuredangle A_tO_tB_t \neq 0$ or π for any $t \in [0,1]$.

Therefore by Axiom IIc and Exercise 1.13, f is a continuous function.

Further, by Intermediate value theorem, $f(0)$ and $f(1)$ have the same sign; hence the result follows. ☐

Same sign lemmas

3.6. Lemma. *Assume $Q' \in [PQ)$ and $Q' \neq P$. Then for any $X \notin (PQ)$ the angles $\angle PQX$ and $\angle PQ'X$ have the same sign.*

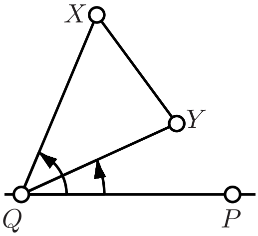


Proof. By Proposition 2.4, for any $t \in [0,1]$ there is unique point $Q_t \in [PQ)$ such that $PQ_t = (1-t)\cdot PQ + t\cdot PQ'$. Note that the map $t \mapsto Q_t$ is continuous, $Q_0 = Q$ and $Q_1 = Q'$ and for any $t \in [0,1]$, we have $P \neq Q_t$.

Applying Corollary 3.5, for $P_t = P$, Q_t and $X_t = X$, we get that $\angle PQX$ has the same sign as $\angle PQ'X$. ☐

3.7. Lemma. *Assume $[XY]$ does not intersect (PQ) then the angles $\angle PQX$ and $\angle PQY$ have the same sign.*

The proof is nearly identical to the one above.



Proof. According to Proposition 2.4, for any $t \in [0,1]$ there is a point $X_t \in [XY]$ such that $XX_t = t{\cdot}XY$. Note that the map $t \mapsto X_t$ is continuous, $X_0 = X$ and $X_1 = Y$ and for any $t \in [0,1]$, we have $Q \neq Y_t$.

Applying Corollary 3.5, for $P_t = P$, $Q_t = Q$ and X_t, we get that $\angle PQX$ has the same sign as $\angle PQY$. □

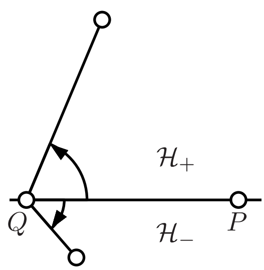
Half-planes

3.8. Proposition. *The complement of a line (PQ) in the plane can be presented in the unique way as a union of two disjoint subsets called* half-planes *such that*

(a) *Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if the angles $\angle PQX$ and $\angle PQY$ have the same sign.*

(b) *Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if $[XY]$ does not intersect (PQ).*

Further we say that X and Y lie on *one side from* (PQ) if they lie in one of the half-planes of (PQ) and we say that P and Q lie on the *opposite sides from* ℓ if they lie in the different half-planes of ℓ.



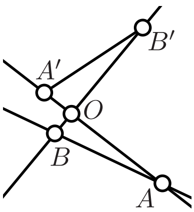
Proof. Let us denote by $\mathcal{H}_+$ (correspondingly $\mathcal{H}_-$) the set of points X in the plane such that $\angle PQX$ is positive (correspondingly negative).

According to Theorem 2.8, $X \in (PQ)$ if and only if $\measuredangle PQX \neq 0$ nor π. Therefore $\mathcal{H}_+$ and $\mathcal{H}_-$ give the unique subdivision of the complement of (PQ) which satisfies property (a).

Now let us prove that the this subdivision depends only on the line (PQ); i.e., if $(P'Q') = (PQ)$ and $X, Y \notin (PQ)$ then the angles $\angle PQX$ and $\angle PQY$ have the same sign if and only if the angles $\angle P'Q'X$ and $\angle P'Q'Y$ have the same sign.

Applying Exercise 3.2, we can assume that $P = P'$ and $Q' \in [PQ)$. It remains to apply Lemma 3.6.

(b). Assume $[XY]$ intersects (PQ). Since the subdivision depends only on the line (PQ), we can assume that $Q \in [XY]$. In this case, by Exercise 3.2, the angles $\angle PQX$ and $\angle PQY$ have opposite signs.



Now assume $[XY]$ does not intersects (PQ). In this case, by Lemma 3.7, $\angle PQX$ and $\angle PQY$ have the same sign. □

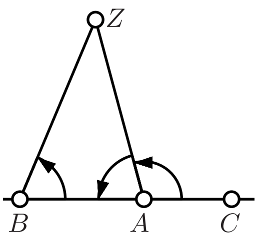
3.9. Exercise. *Assume that the angles $\angle AOB$ and $\angle A'OB'$ are vertical. Show that the line (AB) does not intersect the segment $[A'B']$.*

Consider triangle $\triangle ABC$. The segments $[AB]$, $[BC]$ and $[CA]$ are called *sides of the triangle*.

The following theorem is a corollary of Proposition 3.8.

3.10. Pasch's theorem. *Assume line ℓ does not pass through any vertex a triangle. Then it intersects either two or zero sides of the triangle.*

3.11. Signs of angles of triangle. *In any nondegenerate triangle $\triangle ABC$ the angles $\angle ABC$, $\angle BCA$ and $\angle CAB$ have the same sign.*



Proof. Choose a point $Z \in (AB)$ so that A lies between B and Z.

According to Lemma 3.6, the angles $\angle ZBC$ and $\angle ZAC$ have the same sign.

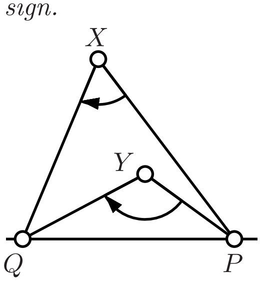
Note that $\measuredangle ABC = \measuredangle ZBC$ and

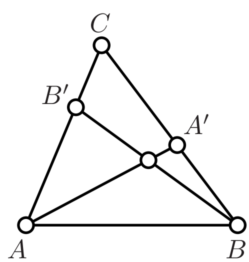
$$\measuredangle ZAC + \measuredangle CAB \equiv \pi.$$

Therefore $\angle CAB$ has the same sign as $\angle ZAC$ which in turn has the same sign as $\measuredangle ABC = \measuredangle ZBC$.

Repeating the same argument for $\angle BCA$ and $\angle CAB$, we get the result. □

3.12. Exercise. *Show that two points $X, Y \notin (PQ)$ lie on the same side from (PQ) if and only if the angles $\angle PXQ$ and $\angle PYQ$ have the same sign.*





3.13. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle, $A' \in [BC]$ and $B' \in [AC]$. Show that the segments $[AA']$ and $[BB']$ intersect.*

3.14. Exercise. *Assume that the points X and Y lie on the opposite sides from the line (PQ). Show that the half-line $[PX)$ does not interests $[QY)$.*

3.15. Advanced exercise. *Note that the following quantity*

$$\tilde{\measuredangle} ABC = \left[\begin{array}{lll} \pi & \text{if} & \measuredangle ABC = \pi \\ -\measuredangle ABC & \text{if} & \measuredangle ABC < \pi \end{array}\right.$$

can serve as the angle measure; i.e., the axioms hold if one changes everywhere $\measuredangle$ to $\tilde{\measuredangle}$.

Show that $\measuredangle$ and $\tilde{\measuredangle}$ are the only possible angle measures on the plane.

Show that without Axiom IIc, this is not longer true.

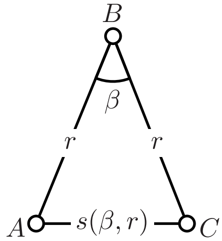
Triangle with the given sides

Consider triangle $\triangle ABC$. Let $a = BC$, $b = CA$ and $c = AB$. Without loss of generality we may assume that $a \leqslant b \leqslant c$. Then all three triangle inequalities for $\triangle ABC$ hold if and only if $c \leqslant a + b$. The following theorem states that this is the only restriction on a, b and c.

3.16. Theorem. *Assume that $0 < a \leqslant b \leqslant c \leqslant a + b$. Then there is a triangle $\triangle ABC$ such that $a = BC$, $b = CA$ and $c = AB$.*

The proof requires some preparation.

Assume $r > 0$ and $\pi > \beta > 0$. Consider triangle $\triangle ABC$ such that $AB = BC = r$ and $\measuredangle ABC = \beta$. The existence of such triangle follow from Axiom IIa and Proposition 2.4.



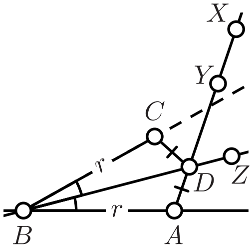
Note that according to Axiom III, the values β and r define the triangle up to congruence. In particular the distance AC depends only on β and r. Set

$$s(\beta, r) \stackrel{\text{def}}{=\!=} AC.$$

3.17. Proposition. *Given $r > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < \beta < \delta$ then $s(r, \beta) < \varepsilon$.*

Proof. Fix two point A and B such that $AB = r$.

Choose a point X such that $\measuredangle ABX$ is positive. Let $Y \in [AX)$ be the point such that $AY = \frac{\varepsilon}{8}$; it exists by Proposition 2.4.



Note that X and Y lie in the same side from (AB); therefore $\angle ABY$ is positive. Set $\delta = \measuredangle ABY$.

Assume $0 < \beta < \delta$, $\measuredangle ABC = \beta$ and $BC = r$.

Applying Axiom IIa, we can choose a half-line $[BZ)$ such that $\measuredangle ABZ = \frac{1}{2}\cdot\beta$. Note that A and Y lie on the opposite sides from (BZ). Therefore (BZ) intersects $[AY]$; denote by D the point of intersection.

Since $D \in (BZ)$, we get $\measuredangle ABD = \frac{\beta}{2}$ or $\frac{\beta}{2} - \pi$. The later is impossible since D and Y lie on the same side from (AB). Therefore

$$\measuredangle ABD = \measuredangle DBC = \tfrac{\beta}{2}.$$

By Axiom III, $\triangle ABD \cong \triangle DBD$. In particular

$$\begin{aligned} AC &\leqslant AD + DC = \\ &= 2\cdot AD \leqslant \\ &\leqslant 2\cdot AY = \\ &= \tfrac{\varepsilon}{4}. \end{aligned}$$

Hence the result follows. □

3.18. Corollary. *Fix a real number $r > 0$ and two distinct points A and B. Then for any real number $\beta \in [0, \pi]$, there is unique point C_β such that $BC_\beta = r$ and $\measuredangle ABC_\beta = \beta$. Moreover, the map $\beta \mapsto C_\beta$ is a continuous map from $[0, \pi]$ to the plane.*

Proof. The existence and uniqueness of C_β follows from Axiom IIa and Proposition 2.4.

Note that if $\beta_1 \neq \beta_2$ then

$$C_{\beta_1}C_{\beta_2} = s(r, |\beta_1 - \beta_2|).$$

Therefore Proposition 3.17 implies that the map $\beta \mapsto C_\beta$ is continuous. □

Proof of Theorem 3.16. Fix points A and B such that $AB = c$. Given $\beta \in [0, \pi]$, denote by C_β the point in the plane such that $BC_\beta = a$ and $\measuredangle ABC = \beta$.

According to Corollary 3.18, the map $\beta \mapsto C_\beta$ is a continuous. Therefore function $b(\beta) = AC_\beta$ is continuous (formally it follows from Exercise 1.12 and Exercise 1.13).

Note that $b(0) = c - a$ and $b(\pi) = c + a$. Since $c - a \leqslant b \leqslant c + a$, by Intermediate value theorem (3.4) there is $\beta_0 \in [0, \pi]$ such that $b(\beta_0) = b$. Hence the result follows. □

Chapter 4

Congruent triangles

Side-angle-side condition

Our next goal is to give conditions which guarantee congruence of two triangles. One of such conditions is Axiom III, it is also called *side-angle-side condition* or briefly *SAS condition*.

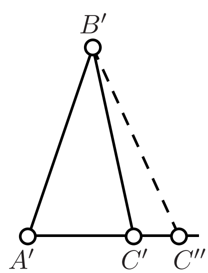
Angle-side-angle condition

4.1. ASA condition. *Assume that* $AB = A'B'$, $\measuredangle ABC \equiv \pm\measuredangle A'B'C'$, $\measuredangle CAB \equiv \pm\measuredangle C'A'B'$ *and* $\triangle A'B'C'$ *is nondegenerate. Then*

$$\triangle ABC \cong \triangle A'B'C'.$$

Note that for degenerate triangles the statement does not hold, say consider one triangle with sides 1, 4, 5 and the other with sides 2, 3, 5.

Proof. According to Theorem 3.11, either



❶ $$\begin{aligned}\measuredangle ABC &\equiv \measuredangle A'B'C',\\ \measuredangle CAB &\equiv \measuredangle C'A'B'\end{aligned}$$

or

❷ $$\begin{aligned}\measuredangle ABC &\equiv -\measuredangle A'B'C',\\ \measuredangle CAB &\equiv -\measuredangle C'A'B'.\end{aligned}$$

Further we assume that ❶ holds; the case ❷ is analogous.

Let C'' be the point on the half-line $[A'C')$ such that that $A'C'' = AC$.

By Axiom III, $\triangle A'B'C'' \cong \triangle ABC$. Applying Axiom III again, we get

$$\measuredangle A'B'C'' \equiv \measuredangle ABC \equiv \measuredangle A'B'C'.$$

By Axiom IIa, $[B'C') = [BC'')$. Hence C'' lies on $(B'C')$ as well as on $(A'C')$.

Since $\triangle A'B'C'$ is not degenerate, $(A'C')$ is distinct from $(B'C')$. Applying Axiom I, we get $C'' = C'$.

Therefore $\triangle A'B'C' = \triangle A'B'C'' \cong \triangle ABC$. □

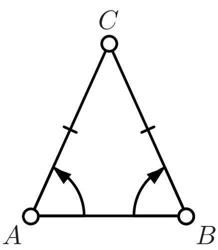
Isosceles triangles

A triangle with two equal sides is called *isosceles*; the remaining side is called *base* of isosceles triangle.

4.2. Theorem. *Assume $\triangle ABC$ is isosceles with base $[AB]$. Then*

$$\measuredangle ABC \equiv -\measuredangle BAC.$$

Moreover, the converse holds if $\triangle ABC$ is nondegenerate.



The following proof is due to Pappus of Alexandria.

Proof. Note that

$$CA = CB, \ CB = CA, \ \measuredangle ACB \equiv -\measuredangle BCA.$$

Therefore by Axiom III,

$$\triangle CAB \cong \triangle CBA.$$

Applying the theorem on the signs of angles of triangles (3.11) and Axiom III again, we get

$$\measuredangle CAB \equiv -\measuredangle CBA.$$

To prove the converse, we assume $\measuredangle CAB \equiv -\measuredangle CBA$. By ASA condition 4.1, $\triangle CAB \cong \triangle CBA$. Therefore $CA = CB$. □

Side-side-side condition

4.3. SSS condition. $\triangle ABC \cong \triangle A'B'C'$ *if*

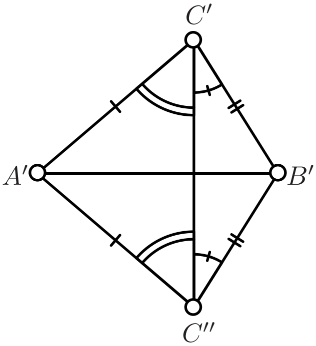
$$A'B' = AB, \quad B'C' = BC \quad \textit{and} \quad C'A' = CA.$$

Proof. Choose C'' so that $A'C'' = A'C'$ and $\measuredangle B'A'C'' \equiv \measuredangle BAC$. According to Axiom III,

$$\triangle A'B'C'' \cong \triangle ABC.$$

It will suffice to prove that

❸ $\triangle A'B'C' \cong \triangle A'B'C''$.



The condition ❸ trivially holds if $C'' = C'$. Thus it remains to consider the case $C'' \neq C'$.

Clearly, the corresponding sides of $\triangle A'B'C'$ and $\triangle A'B'C''$ are equal.

Note that triangles $\triangle C'A'C''$ and $\triangle C'B'C''$ are isosceles. By Theorem 4.2, we have

$$\measuredangle A'C''C' \equiv -\measuredangle A'C'C'',$$
$$\measuredangle C'C''B' \equiv -\measuredangle C''C'B'.$$

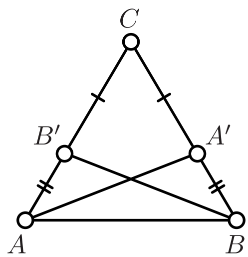
Whence by addition

$$\measuredangle A'C'B' \equiv -\measuredangle A'C''B'.$$

Applying Axiom III again, we get ❸. □

4.4. Exercise. *Let M be the midpoint of side $[AB]$ of a triangle $\triangle ABC$ and M' be the midpoint of side $[A'B']$ of a triangle $\triangle A'B'C'$. Assume $C'A' = CA$, $C'B' = CB$ and $C'M' = CM$. Prove that $\triangle A'B'C' \cong \triangle ABC$.*

4.5. Exercise. *Let $\triangle ABC$ be isosceles with base $[AB]$ and the points $A' \in [BC]$ and $B' \in [AC]$ be such that $CA' = CB'$. Show that*
(a) $\triangle AB'C \cong \triangle BA'C$;
(b) $\triangle ABB' \cong \triangle BAA'$.



4.6. Exercise. *Show that if $AB + BC = AC$ then $B \in [AC]$.*

4.7. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle and let ι be a motion of the plane such that*

$$\iota(A) = A, \quad \iota(B) = B \quad \text{and} \quad \iota(C) = C.$$

Show that ι is the identity; i.e. $\iota(X) = X$ for any point X on the plane.

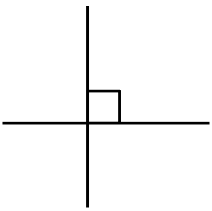
Chapter 5

Perpendicular lines

Right, acute and obtuse angles

⋄ If $|\measuredangle AOB| = \frac{\pi}{2}$, we say that the angle $\angle AOB$ is *right*;
⋄ If $|\measuredangle AOB| < \frac{\pi}{2}$, we say that the angle $\angle AOB$ is *acute*;
⋄ If $|\measuredangle AOB| > \frac{\pi}{2}$, we say that the angle $\angle AOB$ is *obtuse*.

On the diagrams, the right angles will be marked with a little square.



If $\angle AOB$ is right, we say also that $[OA)$ is *perpendicular* to $[OB)$; it will be written as $[OA) \perp [OB)$.

From Theorem 2.8, it follows that two lines (OA) and (OB) are appropriately called *perpendicular*, if $[OA) \perp [OB)$. In this case we also write $(OA) \perp (OB)$.

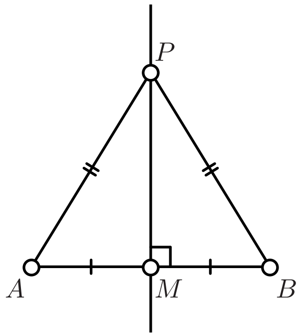
5.1. Exercise. *Assume point O lies between A and B. Show that for any point X the angle $\angle XOA$ is acute if and only if $\angle XOB$ is obtuse.*

Perpendicular bisector

Assume M is the midpoint of the segment $[AB]$; i.e., $M \in (AB)$ and $AM = MB$.

The line ℓ passing through M and perpendicular to (AB) passing through M is called *perpendicular bisector* to the segment $[AB]$.

5.2. Theorem. *Given distinct points A and B, all points equidistant from A and B and no others lie on the perpendicular bisector to $[AB]$.*



Proof. Let M be the midpoint of $[AB]$.

Assume $PA = PB$ and $P \neq M$. According to SSS-condition (4.3), $\triangle AMP \cong \triangle BMP$. Hence

$$\measuredangle AMP \equiv \pm\measuredangle BMP.$$

Since $A \neq B$, we have "$-$" in the above formula. Further,

$$\begin{aligned}\pi &\equiv \measuredangle AMB \equiv \\ &\equiv \measuredangle AMP + \measuredangle PMB \equiv \\ &\equiv 2\cdot\measuredangle AMP.\end{aligned}$$

I.e. $\measuredangle AMP \equiv \pm\frac{\pi}{2}$ and therefore P lies on the perpendicular bisector.

To prove converse, suppose $P \neq M$ is any point in the perpendicular bisector to $[AB]$. Then $\measuredangle AMP \equiv \pm\frac{\pi}{2}$, $\measuredangle BMP \equiv \pm\frac{\pi}{2}$ and $AM = BM$. Therefore $\triangle AMP \cong \triangle BMP$; in particular $AP = BP$. □

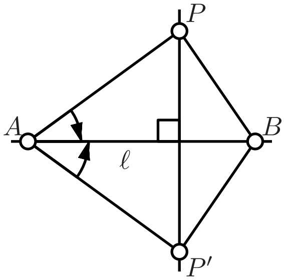
5.3. Exercise. *Let ℓ be the perpendicular bisector the the segment $[AB]$ and X be an arbitrary point on the plane.*

Show that $AX < BX$ if and only if X and A lie on the same side from ℓ.

5.4. Exercise. *Let $\triangle ABC$ be nondegenerate. Show that $AB > BC$ if and only if $|\measuredangle BCA| > |\measuredangle ABC|$.*

Uniqueness of perpendicular

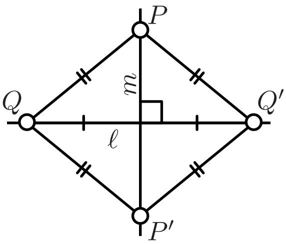
5.5. Theorem. *There is one and only one line which pass through a given point P and perpendicular to a given line ℓ.*



According to the above theorem, there is unique point $Q \in \ell$ such that $(QP) \perp \ell$. This point Q is called *foot point* of P on ℓ.

Proof. If $P \in \ell$ then both statements follows from Axiom II.

Existence for $P \notin \ell$. Let A, B be two distinct points of ℓ. Choose P' so that $AP' = AP$ and $\measuredangle P'AB \equiv -\measuredangle PAB$. According to Axiom III, $\triangle AP'B \cong \triangle APB$. Therefore $AP = AP'$ and $BP = BP'$.



According to Theorem 5.2, A and B lie on perpendicular bisector to $[PP']$. In particular $(PP') \perp (AB) = \ell$.

Uniqueness for $P \notin \ell$. We will apply the theorem on perpendicular bisector (5.2) few times. Assume $m \perp \ell$ and $m \ni P$. Then m a perpendicular bisector to some segment $[QQ']$ of ℓ; in particular, $PQ = PQ'$.

Since ℓ is perpendicular bisector to $[PP']$, we get $PQ = P'Q$ and $PQ' = P'Q'$. Therefore

$$PQ = P'Q = PQ' = P'Q'.$$

I.e. P' lies on the perpendicular bisector to $[QQ']$ which is m. By Axiom I, $m = (PP')$. □

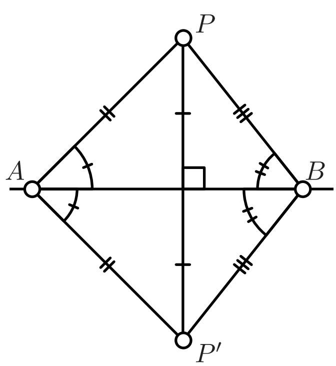
Reflection

To find the *reflection* P' through the line (AB) of a point P, one drops a perpendicular from P onto (AB), and continues it to the same distance on the other side.

According to Theorem 5.5, P' is uniquely determined by P.

Note that $P = P'$ if and only if $P \in (AB)$.

5.6. Proposition. *Assume P' is a reflection of the point P in the line (AB). Then $AP' = AP$ and if $A \neq P$ then $\measuredangle BAP' \equiv -\measuredangle BAP$.*



Proof. Note that if $P \in (AB)$ then $P = P'$ and by Corollary 2.9 $\measuredangle BAP = 0$ or π. Hence the statement follows.

If $P \notin (AB)$, then $P' \neq P$. By construction (AB) is perpendicular bisector of $[PP']$. Therefore, according to Theorem 5.2, $AP' = AP$ and $BP' = BP$.

In particular, $\triangle ABP' \cong \triangle ABP$. Therefore $\measuredangle BAP' \equiv \pm\measuredangle BAP$. Since $P' \neq P$ and $AP' = AP$, we get $\measuredangle BAP' \neq \measuredangle BAP$. I.e., we are left with the case

$$\measuredangle BAP' \equiv -\measuredangle BAP.$$

□

5.7. Corollary. *Reflection through the line is a motion of the plane. More over if $\triangle P'Q'R'$ is the reflection of $\triangle PQR$ then*

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$

Proof. From the construction it follows that the composition of two reflections through the same line, say (AB), is the identity map. In particular reflection is a bijection.

Assume P', Q' and R' denote the reflections of the points P, Q and R through (AB). Let us first show that

❶ $$P'Q' = PQ \quad \text{and} \quad \measuredangle AP'Q' \equiv -\measuredangle APQ.$$

Without loss of generality we may assume that the points P and Q are distinct from A and B. By Proposition 5.6,

$$\begin{aligned} \measuredangle BAP' &\equiv -\measuredangle BAP, & \measuredangle BAQ' &\equiv -\measuredangle BAQ, \\ AP' &= AP, & AQ' &= AQ. \end{aligned}$$

It follows that $\measuredangle P'AQ' \equiv -\measuredangle PAQ$. Therefore $\triangle P'AQ' \cong \triangle PAQ$ and ❶ follows.

Repeating the same argument for P and R, we get

$$\measuredangle AP'R' \equiv -\measuredangle APR.$$

Subtracting the second identity in ❶, we get

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$ □

5.8. Exercise. *Show that any motion of the plane can be presented as a composition of at most three reflections.*

Applying the exercise above and Corollary 5.7, we can divide the motions of the plane in two types, *direct* and *indirect motions*. The motion m is direct if

$$\measuredangle Q'P'R' = \measuredangle QPR$$

for any $\triangle PQR$ and $P' = m(P)$, $Q' = m(Q)$ and $R' = m(R)$; if instead we have

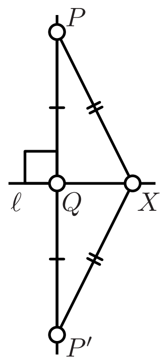
$$\measuredangle Q'P'R' \equiv -\measuredangle QPR$$

for any $\triangle PQR$ then the motion m is called indirect.

5.9. Lemma. *Let Q be the foot point of P on line ℓ. Then*

$$PX > PQ$$

for any point X on ℓ distinct from Q.



Proof. If $P \in \ell$ then the result follows since $PQ = 0$. Further we assume that $P \notin \ell$.

Let P' be the reflection of P in ℓ. Note that Q is the midpoint of $[PP']$ and ℓ is perpendicular bisector of $[PP']$. Therefore

$$PX = P'X \quad \text{and} \quad PQ = P'Q = \tfrac{1}{2}\cdot PP'$$

Note that ℓ meets $[PP']$ at the point Q only. Therefore by the triangle inequality and Exercise 4.6,

$$PX + P'X > PP'.$$

Hence the result follows. □

5.10. Exercise. *Let X and Y be the reflections of P through the lines (AB) and (BC) correspondingly. Show that*

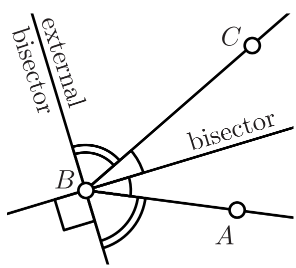
$$\measuredangle XBY \equiv 2\cdot\measuredangle ABC.$$

Angle bisectors

If $\measuredangle ABX \equiv -\measuredangle CBX$ then we say that line (BX) *bisects angle* $\angle ABC$, or line (BX) is a *bisector* of $\angle ABC$. If $\measuredangle ABX \equiv \pi - \measuredangle CBX$ then the line (BX) is called *external bisector* of $\angle ABC$.

Note that bisector and external bisector are uniquely defined by the angle.

Note that if $\measuredangle ABA' = \pi$, i.e., if B lies between A and A', then bisector of $\angle ABC$ is the external bisector of $\angle A'BC$ and the other way around.

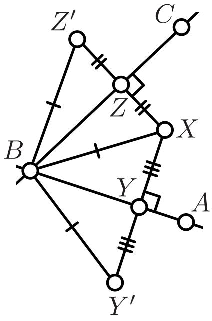


5.11. Exercise. *Show that for any angle, its bisector and external bisector are orthogonal.*

5.12. Lemma. *Given angle $\angle ABC$ and a point X, consider foot points Y and Z of X on (AB) and (BC). Assume $\measuredangle ABC \not\equiv \pi, 0$.*

Then $XY = XZ$ if and only if X lies on the bisector or external bisector of $\angle ABC$.

Proof. Let Y' and Z' be the reflections of X through (AB) and (BC) correspondingly. By Proposition 5.6, $XB = Y'B = Z'B$.



Note that

$$XY' = 2\cdot XY \quad \text{and} \quad XZ' = 2\cdot XZ.$$

Applying SSS and then SAS congruence conditions, we get

$$\begin{aligned} XY = XZ \;&\Leftrightarrow \\ \Leftrightarrow\; XY' = XZ' \;&\Leftrightarrow \\ \Leftrightarrow\; \triangle BXY' \cong \triangle BXZ' \;&\Leftrightarrow \\ \Leftrightarrow\; \measuredangle XBY' \equiv \pm\measuredangle BXZ'&. \end{aligned} \quad ❷$$

According to Proposition 5.6,

$$\begin{aligned} \measuredangle XBA &\equiv -Y'BA, \\ \measuredangle XBC &\equiv -Z'BC. \end{aligned}$$

Therefore

$$2\cdot\measuredangle XBA \equiv \measuredangle XBY' \quad \text{and} \quad 2\cdot\measuredangle XBC \equiv -XBZ'.$$

I.e., we can continue the chain of equivalence conditions ❷ the following way

$$\measuredangle XBY' \equiv \pm\measuredangle BXZ' \;\Leftrightarrow\; 2\cdot\measuredangle XBA \equiv \pm 2\cdot\measuredangle XBC.$$

Since $(AB) \neq (BC)$, we have

$$2\cdot\measuredangle XBA \not\equiv 2\cdot\measuredangle XBC$$

(compare to Exercise 2.11). Therefore

$$XY = XZ \;\Leftrightarrow\; 2\cdot\measuredangle XBA \equiv -2\cdot\measuredangle XBC.$$

The last identity means either

$$\measuredangle XBA + \measuredangle XBC \equiv 0$$

or

$$\measuredangle XBA + \measuredangle XBC \equiv \pi.$$

Hence the result follows. □

Circles

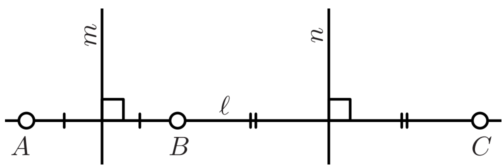
Given a positive real number r and a point O, the set Γ of all points on distant r from O is called *circle* with *radius* r and *center* O.

We say that a point P lies *inside* Γ if $OP < r$ and if $OP > r$, we say that P lies *outside* Γ.

A segment between two points on Γ is called *chord* of Γ. A chord passing through the center is called *diameter*.

5.13. Exercise. *Assume two distinct circles* Γ *and* Γ' *have a common chord* $[AB]$*. Show that the line between centers of* Γ *and* Γ' *forms a perpendicular bisector to* $[AB]$*.*

5.14. Lemma. *A line and a circle can have at most two points of intersection.*



Proof. Assume A, B and C are distinct points which lie on a line ℓ and a circle Γ with center O.

Then $OA = OB = OC$; in particular O lies on the perpendicular bisectors m and n to $[AB]$ and $[BC]$ correspondingly.

Note that the midpoints of $[AB]$ and $[BC]$ are distinct. Therefore m and n are distinct. The later contradicts the uniqueness of perpendicular (Theorem 5.5). □

5.15. Exercise. *Show that two distinct circles can have at most two points of intersection.*

In consequence of the above lemma, a line ℓ and a circle Γ might have 2, 1 or 0 points of intersections. In the first case the line is called *secant line*, in the second case it is *tangent line*; if P is the only point of intersection of ℓ and Γ, we say that ℓ *is tangent to* Γ *at* P.

Similarly, according Exercise 5.15, two circles might have 2, 1 or 0 points of intersections. If P is the only point of intersection of circles Γ and Γ', we say that Γ *is tangent to* Γ *at* P.

5.16. Lemma. *Let* ℓ *be a line and* Γ *be a circle with center* O*. Assume* P *is a common point of* ℓ *and* Γ*. Then* ℓ *is tangent to* Γ *at* P *if and only if and only if* $(PO) \perp \ell$*.*

Proof. Let Q be the foot point of O on ℓ.

Assume $P \neq Q$. Denote by P' the reflection of P through (OQ).

Note that $P' \in \ell$ and (OQ) is perpendicular bisector of $[PP']$. Therefore $OP = OP'$. Hence $P, P' \in \Gamma \cap \ell$; i.e., ℓ is secant to Γ.

If $P = Q$ then according to Lemma 5.9, $OP < OX$ for any point $X \in \ell$ distinct from P. Hence P is the only point in the intersection $\Gamma \cap \ell$; i.e., ℓ is tangent to Γ at P. □

5.17. Exercise. *Let Γ and Γ' be two circles with centers at O and O' correspondingly. Assume Γ and Γ' intersect at point P. Show that Γ is tangent to Γ' if and only if O, O' and P lie on one line.*

5.18. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' and radii r and r'.*

(a) Show that Γ is tangent to Γ' if and only if

$$OO' = r + r' \quad \text{or} \quad OO' = |r - r'|.$$

(b) Show that Γ intersects Γ' if and only if

$$|r - r'| \leqslant OO' \leqslant r + r'.$$

5.19. Exercise. *Assume three circles intersect at two points. Show that the centers of these circles lie on one line.*

Chapter 6

Parallel lines and similar triangles

Parallel lines

In consequence of Axiom I, any two distinct lines ℓ and m have either one point in common or none. In the first case they are *intersecting*; in the second case, ℓ and m are said to be *parallel* (briefly $\ell \parallel m$); in addition, a line is always regarded as parallel to itself.

6.1. Proposition. *Let ℓ, m and n be the lines in the plane. Assume that $n \perp m$ and $m \perp \ell$. Then $\ell \parallel n$.*

Proof. Assume contrary; i.e., $\ell \nparallel n$. Then there is a point, say Z, of intersection of ℓ and n. Then by Theorem 5.5, $\ell = n$. In particular $\ell \parallel n$, a contradiction. □

6.2. Theorem. *Given a point P and line ℓ in the Euclidean plane there is unique line m which pass though P and parallel to ℓ.*

The above theorem has two parts, existence and uniqueness. In the proof of uniqueness we will use Axiom IV for the first time.

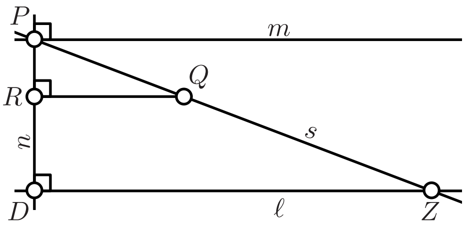
Proof; existence. Apply Theorem 5.5 two times, first to construct line m through P which is perpendicular to ℓ and second to construct line n through P which is perpendicular to m. Then apply Proposition 6.1.

Uniqueness. If $P \in \ell$ then $m = \ell$ by the definition of parallel lines. Further we assume $P \notin \ell$.

Let us construct lines $n \ni P$ and $m \ni P$ as in the proof of existence, so $m \parallel \ell$.

Assume there is yet an other line $s \ni P$ which is distinct from m and parallel to ℓ. Choose a point $Q \in s$ which lies with ℓ on the same side from m. Let R be the foot point of Q on n.

Let D be the point of intersection of n and ℓ. According to Proposition 6.1 $(QR) \parallel m$. Therefore Q, R and ℓ lie on the same side from m. In particular, $R \in [PD)$.



Choose $Z \in [PQ)$ such that

$$\frac{PZ}{PQ} = \frac{PD}{PR}.$$

Then by Axiom IV, $(ZD) \perp (PD)$; i.e. $Z \in \ell \cap s$, a contradiction. □

6.3. Corollary. *Assume ℓ, m and n are lines in the Euclidean plane such that $\ell \parallel m$ and $m \parallel n$. Then $\ell \parallel n$.*

Proof. Assume contrary; i.e. $\ell \nparallel n$. Then there is a point $P \in \ell \cap n$. By Theorem 6.2, $n = \ell$, a contradiction. □

Note that from the definition, we have $\ell \parallel m$ if and only if $m \parallel \parallel \ell$. Therefore according to the above corollary "$\parallel$" is an equivalence relation.

6.4. Exercise. *Let k, ℓ, m and n be the lines in Euclidean plane. Assume that $k \perp \ell$ and $m \perp n$. Show that if $k \parallel m$ then $\ell \parallel n$.*

Similar triangles

Two triangles $\triangle A'B'C'$ and $\triangle ABC$ are *similar* (briefly $\triangle A'B'C' \sim \sim \triangle ABC$) if their sides are proportional, i.e.,

❶ $$A'B' = k{\cdot}AB, \quad B'C' = k{\cdot}BC \quad \text{and} \quad C'A' = k{\cdot}CA$$

for some $k > 0$ and

❷ $$\begin{aligned} \measuredangle A'B'C' &= \pm\measuredangle ABC, \\ \measuredangle B'C'A' &= \pm\measuredangle BCA, \\ \measuredangle C'A'B' &= \pm\measuredangle CAB. \end{aligned}$$

Remarks.

- ⋄ According to 3.11, in the above three equalities the signs can be assumed to me the same.
- ⋄ If $\triangle A'B'C' \sim \triangle ABC$ with $k = 1$ in ❶, then $\triangle A'B'C' \cong \cong \triangle ABC$.
- ⋄ Note that "$\sim$" is an *equivalence relation.*
  I.e., if $\triangle A'B'C' \sim \triangle ABC$ then
  $$\triangle ABC \sim \triangle A'B'C'$$
  and if $\triangle A''B''C'' \sim \triangle A'B'C'$ and $\triangle A'B'C' \sim \triangle ABC$ then
  $$\triangle A''B''C'' \sim \triangle ABC.$$

Using "$\sim$", the Axiom IV can be formulated the following way.

6.5. Reformulation of Axiom IV. *If for two triangles $\triangle ABC$, $\triangle AB'C'$ and $k > 0$ we have $B' \in [AB)$, $C' \in [AC)$, $AB' = k{\cdot}AB$ and $AC' = k{\cdot}AC$ then $\triangle ABC \sim \triangle AB'C'$.*

In other words, the Axiom IV provides a condition which guarantee that two triangles are similar. Let us formulate yet three such conditions.

6.6. Similarity conditions. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ in the Euclidean plane are similar if one of the following conditions hold.*

(SAS) For some constant $k > 0$ we have

$$AB = k{\cdot}A'B',\ AC = k{\cdot}A'C'$$
$$\text{and}\ \ \measuredangle BAC = \pm\measuredangle B'A'C'.$$

(AA) The triangle $\triangle A'B'C'$ is nondegenerate and

$$\measuredangle ABC = \pm\measuredangle A'B'C',\ \measuredangle BAC = \pm\measuredangle B'A'C'.$$

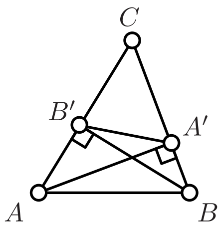
(SSS) For some constant $k > 0$ we have

$$AB = k{\cdot}A'B',\ AC = k{\cdot}A'C',\ CB = k{\cdot}C'B'.$$

Each of these conditions is proved by applying the Axiom IV with SAS, ASA and SSS congruence conditions correspondingly (see Axiom III and the conditions 4.1, 4.3).

Proof. Set $k = \frac{AB}{A'B'}$. Choose points $B'' \in [A'B')$ and $C'' \in [A'C')$ so that $A'B'' = k{\cdot}A'B'$ and $A'C'' = k{\cdot}A'C'$. By Axiom IV, $\triangle A'B'C' \sim \sim \triangle A'B''C''$.

Applying SAS, ASA or SSS congruence condition, depending on the case, we get $\triangle A'B''C'' \cong \triangle ABC$. Hence the result follows. ☐



A triangle with all acute angles is called *acute*.

6.7. Exercise. *Let $\triangle ABC$ be an acute triangle in the Euclidean plane. Denote by A' the foot point of A on (BC) and by B' the foot point of B on (AC). Prove that $\triangle A'B'C \sim \triangle ABC$.*

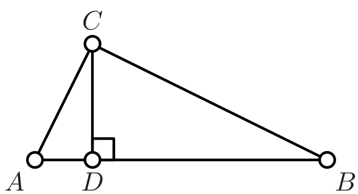
Pythagorean theorem

A triangle is called *right* if one of its angles is right. The side opposite the right angle is called the *hypotenuse*. The sides adjacent to the right angle are called *legs*.

6.8. Theorem. *Assume $\triangle ABC$ be a right triangle in the Euclidean plane with right angle at C. Then*

$$AC^2 + BC^2 = AB^2.$$

Proof. Let D be the foot point of C on (AB).



According to Lemma 5.9,

$$AD < AC < AB$$

and

$$BD < BC < AB.$$

Therefore D lies between A and B; in particular,

❸ $$AD + BD = AB.$$

Note that by AA similarity condition, we have

$$\triangle ADC \sim \triangle ACB \sim \triangle CDB.$$

In particular

❹ $$\frac{AD}{AC} = \frac{AC}{AB} \quad \text{and} \quad \frac{BD}{BC} = \frac{BC}{BA}.$$

Let us rewrite identities ❹ on an other way:

$$AC^2 = AB \cdot AD \quad \text{and} \quad BC^2 = AB \cdot BD.$$

summing up above two identities and applying ❸, we get

$$AC^2 + BC^2 = AB \cdot (AD + BD) = AB^2.$$

□

Angles of triangle

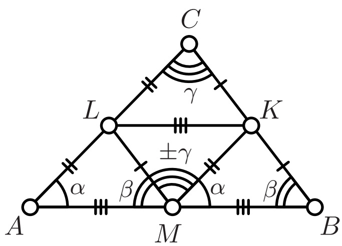
6.9. Theorem. *In any triangle $\triangle ABC$ in the Euclidean plane, we have*

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB \equiv \pi.$$

Proof. First note that if $\triangle ABC$ is degenerate then the equality follows from Lemma 2.8. Further we assume that $\triangle ABC$ is nondegenerate.

Set

$$\begin{aligned}\alpha &= \measuredangle CAB,\\ \beta &= \measuredangle ABC,\\ \gamma &= \measuredangle BCA.\end{aligned}$$



We need to prove that

❺ $$\alpha + \beta + \gamma \equiv \pi.$$

Let K, L, M be the midpoints of the sides $[BC]$, $[CA]$, $[AB]$ respectively. Observe that according to Axiom IV,

$$\begin{aligned}\triangle AML &\sim \triangle ABC,\\ \triangle MBK &\sim \triangle ABC,\\ \triangle LKC &\sim \triangle ABC\end{aligned}$$

and

$$LM = \tfrac{1}{2}\!\cdot\! BC, \quad MK = \tfrac{1}{2}\!\cdot\! CA, \quad KL = \tfrac{1}{2}\!\cdot\! AB.$$

According to SSS-condition (6.6), $\triangle KLM \sim \triangle ABC$. Thus,

❻ $$\measuredangle MKL = \pm\alpha, \quad \measuredangle KLM = \pm\beta, \quad \measuredangle BCA = \pm\gamma.$$

According to 3.11, the "+" or "−" sign is to be the same throughout ❻.

If in ❻ we have "+" then ❺ follows since

$$\beta + \gamma + \alpha \equiv \measuredangle AML + \measuredangle LMK + \measuredangle KMB \equiv \measuredangle AMB \equiv \pi$$

It remains to show that we can not have "−" in ❻. In this case the same argument as above gives

$$\alpha + \beta - \gamma \equiv \pi.$$

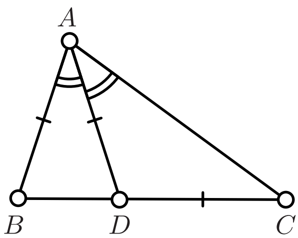
The same way we get

$$\alpha - \beta + \gamma \equiv \pi$$

Adding last two identities we get

$$2\cdot\alpha \equiv 0.$$

Equivalently $\alpha \equiv \pi$ or 0; i.e. $\triangle ABC$ is degenerate, a contradiction. □



6.10. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle. Assume there is a point $D \in [BC]$ such that (AD) bisects $\angle BAC$ and $BA = AD = DC$. Find the angles of $\triangle ABC$.*

6.11. Exercise. *Show that*

$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| = \pi.$$

for any $\triangle ABC$ in the Euclidean plane.

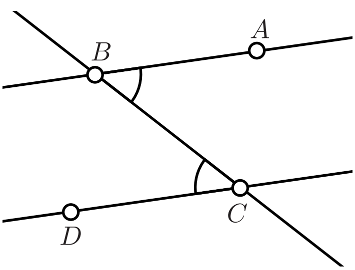
6.12. Corollary. *In the Euclidean plane, $(AB) \parallel (CD)$ if and only if*

❼ $$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 0.$$

Equivalently

$$\measuredangle ABC + \measuredangle BCD \equiv 0 \quad or \quad \measuredangle ABC + \measuredangle BCD \equiv \pi;$$

in the first case A and D lie on the opposite sides of (BC), in the second case A and D lie on the same sides of (BC).



Proof. If $(AB) \nparallel (CD)$ then there is $Z \in (AB) \cap (CD)$ and $\triangle BCZ$ is nondegenerate.

According to Theorem 6.9,

$$\measuredangle ZBC + \measuredangle BCZ \equiv \pi - \measuredangle CZB \not\equiv 0 \text{ or } \pi.$$

Note that $2\cdot\measuredangle ZBC \equiv 2\cdot\measuredangle ABC$ and $2\cdot\measuredangle BCZ \equiv 2\cdot\measuredangle BCD$. Therefore

$$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 2\cdot\measuredangle ZBC + 2\cdot\measuredangle BCZ \not\equiv 0;$$

i.e., ❼ does not hold.

It remains to note that the identity ❼ uniquely defines line (CD). Therefore by Theorem 6.2, if $(AB) \parallel (CD)$ then equality ❼ holds.

Applying Proposition 3.8, we get the last part of the corollary. □

Parallelograms

A *quadrilateral* is an ordered quadruple of pairwise distinct points in the plane. A quadrilateral formed by quadruple (A, B, C, D) will be called *quadrilateral* $ABCD$.

Given a quadrilateral $ABCD$, the four segments $[AB]$, $[BC]$, $[CD]$ and $[DA]$ are called *sides of* $ABCD$; the remaining two segments $[AC]$ and $[BD]$ are called *diagonals of* $ABCD$.

6.13. Exercise. *Show for any quadrilateral $ABCD$ in the Euclidean plane we have*

$$\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB \equiv 0$$

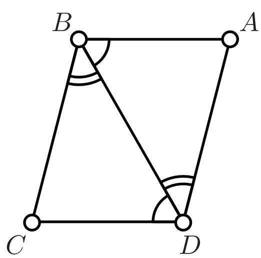
A quadrilateral $ABCD$ in the Euclidean plane is called *nondegenerate* if any three points from A, B, C, D do not lie on one line.

The nondegenerate quadrilateral $ABCD$ is called *parallelogram* if $(AB) \parallel (CD)$ and $(BC) \parallel (DA)$.

6.14. Lemma. *If $ABCD$ is a parallelogram then*

(a) $\measuredangle DAB = \measuredangle BCD$*;*

(b) $AB = CD$*.*

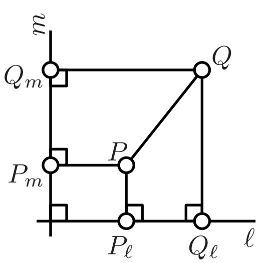


Proof. Since $(AB) \parallel (CD)$, the points C and D lie on the same side from (AB). Hence $\angle ABD$ and $\angle ABC$ have the same sign. Analogously, $\angle CBD$ and $\angle CBA$ have the same sign. Since $\angle ABC \equiv -\angle CBA$, we get that the angles $\angle DBA$ and $\angle DBC$ have opposite signs; i.e., A and C lie on the opposite sides of (BD).

According to Corollary 6.12,

$$\measuredangle BDC \equiv -\measuredangle DBA \quad \text{and} \quad \measuredangle DBC \equiv -\measuredangle BDA.$$

By angle-side-angle condition $\triangle ABD \cong \triangle CDB$. Which implies both statements in the lemma. □



6.15. Exercise. *Let ℓ and m be perpendicular lines in the Euclidean plane. Given a points P denote by P_ℓ and P_m the foot points of P on ℓ and m correspondingly.*

(a) Show that for any $X \in \ell$ and $Y \in m$ there is unique point P such that $P_\ell = X$ and $P_m = Y$.

(b) *Show that* $PQ^2 = P_\ell Q_\ell^2 + P_m Q_m^2$ *for any pair of points* P *and* Q.

(c) *Conclude that Euclidean plane is isometric to* $(\mathbb{R}^2, d_2)$ *defined on page 12.*

6.16. Exercise. *Use the Exercise 6.15, to give an alternative proof of Theorem 3.16 in the Euclidean plane.*

I.e., prove that given real numbers a, b *and* c *such that*

$$0 < a \leqslant b \leqslant c \leqslant a + c,$$

there is a triangle $\triangle ABC$ *such that* $a = BC$, $b = CA$ *and* $c = AB$.

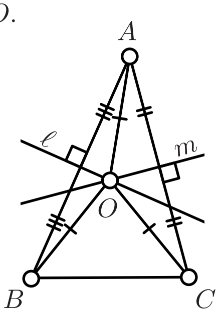
Chapter 7

Triangle geometry

Circumcircle and circumcenter

7.1. Theorem. *Perpendicular bisectors to the sides of any nondegenerate triangle in the Euclidean plane intersect at one point.*

The point of the intersection of the perpendicular bisectors is called circumcenter. It is the center of the circumcircle of the triangle; i.e., the circle which pass through all three vertices of the triangle. The circumcenter of the triangle is usually denoted by O.



Proof. Let $\triangle ABC$ be nondegenerate. Let ℓ and m be perpendicular bisectors to sides $[AB]$ and $[AC]$ correspondingly.

Assume ℓ and m intersect, let $O = \ell \cap n$. Since $O \in \ell$, we have $OA = OB$ and since $O \in m$, we have $OA = OC$. It follows that $OB = OC$; i.e. O lies on the perpendicular bisector to $[BC]$.

It remains to show that $\ell \nparallel m$; assume contrary. Since $\ell \perp (AB)$ and $m \perp (AC)$, we get $(AC) \parallel \parallel (AB)$ (see Exercise 6.4). Therefore by Theorem 5.5, $(AC) = (AB)$; i.e., $\triangle ABC$ is degenerate, a contradiction. $\square$

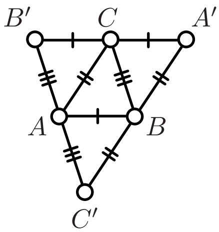
7.2. Corollary. *There is unique circle which pass through vertices of a given nondegenerate triangle in the Euclidean plane.*

Altitudes and orthocenter

An *altitude* of a triangle is a line through a vertex and perpendicular to the line containing the opposite side. The term *altitude* maybe also used for the distance from the vertex to its foot point on the line containing opposite side.

7.3. Theorem. *The three altitudes of any nondegenerate triangle in the Euclidean plane intersect in a single point.*

The point of intersection of altitudes is called *orthocenter*; it is usually denoted as H.



Proof. Let $\triangle ABC$ be nondegenerate.

Consider three lines $\ell \parallel (BC)$ through A, $m \parallel (CA)$ through B and $n \parallel (AB)$ through C. Since $\triangle ABC$ is nondegenerate, the lines ℓ, m and n are not parallel. Set $A' = m \cap n$, $B' = n \cap \ell$ and $C' = \ell \cap m$.

Note that $ABA'C$, $BCB'A$ and $CBC'A$ are parallelograms. Applying Lemma 6.14 we get that $\triangle ABC$ is the median triangle of $\triangle A'B'C'$; i.e., A, B and C are the midpoints of $[B'C']$, $[C'A']$ and $[A'B']$ correspondingly. By Exercise 6.4, $(B'C') \parallel (BC)$, the altitudes from A is perpendicular to $[B'C']$ and from above it bisects $[B'C']$.

Thus altitudes of $\triangle ABC$ are also perpendicular bisectors of the triangle $\triangle A'B'C'$. Applying Theorem 7.1, we get that altitudes of $\triangle ABC$ intersect at one point. □

7.4. Exercise. *Assume H is the orthocenter of an acute triangle $\triangle ABC$ in the Euclidean plane. Show that A is orthocenter of $\triangle HBC$.*

Medians and centroid

A median of a triangle is a segment joining a vertex to the midpoint of the opposing side.

7.5. Theorem. *The three medians of any nondegenerate triangle in the Euclidean plane intersect in a single point. Moreover the point of intersection divides each median in ratio 2:1.*

The point of intersection of medians is called *centroid*; it is usually denoted by M.

Proof. Consider a nondegenerate triangle $\triangle ABC$. Let $[AA']$ and $[BB']$ be its medians.

According to Exercise 3.13, $[AA']$ and $[BB']$ are intersecting. Let us denote by M the point of intersection.

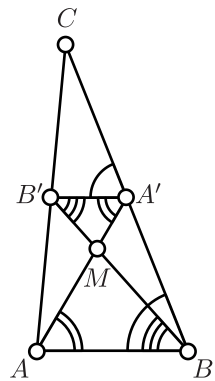
By side-angle-side condition, $\triangle B'A'C \sim \triangle ABC$ and $A'B' = \frac{1}{2}\cdot AB$. In particular $\measuredangle ABC \equiv \measuredangle B'A'C$.

Since A' lies between B and C, we get $\measuredangle BA'B' + \measuredangle B'A'C = \pi$. Therefore

$$\measuredangle B'A'B + \measuredangle A'BC = \pi.$$

By Corollary 6.12 $(AB) \parallel (A'B')$.

Note that A' and A lie on the opposite sides from (BB'). Therefore by Corollary 6.12 we get



$$\measuredangle B'A'M = \measuredangle BAM.$$

The same way we get,

$$\measuredangle A'B'M = \measuredangle ABM.$$

By AA condition, $\triangle ABM \sim \triangle A'B'M$.

Since $A'B' = \frac{1}{2}\cdot AB$, we have

$$\frac{A'M}{AM} = \frac{B'M}{BM} = \frac{1}{2}.$$

In particular M divides medians $[AA']$ and $[BB']$ in ratio 2:1.

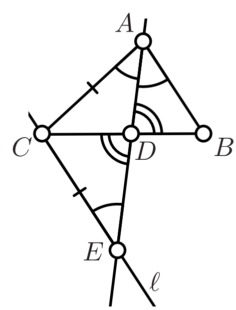
Note that M is unique point on $[BB']$ such that

$$\frac{B'M}{BM} = \frac{1}{2}.$$

Repeating the same argument for vertices B and C we get that all medians $[CC']$ and $[BB']$ intersect in M. □

Bisector of triangle

7.6. Lemma. *Let $\triangle ABC$ be a nondegenerate triangle in the Euclidean plane. Assume that the bisector of $\angle BAC$ intersects $[BC]$ at the point D. Then*



❶ $$\frac{AB}{AC} = \frac{DB}{DC}.$$

Proof. Let ℓ be the line through C parallel to (AB). Note that $\ell \nparallel (AD)$; set $E = \ell \cap (AD)$.

Note that B and C lie on the opposite sides of (AD). Therefore by Corollary 6.12,

❷ $$\measuredangle BAD = \measuredangle CED.$$

Further, note that $\angle ADB$ and $\angle EDC$ are vertical; in particular, by 2.12

$$\measuredangle ADB = \measuredangle EDC.$$

By AA-similarity condition, $\triangle ABD \sim \triangle ECD$. In particular,

❸ $$\frac{AB}{EC} = \frac{DB}{DC}.$$

Since (AD) bisects $\angle BAC$, we get $\measuredangle BAD = \measuredangle DAC$. Together with ❷, it implies that $\measuredangle CEA = \measuredangle EAC$. By Theorem 4.2, $\triangle ACE$ is isosceles; i.e.

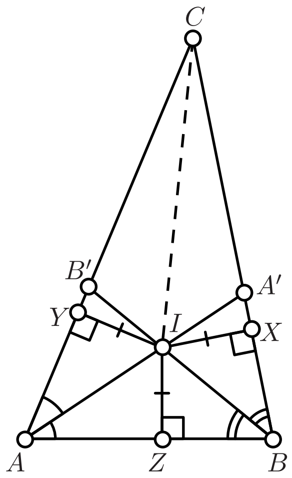
$$EC = AC.$$

The later together with ❸ implies ❶. □

7.7. Exercise. *Prove an analog of Lemma 7.6 for the external bisector.*

Incenter

7.8. Theorem. *The angle bisectors of any nondegenerate triangle intersect at one point.*



The point of intersection of bisectors is called *incenter*; it is usually denoted as I. The point I lies on the same distance from each side, it is the center of a circle tangent to each side of triangle. This circle is called *incircle* and its radius is called *inradius* of the triangle.

Proof. Let $\triangle ABC$ be a nondegenerate triangle.

Note that points B and C lie on the opposite sides from the bisector of $\angle BAC$. Hence this bisector intersects $[BC]$ at a point, say A'.

Analogously, there is $B' \in [AC]$ such the (BB') bisects $\angle ABC$.

Applying Pasch's theorem (3.10), twice for the triangles $\triangle AA'C$ and $\triangle BB'C$, we get that $[AA']$ and $[BB']$ intersect. Let us denote by I the point of intersection.

Let X, Y and Z be the foot points of I on (BC), (CA) and (AB) correspondingly. Applying Lemma 5.12, we get

$$IY = IZ = IX.$$

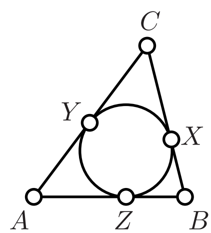
From the same lemma we get that I lies on a bisector or exterior bisector of $\angle BCA$.

The line (CI) intersects $[BB']$, the points B and B' lie on opposite sides of (CI). Therefore the angles $\angle ICA = \angle ICB'$ and $\angle ICB$ have opposite signs. I.e., (CI) can not be exterior bisector of $\angle BCA$. Hence the result follows. □

More exercises

7.9. Exercise. *Assume that bisector at one vertex of a nondegenerate triangle bisects the opposite side. Show that the triangle is isosceles.*

7.10. Exercise. *Assume that at one vertex of a nondegenerate triangle bisector coincides with the altitude. Show that the triangle is isosceles.*



7.11. Exercise. *Assume sides $[BC]$, $[CA]$ and $[AB]$ of $\triangle ABC$ are tangent to incircle at X, Y and Z correspondingly. Show that*

$$AY = AZ = \tfrac{1}{2}\cdot(AB + AC - BC).$$

By the definition, the *orthic triangle* is formed by the base points of its altitudes of the given triangle.

7.12. Exercise. *Prove that orthocenter of an acute triangle coincides with incenter of its orthic triangle.*

What should be an analog of this statement for an obtuse triangle?

Inversive geometry

Chapter 8

Inscribed angles

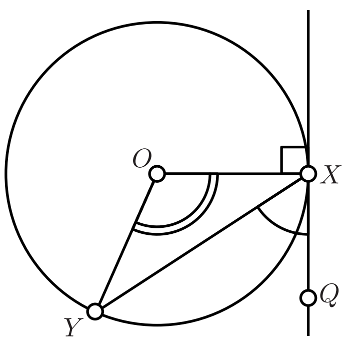
Angle between a tangent line and a chord

8.1. Theorem. *Let Γ be a circle with center O in the Euclidean plane. Assume line (XQ) is tangent to Γ at X and $[XY]$ is a chord of Γ. Then*

❶ $$2\cdot\measuredangle QXY \equiv \measuredangle XOY.$$

Equivalently,

$$\measuredangle QXY \equiv \tfrac{1}{2}\cdot\measuredangle XOY \quad \textit{or} \quad \measuredangle QXY \equiv \tfrac{1}{2}\cdot\measuredangle XOY + \pi.$$



Proof. Note that $\triangle XOY$ is isosceles. Therefore $\measuredangle YXO = \measuredangle OYX$.

Applying Theorem 6.9 to $\triangle XOY$, we get

$$\begin{aligned}\pi &\equiv \measuredangle YXO + \measuredangle OYX + \measuredangle XOY \equiv \\ &\equiv 2\cdot\measuredangle YXO + \measuredangle XOY.\end{aligned}$$

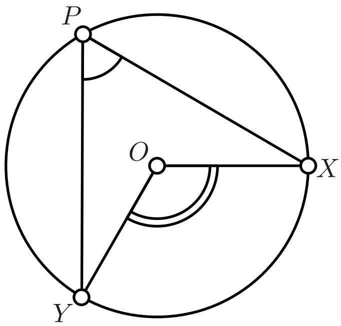
By Lemma 5.16, $(OX) \perp (XQ)$. Therefore

$$\measuredangle QXY + \measuredangle YXO \equiv \pm\tfrac{\pi}{2}.$$

Therefore

$$2\cdot\measuredangle QXY \equiv \pi - 2\cdot\measuredangle YXO \equiv \measuredangle XOY.$$ □

Inscribed angle



We say that triangle is *inscribed* in the circle Γ if all its vertices lie on Γ.

8.2. Theorem. *Let Γ be a circle with center O in the Euclidean plane, and X, Y be two distinct points on Γ. Then $\triangle XPY$ is inscribed in Γ if and only if*

❷ $$2{\cdot}\measuredangle XPY \equiv \measuredangle XOY.$$

Equivalently, if and only if

$$\measuredangle XPY \equiv \tfrac{1}{2}{\cdot}\measuredangle XOY \quad \textit{or} \quad \measuredangle XPY \equiv \tfrac{1}{2}{\cdot}\measuredangle XOY + \pi.$$

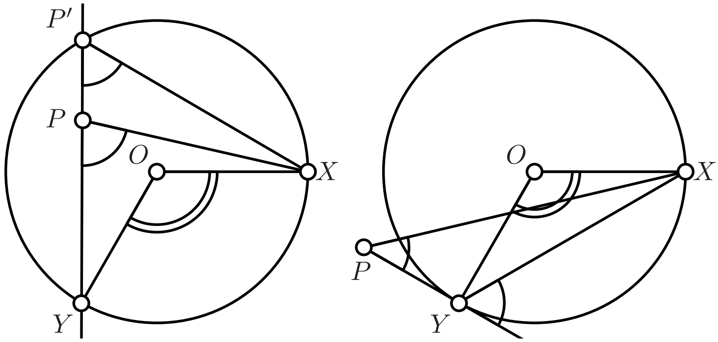
Proof. Choose a point Q such that $(PQ) \perp (OP)$. By Lemma 5.16, (PQ) is tangent to Γ.

According to Theorem 8.1,

$$2{\cdot}\measuredangle QPX \equiv \measuredangle POX,$$
$$2{\cdot}\measuredangle QPY \equiv \measuredangle POY.$$

Subtracting one identity from the other we get ❷.

To prove the converse, let us argue by contradiction. Assume that ❷ holds for some $P \notin \Gamma$. Note that $\measuredangle XOY \neq 0$ and therefore $\measuredangle XPY$ is distinct from 0 and π; i.e., $\triangle PXY$ is nondegenerate.



If the line (PY) is secant to Γ, denote by P' the point of intersection of Γ and (PY) which is distinct from Y. From above we get

$$2{\cdot}\measuredangle XP'Y \equiv \measuredangle XOY.$$

In particular,

$$2 \cdot \measuredangle XP'Y \equiv 2 \cdot \measuredangle XPY.$$

By Corollary 6.12, $(P'X) \parallel (PX)$. Since $\triangle PXY$ is nondegenerate, the later implies $P = P'$, which contradicts $P \notin \Gamma$.

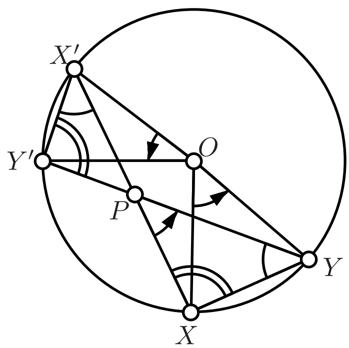
In the remaining case, if (PX) is tangent to Γ, the proof goes along the same lines. Namely, by Theorem 8.1,

$$2 \cdot \measuredangle PYX \equiv \measuredangle XOY.$$

In particular,

$$2 \cdot \measuredangle PYX \equiv 2 \cdot \measuredangle XPY.$$

By Corollary 6.12, $(PY) \parallel (XY)$; therefore $(PY) = (XY)$. I.e., $\triangle PXY$ is degenerate, a contradiction. □



8.3. Exercise. *Let $[XX']$ and $[YY']$ be two chords of circle Γ with center O and radius r in the Euclidean plane. Assume (XX') and (YY') intersect at point P. Show that*

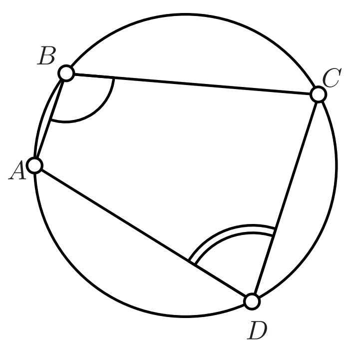
(a) $2 \cdot \measuredangle XPY = \measuredangle XOY + \measuredangle X'OY'$;
(b) $\triangle PXY \sim \triangle PY'X'$;
(c) $PX \cdot PX' = |OP^2 - r^2|$.

8.4. Exercise. *Assume that the chords $[XX']$, $[YY']$ and $[ZZ']$ of the circle Γ in the Euclidean plane intersect at one point. Show that*

$$XY' \cdot ZX' \cdot YZ' = X'Y \cdot Z'X \cdot Y'Z.$$

Inscribed quadrilateral

A quadrilateral $ABCD$ is called *inscribed* if all the points A, B, C and D lie on a circle or a line.



8.5. Theorem. *A quadrilateral $ABCD$ in the Euclidean plane is inscribed if and only if*

❸ $\quad 2 \cdot \measuredangle ABC + 2 \cdot \measuredangle CDA \equiv 0.$

Equivalently, if and only if

$$\measuredangle ABC + \measuredangle CDA \equiv \pi \quad \text{or} \quad \measuredangle ABC \equiv -\measuredangle CDA.$$

Proof. Assume $\triangle ABC$ is degenerate. By Corollary 2.9,

$$2\cdot\measuredangle ABC \equiv 0;$$

From the same corollary, we get

$$2\cdot\measuredangle CDA \equiv 0$$

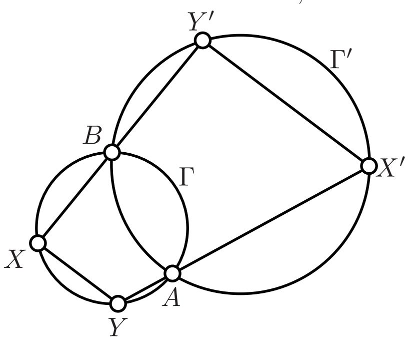
if and only if $D \in (AB)$; hence the result follows.

It remains to consider the case if $\triangle ABC$ is nondegenerate.

Denote by Γ the circumcircle of $\triangle ABC$ and let O be the center of Γ. According to Theorem 8.2,

❹ $$2\cdot\measuredangle ABC \equiv \measuredangle AOB.$$

From the same theorem, $D \in \Gamma$ if and only if



❺ $$2\cdot\measuredangle CDA \equiv \measuredangle BOA.$$

Adding ❹ and ❺, we get the result. □

8.6. Exercise. *Let Γ and Γ' be two circles which intersect at two distinct points A and B. Assume $[XY]$ and $[X'Y']$ be the chords of Γ and Γ' correspondingly such that A lies between X and X' and B lies between Y and Y'. Show that $(XY) \parallel (X'Y')$.*

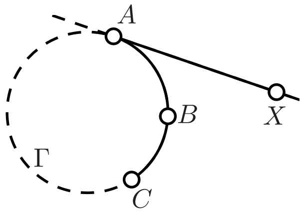
8.7. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle in the Euclidean plane, A' and B' be foot points of altitudes from A and B. Show that A, B, A' and B' lie on one circle.*

What is the center of this circle?

Arcs

A subset of a circle bounded by two points is called a circle arc.

More precisely, let Γ be a circle and $A, B, C \in \Gamma$ be three distinct points. The subset which includes the points A, C as well as all the points on Γ which lie with B on the same side from (AC) is called *circle arc ABC*.



For the circle arc ABC, the points A and C are called *endpoints*. Note that given two distinct points A and C there are two circle arcs of Γ with the endpoints at A and C.

A half-line $[AX)$ is called *tangent* to arc ABC at A if the line (AX) is tangent to Γ and the points X and B lie on the same side from the line (AC).

If B lies on the line (AC), the arc ABC degenerates to one of two following a subsets of line (AC).

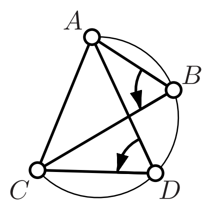
- ◇ If B lies between A and C then we define the arc ABC as the segment $[AC]$. In this case the half-line $[AC)$ is tangent to the arc ABC at A.
- ◇ If $B \in (AC)\backslash[AC]$ then we define the arc ABC as the line (AC) without all the points between A and C. If we choose points X and $Y \in (AC)$ such that the points X, A, C and Y appear in the same order on the line then the arc ABC is formed by two half-lines in $[AX)$ and $[CY)$. The half-line $[AX)$ is tangent to the arc ABC at A.
- ◇ In addition, any half-line $[AB)$ will be regarded as an arc. This degenerate arc has only one end point A and it assumed to be tangent to itself at A.

The circle arcs together with the degenerate arcs will be called *arcs*.

8.8. Proposition. *In the Euclidean plane, a point D lies on the arc ABC if and only if*

$$\measuredangle ADC = \measuredangle ABC$$

or D coincides with A or C.



Proof. Note that if A, B and C lie on one line then the statement is evident.

Assume Γ be the circle passing through A, B and C.

Assume D is distinct from A and C. According to Theorem 8.5, $D \in \Gamma$ if and only if

$$\measuredangle ADC = \measuredangle ABC \quad \text{or} \quad \measuredangle ADC \equiv \measuredangle ABC + \pi.$$

By Exercise 3.12, the first identity holds then B and D lie on one side of (AC); i.e., D belongs to the arc ABC. If the second identity holds then the points B and D lie on the opposite sides from (AC), in this case D does not belong to the arc ABC. □

8.9. Proposition. *In the Euclidean plane, a half-lines $[AX)$ is tangent to the arc ABC if and only if*

$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$

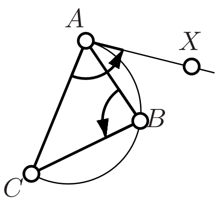
Proof. Note that for a degenerate arc ABC the statement is evident. Further we assume the arc ABC is nondegenerate.

Applying theorems 8.1 and 8.2, we get

$$2{\cdot}\measuredangle ABC + 2{\cdot}\measuredangle CAX \equiv 0.$$

Therefore either

$$\measuredangle ABC + \measuredangle CAX \equiv \pi \quad \text{or} \quad \measuredangle ABC + \measuredangle CAX \equiv 0.$$



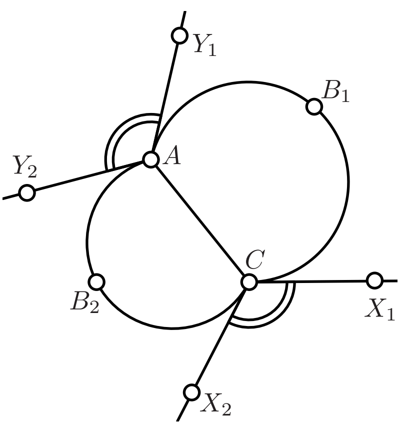
Since $[AX)$ is the tangent half-line to the arc ABC, X and B lie on the same side from (AC). Therefore the angles $\angle CAX$, $\angle CAB$ and $\angle ABC$ have the same sign. In particular $\measuredangle ABC + \measuredangle CAX \not\equiv 0$; i.e., we are left with the case

$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$

□

8.10. Exercise. *Assume that in the Euclidean plane, the half-lines $[AX)$ and $[AY)$ are tangent to the arcs ABC and ACB correspondingly. Show that $\angle XAY$ is straight.*

8.11. Exercise. *Show that in the Euclidean plane, there is unique arc with endpoints at the given points A and C which is tangent at A to the given half line $[AX)$.*



8.12. Exercise. *Consider two arcs AB_1C and AB_2C in the Euclidean plane. Let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to arcs AB_1C and AB_2C at A and $[CY_1)$ and $[CY_2)$ be the half-lines tangent to arcs AB_1C and AB_2C at C. Show that*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1CY_2.$$

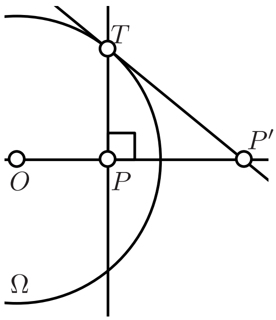
Chapter 9

Inversion

Let Ω be the circle with center O and radius r. The *inversion* of a point P with respect to Ω is the point $P' \in [OP)$ such that

$$OP \cdot OP' = r^2.$$

In this case the circle will be called the *circle of inversion* and its center is called *center of inversion*.



The inversion of O is undefined. If P is inside Ω then P' is outside and the other way around. Further, $P = P'$ if and only if $P \in \Omega$.

Note that the inversion takes P' back to P.

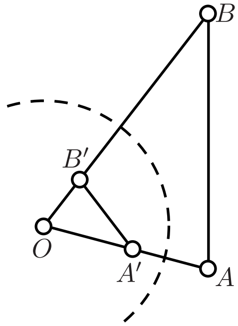
9.1. Exercise. *Let P be a point inside of a circle Ω centered at O in the Euclidean plane. Let T be a point where the perpendicular to (OP) from P intersects Ω. Let P' be the point where the tangent to Ω at T intersects (OP). Show that P' is the inversion of P in the circle Ω.*

9.2. Lemma. *Let A' and B' be inversions of A and B with respect to a circle of center O in the Euclidean plane. Then*

$$\triangle OAB \sim \triangle OB'A'.$$

Moreover,

$$\measuredangle AOB \equiv -\measuredangle B'OA',$$
❶
$$\measuredangle OBA \equiv -\measuredangle OA'B',$$
$$\measuredangle BAO \equiv -\measuredangle A'B'O.$$



Proof. Let r be the radius of the circle of the inversion.

From the definition of inversion, we get

$$OA{\cdot}OA' = OB{\cdot}OB' = r^2.$$

Therefore

$$\frac{OA}{OB'} = \frac{OB}{OA'}.$$

Clearly

$$❷ \qquad \measuredangle AOB = \measuredangle A'OB' \equiv -\measuredangle B'OA'.$$

From SAS, we get

$$\triangle OAB \sim \triangle OB'A'.$$

Applying Theorem 3.11 and ❷, we get ❶. □

9.3. Exercise. *Let P' be the inversion of P in the circle Γ. Assume that $P \ne P'$. Show that the value $\frac{PX}{P'X}$ is the same for all $X \in \Gamma$.*

The converse to the above exercise also holds. Namely, given positive real number $k \ne 1$ and two distinct points P and P' in the Euclidean plane the locus of points X such that $\frac{PX}{P'X} = k$ forms a circle which is called *circle of Apollonius*. In this case P' is inverse of P in the circle of Apollonius.

9.4. Exercise. *Let A', B', C' be the images of A, B, C under inversion in the incircle of $\triangle ABC$ in the Euclidean plane. Show that the incenter of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$.*

Cross-ratio

Although inversion changes the distances and angles, some quantities expressed in distances or angles do not change after inversion. The following theorem gives the simplest examples of such quantities.

9.5. Theorem. *Let $ABCD$ and $A'B'C'D'$ be two quadrilaterals in the Euclidean plane such that the points A', B', C' and D' are inversions of A, B, C, and D correspondingly.*

Then

(a)

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(b)

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle A'B'C' + \measuredangle C'D'A').$$

(c) If quadrilateral $ABCD$ is inscribed then so is $A'B'C'D'$.

Proof; (a). Let O be the center of inversion. According to Lemma 9.2, $\triangle AOB \sim \triangle B'OA'$. Therefore

$$\frac{AB}{A'B'} = \frac{OA}{OB'}.$$

Analogously,

$$\frac{BC}{B'C'} = \frac{OC}{OB'}, \qquad \frac{CD}{C'D'} = \frac{OC}{OD'}, \qquad \frac{DA}{D'A'} = \frac{OA}{OD'}.$$

Therefore

$$\frac{AB}{A'B'} \cdot \frac{B'C'}{BC} \cdot \frac{CD}{C'D'} \cdot \frac{D'A'}{DA} = \frac{OA}{OB'} \cdot \frac{OB'}{OC} \cdot \frac{OC}{OD'} \cdot \frac{OD'}{OA} = 1.$$

Hence (a) follows.

(b). According to Lemma 9.2,

$$\begin{aligned} \measuredangle ABO &\equiv -\measuredangle B'A'O, & \measuredangle OBC &\equiv -\measuredangle OA'B', \\ \measuredangle CDO &\equiv -\measuredangle D'C'O, & \measuredangle ODA &\equiv -\measuredangle OA'D'. \end{aligned}$$

Summing these four identities we get

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle D'C'B' + \measuredangle B'A'D').$$

Applying Axiom IIb and Exercise 6.13, we get

$$\begin{aligned} \measuredangle A'B'C' + \measuredangle C'D'A' &\equiv -(\measuredangle B'C'D' + \measuredangle D'A'B') \equiv \\ &\equiv \measuredangle D'C'B' + \measuredangle B'A'D'. \end{aligned}$$

Hence (b) follows.

(c). Follows from (b) and Theorem 8.5. □

Inversive plane and clines

Let Ω be a circle with center O and radius r. Consider the inversion in Ω.

Recall that inversion of O is not defined. To deal with this problem it is useful to add to the plane an extra point; it will be called *the point at infinity* and we will denote it as ∞. W e can assume that ∞ is inversion of O and the other way around.

The Euclidean plane with added a point at infinity is called *inversive plane.*

We will always assume that any line and half-line contains ∞.

It will be convenient to use notion of *cline*, which means *circle or line*; for example we may say *if cline contains* ∞ *then it is a line* or *cline which does not contain* ∞ *is a circle.*

Note that according to Theorem 7.1, for any $\triangle ABC$ there is unique cline which pass through A, B and C.

9.6. Theorem. *In the inversive plane, inversion of a cline is a cline.*

Proof. Denote by O the center of inverse.

Let Γ be a cline. Choose three distinct points A, B and C on Γ. (If $\triangle ABC$ is nondegenerate then Γ is the circumcircle of $\triangle ABC$; if $\triangle ABC$ is degenerate then Γ is the line passing through A, B and C.)

Denote by A', B' and C' the inversions of A, B and C correspondingly. Let Γ' be the cline which pass though A', B' and C'. According to 7.1, Γ' is well defined.

Assume D is a point of inversive plane which is distinct from A, C, O and ∞. According to Theorem 8.5, $D \in \Gamma$ if and only if

$$2\cdot\measuredangle CDA + 2\cdot\measuredangle ABC \equiv 0.$$

According to Theorem 9.5b, the later is equivalent to

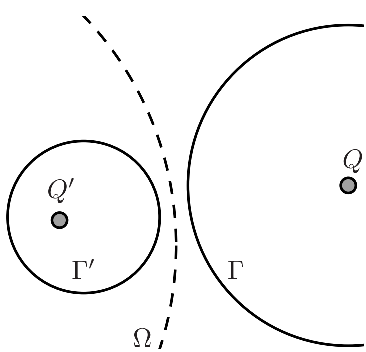
$$2\cdot\measuredangle C'D'A' + 2\cdot\measuredangle A'B'C' \equiv 0.$$

Applying Theorem 8.5 again, we get that the later is equivalent to $D' \in \Gamma'$. Hence the result follows.

It remains to prove that $O \in \Gamma \ \Leftrightarrow\ \infty \in \Gamma'$ and $\infty \in \Gamma \ \Leftrightarrow\ O \in \Gamma'$. Since Γ is inversion of Γ' it is sufficient to prove only

$$\infty \in \Gamma \ \Leftrightarrow\ O \in \Gamma'.$$

Since $\infty \in \Gamma$ we get that Γ is a line. Therefore for any $\varepsilon > 0$, the line Γ contains point P with $OP > r^2/\varepsilon$. For the inversion $P' \in \Gamma'$ of P, we have $OP' = r^2/OP < \varepsilon$. I.e., the cline Γ' contains points arbitrary close to O. It follows that $O \in \Gamma'$. ☐



9.7. Exercise. *Assume that if circle* Γ' *is the inversion of circle* Γ *in* *the Euclidean plane. Denote by Q the center of Γ and by Q' the inversion of Q.*

Show that Q' is not the center of Γ'.

9.8. Exercise. *Show that for any pair of tangent circles in the inversive plane there is an inversion which sends them to a pair of parallel lines.*

9.9. Theorem. *Consider inversion with respect to circle Ω with center O in the inversive plane. Then*

(a) Line passing through O is inverted into itself.

(b) Line not passing through O is inverted into a circle which pass through O, and the other way around.

(c) A circle not passing through O is inverted into a circle not passing through O.

Proof. In the proof we use Theorem 9.6 without mentioning.

(a). Note that if line passing through O it contains both ∞ and O. Therefore its inversion also contains ∞ and O. In particular image is a line passing through O.

(b). Since any line ℓ pass through ∞, its image ℓ' has to contain O. If the line did not contain O then $\ell' \not\ni \infty$. Therefore ℓ' is a circle which pass through O.

(c). If circle Γ does not contain O then its image Γ' does not contain ∞. Therefore Γ' is a circle. Since $\Gamma \not\ni \infty$ we get $\Gamma' \not\ni O$. Hence the result follows. □

Ptolemy's identity

Here is one application of inversion, which we include as an illustration only.

9.10. Theorem. *Let $ABCD$ be an inscribed quadrilateral in the Euclidean plane. Assume that the points A, B, C and D appear on the cline in the same order. Then*

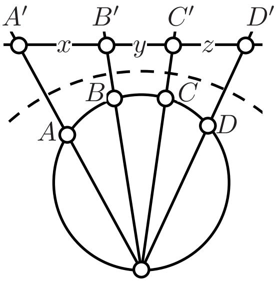
$$AB{\cdot}CD + BC{\cdot}DA = AC{\cdot}BD$$

Proof. Assume the points A, B, C, D lie on one line in this order.

Set $x = AB$, $y = BC$, $z = CD$. Note that

$$x{\cdot}z + y{\cdot}(x + y + z) = (x + y){\cdot}(y + z).$$

Since $AC = x + y$, $BD = y + z$ and $DA = x + y + z$, it proves the identity.



It remains to consider the case when quadrilateral $ABCD$ is inscribed in a circle, say Γ.

The identity can be rewritten as

$$\frac{AB\cdot DC}{BD\cdot CA} + \frac{BC\cdot AD}{CA\cdot DB} = 1.$$

On the left hand side we have two cross-ratios. According to Theorem 9.5(a), the left hand side does not change if we apply an inversion to each point.

Consider an inversion in a circle centered at a point O which lie on Γ between A and D. By Theorem 9.9, this inversion maps Γ to a line. This reduces the problem to the case when A, B, C and D lie on one line, which was already considered. □

Perpendicular circles

Assume two circles Γ and Δ intersect at two points say X and X'. Let ℓ and m be tangent lines at X to Γ and Δ correspondingly. Analogously, ℓ' and m' be tangent lines at X' to Γ and Δ.

From Exercise 8.12, we get that $\ell \perp m$ if and only if $\ell' \perp m'$.

We say that circle Γ is *perpendicular* to circle Δ (briefly $\Gamma \perp \Delta$) if they intersect and the lines tangent to the circle at one point (and therefore both points) of intersection are perpendicular.

Similarly, we say that circle Γ is perpendicular to a line ℓ (briefly $\Gamma \perp \ell$) if $\Gamma \cap \ell \neq \varnothing$ and ℓ perpendicular to the tangent lines to Γ at one point (and therefore both points) of intersection. According to Lemma 5.16, it happens only if the line ℓ pass through the center of Γ.

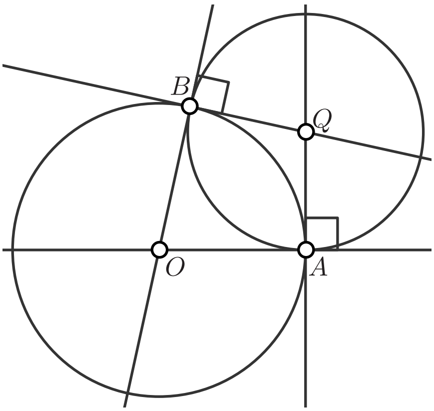
Now we can talk about perpendicular clines.

9.11. Theorem. *Assume Γ and Ω are distinct circles in the Euclidean plane. Then $\Omega \perp \Gamma$ if and only if the circle Γ coincides with its inversion in Ω.*

Proof. Denote by Γ' the inversion of Γ.

($\Rightarrow$) Let O be the center of Ω and Q be the center of Γ. Denote by A and B the points of intersections of Γ and Ω. According to Lemma 5.16, $\Gamma \perp \Omega$ if and only if (OA) and (OB) are tangent to Γ.

Note that Γ' also tangent to (OA) and (OB) at A and B correspondingly. It follows that A and B are the foot points of the center of Γ' on (OA) and (OB). Therefore both Γ' and Γ have the center Q. Finally, $\Gamma' = \Gamma$, since both circles pass through A.



(⇐) Assume $\Gamma = \Gamma'$.

Since $\Gamma \neq \Omega$, there is a point P which lies on Γ, but not on Ω. Let P' be the inversion of P in Ω. Since $\Gamma = \Gamma'$, we have $P' \in \Gamma$, in particular the half-line $[OP)$ intersects Γ at two points; i.e., O lies outside of Γ.

As Γ has points inside and outside Ω, the circles Γ and Ω intersect. The later follows from Exercise 5.18(b). Let A be a point of their intersection; we need to show that A is the only intersection point of (OA) and Γ. Assume X is an other point of intersection. Since O is outside of Γ, the point X lies on the half-line $[OA)$.

Denote by X' the inversion of X in Ω. Clearly the three points X, X', A lie on Γ and (OA). The later contradicts Lemma 5.14. □

9.12. Corollary. *A cline in the inversive plane which is distinct from the circle of inversion inverts to itself if and only if it is perpendicular to the circle of inversion.*

Proof. By Theorem 9.11, it is sufficient to consider the case when the cline is a line. The later follows from Theorem 9.9. □

9.13. Corollary. *Let P and P' be two distinct points in the Euclidean plane such that P' is the inversion of P in the circle Ω. Assume that a cline Γ pass through P and P'. Then $\Gamma \perp \Omega$.*

Proof. Without loss of generality we may assume that P is inside and P' is outside Ω. It follows that Γ intersects Ω; denote by A a point of intersection.[0]

Denote by Γ' the inversion of Γ. Since A is inversion of itself, the points A, P and P' lie on Γ; therefore $\Gamma' = \Gamma$. By Theorem 9.11, $\Gamma \perp \Omega$. □

9.14. Corollary. *Let P and Q be two distinct points inside the circle Ω in the Euclidean plane. Then there is unique cline Γ perpendicular to Ω which pass through P and Q.*

Proof. Let P' be the inversion of point P in a circle Ω. According to Corollary 9.13, the cline passing through P and Q is perpendicular to Ω if and only if it pass though P'.

Note that P' lies outside of Ω. Therefore the points P, P' and Q are distinct.

According to Corollary 7.2, there is unique cline passing through P, Q and P'. Hence the result follows. □

9.15. Exercise. *Let Ω_1 and Ω_2 be two distinct circles in the Euclidean plane. Assume that the point P does not lie on Ω_1 nor on Ω_2. Show that there is unique cline passing through P which is perpendicular Ω_1 and Ω_2.*

9.16. Exercise. *Let P, Q, P' and Q' be points in the Euclidean plane. Assume P' and Q' are inversions of P and Q correspondingly. Show that the quadrilateral $PQP'Q'$ is inscribed.*

9.17. Exercise. *Let Ω_1 and Ω_2 be two perpendicular circles with centers at O_1 and O_2 correspondingly. Show that the inversion of O_1 in Ω_2 coincides with the inversion of O_2 in Ω_1*

Angles after inversion

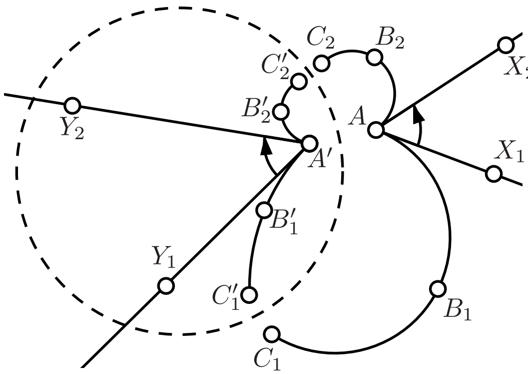
9.18. Proposition. *In the inversive plane, the inversion of an arc is an arc.*

Proof. Consider four distinct points A, B, C and D; let A', B', C' and D' be their inverses. We need to show that D lies on the arc ABC if and only if D' lies on the arc $A'B'C'$. According to Proposition 8.8, the later is equivalent to the following

$$\measuredangle ADC = \measuredangle ABC \quad \Leftrightarrow \quad \measuredangle A'D'C' = \measuredangle A'B'C'.$$

Which follows from Theorem 9.5(b). □

The following theorem roughly says that the angle between arcs changes sign after the inversion. A deeper understanding of this theorem comes from complex analysis.



9.19. Theorem. *Let AB_1C_1, AB_2C_2 be two arcs in the inversive plane and $A'B_1'C_1'$, $A'B_2'C_2'$ be their inversions. Let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to AB_1C_1 and AB_2C_2 at A and $[A'Y_1)$ and $[A'Y_2)$ be the half-lines tangent to $A'B_1'C_1'$ and $A'B_2'C_2'$ at A'. Then*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1A'Y_2.$$

Proof. Applying to Proposition 8.9,

$$\begin{aligned}\measuredangle X_1AX_2 &\equiv \measuredangle X_1AC_1 + \measuredangle C_1AC_2 + \measuredangle C_2AX_2 \equiv \\ &\equiv (\pi - \measuredangle C_1B_1A) + \measuredangle C_1AC_2 + (\pi - \measuredangle AB_2C_2) \equiv \\ &\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_2 + \measuredangle C_2AC_1) \equiv \\ &\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_1) - (\measuredangle C_1B_2C_2 + \measuredangle C_2AC_1).\end{aligned}$$

The same way we get

$$\measuredangle Y_1A'Y_2 \equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1') - (\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1').$$

By Theorem 9.5(b),

$$\begin{aligned}\measuredangle C_1B_1A + \measuredangle AB_2C_1 &\equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1'),\\ \measuredangle C_1B_2C_2 + \measuredangle C_2AC_1 &\equiv -(\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1').\end{aligned}$$

Hence the result follows. □

9.20. Corollary. *Let P', Q' and Γ' be the inversions of points P, Q and circle Γ in a circle Ω of the Euclidean plane. Assume P is inversion of Q in Γ then P' is inversion of Q' in Γ'.*

Proof. If $P = Q$ then $P' = Q' \in \Gamma'$ therefore P' is inversion of Q' in Γ'.

It remains to consider the case $P \neq Q$. Let Δ_1 and Δ_2 be two distinct circles which intersect at P and Q. According to Corollary 9.13, $\Delta_1 \perp \Gamma$ and $\Delta_2 \perp \Gamma$.

Denote by Δ_1' and Δ_2' the inversions of Δ_1 and Δ_2 in Ω. Clearly Δ_1' and Δ_2' intersect at P' and Q'.

From Theorem 9.19, the later is equivalent to $\Delta_1' \perp \Gamma'$ and $\Delta_2' \perp \perp \Gamma'$. By Corollary 9.12 the later implies P' is inversion of Q' in Γ'. $\square$

Non-Euclidean geometry

Chapter 10

Absolute plane

Let us remove Axiom IV from the Definition 2.1. This way we define a new object called *absolute plane* or *neutral plane*. (In the absolute plane, the Axiom IV may or may not hold.)

Clearly any theorem in absolute geometry holds in Euclidean geometry. In other words, Euclidean plane is an example of absolute plane. In the next chapter we will show that there are other examples of absolute plane distinct from the Euclidean plane.

Many theorems in Euclidean geometry which we discussed, still hold in absolute geometry.

In these lectures, the Axiom IV was used for the first time in the proof of uniqueness of parallel line in Theorem 6.2. Therefore all the statements before Theorem 6.2 also hold in absolute plane.

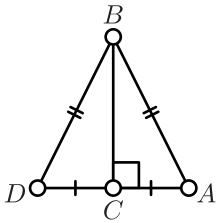
It makes all the discussed results about half-planes, signs of angles, congruence conditions, perpendicular lines and reflections true in absolute plane. If in the formulation of a statement above you do not see words "Euclidean plane" or "inversive plane", it means that the statement holds in absolute plane and the same proof works.

Let us give an example of theorem in absolute geometry, which admits a shorter proof in Euclidean geometry.

10.1. Theorem. *Assume that triangles $\triangle ABC$ and $\triangle A'B'C'$ have right angles at C and C' correspondingly, $AB = A'B'$ and $AC = A'C'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Euclidean proof. By Pythagorean theorem $BC = B'C'$. Then the statement follows from SSS congruence condition. □

Note that the proof of Pythagorean theorem used properties of similar triangles, which in turn used Axiom IV. Hence the above proof is not working in absolute plane.



Absolute proof. Denote by D the reflection of A through (BC) and by D' the reflection of A' through $(B'C')$. Note that

$$AD = 2\cdot AC = 2\cdot A'C' = A'D',$$

$$BD = BA = B'A' = B'D'.$$

By SSS, we get $\triangle ABD \cong \triangle A'B'D'$.

The theorem follows since C is the midpoint of $[AD]$ and C' is the midpoint of $[A'D']$. □

10.2. Exercise. *Give a proof of Exercise 7.9 which works in the absolute plane.*

Two angles of triangle

In this section we will prove a weaker form of Theorem 6.9 which holds in absolute plane.

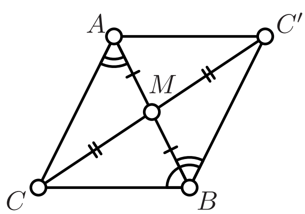
10.3. Proposition. *Let $\triangle ABC$ be nondegenerate triangle in the absolute plane. Then*

$$|\measuredangle CAB| + |\measuredangle ABC| < \pi.$$

Note that in Euclidean plane the theorem follows immediately from Theorem 6.9 and 3.11. In absolute geometry we need to work more.

Proof. Without loss of generality we may assume that $\angle CAB$ and $\angle ABC$ are positive.

Let M be the midpoint of $[AB]$. Chose $C' \in (CM)$ distinct from C so that $C'M = CM$.



Note that the angles $\angle AMC$ and $\angle BMC'$ are vertical; in particular

$$\measuredangle AMC = \measuredangle BMC'.$$

By construction $AM = BM$ and $CM = C'M$. Therefore $\triangle AMC \cong \triangle BMC'$ and according to 3.11, we get

$$\angle CAB = \angle C'BA.$$

In particular,

$$\measuredangle C'BC \equiv \measuredangle C'BA + \measuredangle ABC \equiv$$
$$\equiv \measuredangle CAB + \measuredangle ABC.$$

Finally note that C' and A lie on the same side from (CB). Therefore the angles $\measuredangle CAB$, $\angle ABC$ and $\angle C'BC$ are positive. By Exercise 3.3, the result follows. □

10.4. Exercise. *Assume A, B, C and D be points in absolute plane such that*

$$2\cdot\measuredangle ABC + 2\cdot\measuredangle BCD \equiv 0.$$

Show that $(AB) \parallel (CD)$.

Note that one can not extract the solution of the above exercise from the proof of Corollary 6.12

10.5. Exercise. *Prove* side-angle-angle congruence condition *in absolute plane.*

In other words, let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in absolute plane. Show that $\triangle ABC \cong \triangle A'B'C'$ if

$$AB = A'B', \quad \measuredangle ABC = \pm\measuredangle A'B'C' \quad \text{and} \quad \measuredangle BCA = \pm\measuredangle B'C'A'.$$

Note that in the Euclidean plane, the above exercise follows from ASA and Theorem on sum of angles of triangle (6.9). However, Theorem 6.9 can not be used here since its proof use Axiom IV. Later, in theorem Theorem 12.6, we will show that Theorem 6.9 does not hold in absolute plane.

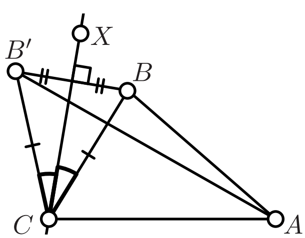
10.6. Exercise. *Assume that point D lies between the vertices A and B of triangle $\triangle ABC$ in the absolute plane. Show that*

$$CD < CA \quad \text{or} \quad CD < CB.$$

Three angles of triangle

10.7. Proposition. *Let $\triangle ABC$ and $\triangle A'B'C$ be two triangles in the absolute plane such that $AC = A'C'$ and $BC = B'C'$. Then*

$$AB < A'B' \quad \text{if and only if} \quad |\measuredangle ACB| < |\measuredangle A'C'B'|.$$



Proof. Without loss of generality, we may assume that $A = A'$ and $C = C'$ and $\measuredangle ACB, \measuredangle ACB' \geqslant 0$. In this case we need to show that

$$AB < AB' \quad \Leftrightarrow \quad \measuredangle ACB < \measuredangle ACB'.$$

Choose point X so that

$$\measuredangle ACX = \tfrac{1}{2}\cdot(\measuredangle ACB + \measuredangle ACB').$$

Note that

- ⋄ (CX) bisects $\angle BCB'$
- ⋄ (CX) is the perpendicular bisector of $[BB']$.
- ⋄ A and B lie on the same side from (CX) if and only if

$$\measuredangle ACB < \measuredangle ACB'.$$

From Exercise 5.3, A and B lie on the same side from (CX) if and only if $AB < AB'$. Hence the result follows. □

10.8. Theorem. *Let $\triangle ABC$ be a triangle in the absolute plane. Then*

$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| \leqslant \pi.$$

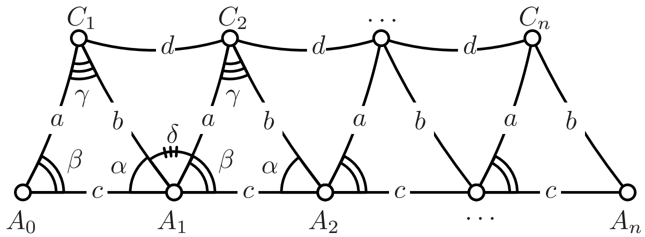
The following proof is due to Legendre [6], earlier proofs were due to Saccheri [9] and Lambert [5].

Proof. Let $\triangle ABC$ be the given triangle. Set

$$a = BC, \qquad b = CA, \qquad c = AB,$$
$$\alpha = \measuredangle CAB \qquad \beta = \measuredangle ABC \qquad \gamma = \measuredangle BCA.$$

Without loss of generality, we may assume that $\alpha, \beta, \gamma \geqslant 0$.

Fix a positive integer n. Consider points A_0, $A_1, \dots, A_n$ on the half-line $[BA)$ so that $BA_i = i\cdot c$ for each i. (In particular, $A_0 = B$ and $A_1 = A$.) Let us construct the points C_1, $C_2, \dots, C_n$, so that $\measuredangle A_iA_{i-1}C_i = \beta$ and $A_{i-1}C_i = a$ for each i.



This way we construct n congruent triangles

$$\begin{aligned}\triangle ABC = \triangle A_1A_0C_1 &\cong \\ \cong \triangle A_2A_1C_2 &\cong \\ \dots& \\ \cong \triangle A_nA_{n-1}C_n&.\end{aligned}$$

Set $d = C_1C_2$ and $\delta = \measuredangle C_2A_1C_1$. Note that

❶ $$\alpha + \beta + \delta = \pi.$$

By Proposition 10.3, $\delta \geqslant 0$.

By construction

$$\triangle A_1C_1C_2 \cong \triangle A_2C_2C_3 \cong \ldots \cong \triangle A_{n-1}C_{n-1}C_n.$$

In particular, $C_iC_{i+1} = d$ for each i.

By repeated application of the triangle inequality, we get that

$$\begin{aligned} n{\cdot}c = A_0A_n &\leqslant \\ &\leqslant A_0C_1 + C_1C_2 + \dots + C_{n-1}C_n + C_nA_n = \\ &= a + (n-1){\cdot}d + b. \end{aligned}$$

In particular,

$$c \leqslant d + \tfrac{1}{n}{\cdot}(a + b - d).$$

Since n is arbitrary positive integer, the later implies

$$c \leqslant d.$$

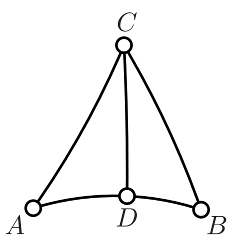
From Proposition 10.7 and SAS, the later is equivalent to

$$\gamma \leqslant \delta.$$

From ❶, the theorem follows. □

The *defect of triangle* $\triangle ABC$ is defined as

$$\operatorname{defect}(\triangle ABC) \stackrel{\text{def}}{=\!=} \pi - |\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB|.$$



Note that Theorem 10.8 sates that, defect of any triangle in absolute plane has to be nonnegative. According to Theorem 6.9, any triangle in Euclidean plane has zero defect.

10.9. Exercise. *Let $\triangle ABC$ be nondegenerate triangle in the absolute plane. Assume D lies between A and B. Show that*

$$\operatorname{defect}(\triangle ABC) = \operatorname{defect}(\triangle ADC) + \operatorname{defect}(\triangle DBC).$$

10.10. Exercise. *Let $ABCD$ be an inscribed quadrilateral in the absolute plane. Show that*

$$\measuredangle ABC + \measuredangle CDA \equiv \measuredangle BCD + \measuredangle DAB.$$

Note that the Theorem 8.5 can not be applied in the above exercise; it use Theorems 8.1 and 8.2; which in turns use Theorem 6.9.

How to prove that something can not be proved?

Many attempts were made to prove that any theorem in Euclidean geometry holds in absolute geometry. The later is equivalent to the statement that Axiom IV is a *theorem* in absolute geometry.

Many these attempts being accepted as proofs for long periods of time until the mistake was found.

There is a number of statements in the geometry of absolute plane which are equivalent to the Axiom IV. It means that if we exchange the Axiom IV in the Definition 2.1 to any of these statements then we will obtain an equivalent axiomatic system.

Here we give a short list of such statements. (We are not going to prove the equivalence in the lectures.)

10.11. Theorem. *An absolute plane is Euclidean if and only if one of the following equivalent conditions hold.*

(a) There is a line ℓ and a point P not on the line such that there is only one line passing through P and parallel to ℓ.
(b) Every nondegenerate triangle can be circumscribed.
(c) There exists a pair of distinct lines which lie on a bounded distance from each other.
(d) There is a triangle with arbitrary large inradius.
(e) There is a nondegenerate triangle with zero defect.

It is hard to imagine an absolute plane, which does not satisfy some of the properties above. That is partly the reason why for the large number of false proofs; each used one of such statements by accident.

Let us formulate the negation of the statement (a) above.

IV_h. For any line ℓ and any point $P \notin \ell$ there are at least two lines which pass through P and have no points of intersection with ℓ.

According to the theorem above, any non-Euclidean absolute plane Axiom IV_h holds.

It opens a way to look for a proof by contradiction. Simply exchange Axiom IV to Axiom IV_h in the Definition 2.1 and start to prove theorems in the obtained axiomatic system. In the case if we arrive to a contradiction, we prove the Axiom IV in absolute plane.

These attempts were unsuccessful as well; instead, this approach led to a new type of geometry.

This idea was growing since 5th century; the most notable result were obtained by Saccheri in [9]. The more this new geometry was developed, it became more and more believable that there will be no contradiction.

The statement that there is no contradiction appears first in private letters of Bolyai, Gauss, Schweikart and Taurinus[1]. They all seem to be afraid to state it in public. Say, in 1818 Gauss writes to Gerling

> *...I am happy that you have the courage to express yourself as if you recognized the possibility that our parallels theory along with our entire geometry could be false. But the wasps whose nest you disturb will fly around your head....*

Lobachevsky came to the same conclusion independently, unlike the others he had courage to state it in public and in print (see [7]). That cost him serious troubles.

Later Beltrami gave a clean proof that if hyperbolic geometry has a contradiction then so is Euclidean geometry. This was done by modeling points, lines, distances and angle measures of hyperbolic geometry using some other objects in Euclidean geometry; this is the subject of the next chapter.

Arguably, the discovery of non-Euclidean geometry was the second main discoveries of 19th century, trailing only the Mendel's laws.

Curvature

In a letter from 1824 Gauss writes:

> *The assumption that the sum of the three angles is less than π leads to a curious geometry, quite different from ours but thoroughly consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of a determination of a constant, which cannot be designated a priori. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen indefinitely large the two coincide. The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle become as small as one wishes, if only the sides are taken large enough; yet the area of the triangle can never exceed a definite limit, regardless how great the sides are taken, nor indeed can it ever reach it.*

In the modern terminology the constant which Gauss mentions, can be expressed as $1/\sqrt{-k}$, where k denotes so called *curvature* of

[1] The oldest surviving letters were the Gauss letter to Gerling 1816 and yet more convincing letter dated by 1818 of Schweikart sent to Gauss via Gerling.

the absolute plane which we are about to introduce.

The identity in the Exercise 10.9 suggests that defect of triangle should be proportional to its area.[2]

In fact for any absolute plane there is a nonpositive real number k such that

$$k\cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any triangle $\triangle ABC$. This number k is called *curvature* of the plane.

For example, by Theorem 6.9, the Euclidean plane has zero curvature. By Theorem 10.8, curvature of any absolute plane is nonpositive.

It turns out that up to isometry, the absolute plane is characterized by its curvature; i.e., two absolute planes are isometric if and only if they have the same curvature.

In the next chapter we will construct hyperbolic plane, this is an example of absolute plane with curvature $k = -1$.

Any absolute planes, distinct from Euclidean, can be obtained by rescaling metric on the hyperbolic plane. Indeed, if we rescale the metric by factor c, the area changes by positive factor c^2, while defect stays the same. Therefore taking $c = \sqrt{-k}$, we can get the absolute plane given curvature $k < 0$. In other words, all the non-Euclidean absolute planes become identical if we use $r = 1/\sqrt{-k}$ as the unit of length.

In the Chapter 13, we briefly discuss the geometry of the unit sphere. Although spheres are not absolute planes, the spherical geometry is a close relative of Euclidean and hyperbolic geometries.

The nondegenerate spherical triangles have negative defect. Moreover if R is the radius of the sphere then

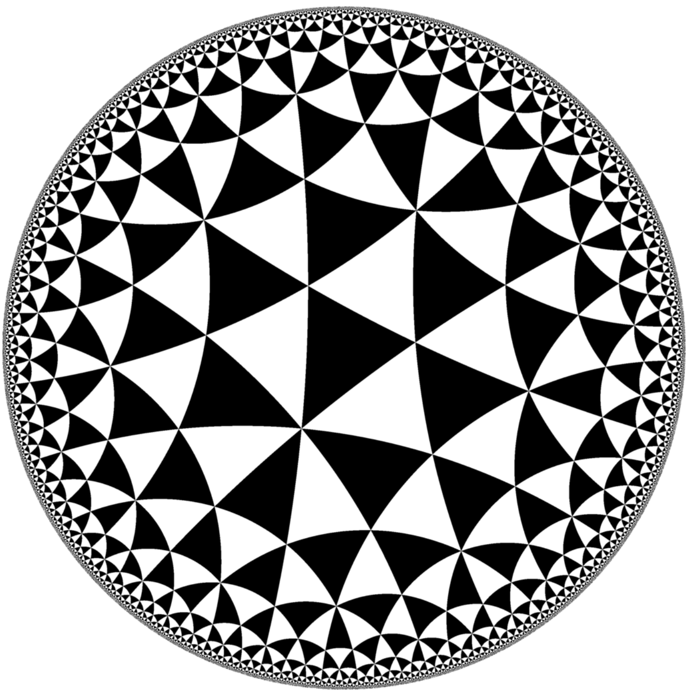
$$\tfrac{1}{R^2}\cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any spherical triangle $\triangle ABC$. In other words, the sphere of radius R has positive curvature $k = \frac{1}{R^2}$.

[2]We did not define *area*; instead we refer to intuitive understanding of area which reader might have. The formal definition of area is quite long and tedious.

Chapter 11

Hyperbolic plane



In this chapter we use inversive geometry to construct the model of hyperbolic plane — an example of absolute plane which is not Euclidean.

Poincaré disk model

Further we will discuss the *Poincaré disk model* of hyperbolic plane; an example of absolute plane in which Axiom IV does not hold, in particular this plane is not Euclidean. This model was discovered by Beltrami in [2] and popularized later by Poincaré.

On the figure above you see the Poincaré disk model of hyperbolic plane which is cut into congruent triangles with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

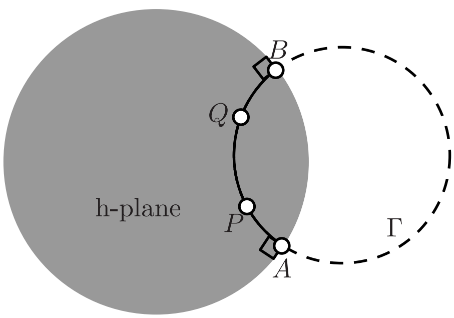
Description of the model

In this section we describe the model; i.e., we give new names for some objects in Euclidean plane which will represent lines, angle measures, distances in the hyperbolic plane.

Hyperbolic plane. Let us fix a circle on the Euclidean plane and call it *absolute*. The set of points inside the absolute will be called *hyperbolic plane* (or *h-plane*). (The absolute itself does *not* lie in the h-plane.)

We will often assume that the absolute is a unit circle.

Hyperbolic lines. The intersections of h-plane with clines perpendicular to the absolute are called *hyperbolic lines* (or *h-lines*).



Note that according to Corollary 9.14, there is unique h-line which pass through given two distinct points P and Q. This h-line will be denoted as $(PQ)_h$.

The arcs of hyperbolic lines will be called *hyperbolic segments* or *h-segments*. An h-segment with endpoints P and Q will be denoted as $[PQ]_h$.

The subset of h-line on one side from a point will be called *hyperbolic half-line* (or *h-half-line*). An h-half-line from P passing through Q will be denoted as $[PQ)_h$.

If Γ is the circle containing the h-line $(PQ)_h$ then the points of intersection of Γ with absolute are called *ideal points* of $(PQ)_h$. (Note that the ideal points of h-line do not belong to the h-line.)

So far $(PQ)_h$ is just a subset of h-plane; below we will introduce h-distance an later we will show that $(PQ)_h$ is a line for the h-distance in the sense of the Definition 1.8.

Hyperbolic distance. Let P and Q be distinct points in h-plane. Denote by A and B be the ideal points of $(PQ)_h$. Without loss of generality, we may assume that on the Euclidean circle containing the h-line $(PQ)_h$, the points A, P, Q, B appear in the same order.

Consider function

$$\delta(P,Q) \stackrel{\text{def}}{=\!=} \frac{AQ{\cdot}BP}{QB{\cdot}PA}.$$

Note that right hand side is the cross-ratio, which appeared in Theorem 9.5. Set $\delta(P,P) = 1$ for any point P in h-plane. Set

$$PQ_h \stackrel{\text{def}}{=\!=} \ln \delta(P,Q).$$

The proof that PQ_h is a metric on h-plane will be given below, for now it is just a function which returns a real value PQ_h for any pair of points P and Q in the h-plane.

Hyperbolic angles. Consider three points P, Q and R in the h-plane such that $P \neq Q$ and $R \neq Q$. The hyperbolic angle $\angle_h PQR$ is ordered pair of h-half-lines $[QP)_h$ and $[QR)_h$.

Let $[QX)$ and $[QY)$ be (Euclidean) half-lines which are tangent to $[QP]_h$ and $[QR]_h$ at Q. Then the *hyperbolic angle measure* (or *h-angle measure*) $\measuredangle_h PQR$ is defined as $\measuredangle XQY$.

What has to be proved?

In the previous section we defined all the notions in the formulation of the axioms. It remains to check that each axiom holds.

Namely we need to show the following statements.

11.1. Statement. *The defined h-distance is a metric on h-plane. I.e., for any three points P, Q and R in the h-plane we have*

(a) $PQ_h \geqslant 0$*;*
(b) $P = Q$ *if and only if* $PQ_h = 0$*;*
(c) $PQ_h = QP_h$*.*
(d) $QR_h \leqslant QP_h + PR_h$*.*

11.2. Statement. *A subset ℓ of h-plane is an h-line if and only if it is a line for h-distance; i.e., if there is a bijection $\iota\colon \ell \to \mathbb{R}$ such that*

$$XY_h = |\iota(X) - \iota(Y)|$$

for any X and $Y \in \ell$.

11.3. Statement. *Each Axiom of absolute plane holds. Namely we have to check the following:*

I. *There is one and only one h-line, that contains any two given distinct points P and Q of h-plane.*

II. *The h-angle measure satisfies the following conditions:*

(a) *Given a h-half-line $[QA)_h$ and $\alpha \in (-\pi, \pi]$ there is unique h-half-line $[QB)_h$ such that $\measuredangle_h AQB = \alpha$*

(b) *For any points A, B and C distinct from Q, we have*

$$\measuredangle_h AQB + \measuredangle_h BQC \equiv \measuredangle_h AQC.$$

(c) *The function*

$$\measuredangle_h \colon (A, Q, B) \mapsto \measuredangle AQB$$

is continuous at any triple of points (A, Q, B) in the h-plane such that $Q \neq A$ and $Q \neq B$ and $\measuredangle_h AQB \neq \pi$.

III. *$\triangle_h ABC \cong \triangle_h A'B'C'$ if and only if $A'B'_h = AB_h$, $A'C'_h = AC_h$ and $\measuredangle_h C'A'B' \equiv \pm\measuredangle_h CAB$.*

Finally we need to prove the following statement in order to show that h-plane is distinct from Euclidean plane.

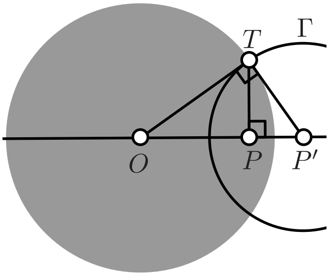
11.4. Statement. *The Axiom IV_h on page 84 holds.*

The proofs of these statements rely on the observation described in the next section.

Auxiliary statements

11.5. Lemma. *Consider h-plane with unit circle as absolute. Let O be the center of absolute and $P \neq O$ be an other point of h-plane. Denote by P' the inversion of P in the absolute.*

Then the circle Γ with center P' and radius $1/\sqrt{1-OP^2}$ is orthogonal to the absolute. Moreover O is the inversion of P in Γ.



Proof. Follows from Exercise 9.17. □

Assume Γ is a cline which is perpendicular to the absolute. Consider the inversion $X \mapsto X'$ in Γ, or if Γ is a line, set $X \mapsto X'$ to be the reflection through Γ.

The following proposition roughly says that the map $X \mapsto X'$ respects all the notions introduced in the previous section. Together with the lemma above, it implies that in any problem which formulated entirely *in h-terms* we can assume that a given point lies in the center of absolute.

11.6. Main observation. *The map $X \mapsto X'$ described above is a bijection of h-plane to itself. Moreover for any points P, Q, R in the h-plane such that $P \neq Q$ and $Q \neq R$ the following conditions hold*

(a) The sets $(PQ)_h$, $[PQ)_h$ and $[PQ]_h$ are mapped to $(P'Q')_h$, $[P'Q')_h$ and $[P'Q']_h$ correspondingly.

(b) $\delta(P', Q') = \delta(P, Q)$ and

$$P'Q'_h = PQ_h.$$

(c)

$$\measuredangle_h P'Q'R' \equiv -\measuredangle_h PQR.$$

Proof. According to Theorem 9.11 the map sends the absolute to itself. Note that the points on Γ do not move, it follows that points inside of absolute remain inside after the mapping and the other way around.

Part (a) follows from 9.6 and 9.19.

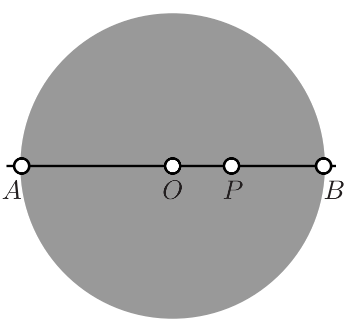
Part (b) follows from Theorem 9.5.

Part (c) follows from Theorem 9.19. □

11.7. Lemma. *Assume that the absolute is a unit circle centered at O. Given a point P in the h-plane, set $x = OP$ and $y = OP_h$. Then*

$$y = \ln \frac{1+x}{1-x} \quad \textit{and} \quad x = \frac{e^y - 1}{e^y + 1}.$$

Proof. Note that h-line $(OP)_h$ lies in a diameter of absolute. Therefore if A and B are points in the definition of h-distance then



$$OA = OB = 1,$$
$$PA = 1 + x,$$
$$PB = 1 - x.$$

Therefore

$$y = \ln \frac{AP \cdot BO}{PB \cdot OA} = \\ = \ln \frac{1+x}{1-x}.$$

Taking exponent of left and right hand side and applying obvious algebra manipulations we get

$$x = \frac{e^y - 1}{e^y + 1}.$$

□

11.8. Lemma. *Assume points P, Q and R appear on one h-line in the same order. Then*

$$PQ_h + QR_h = PR_h$$

Proof. Note that

$$PQ_h + QR_h = PR_h$$

is equivalent to

❶ $$\delta(P,Q){\cdot}\delta(Q,R) = \delta(P,R).$$

Let A and B be the ideal points of $(PQ)_h$. Without loss of generality we can assume that the points A, P, Q, R, B appear in the same order on the cline containing $(PQ)_h$. Then

$$\begin{aligned}\delta(P,Q){\cdot}\delta(Q,R) &= \frac{AQ{\cdot}BP}{QB{\cdot}PA}{\cdot}\frac{AR{\cdot}BQ}{RB{\cdot}QA} = \\ &= \frac{AR{\cdot}BP}{RB{\cdot}PA} = \\ &= \delta(P,R)\end{aligned}$$

Hence ❶ follows. □

Let P be a point in h-plane and $\rho > 0$. The set of all points Q in the h-plane such that $PQ_h = \rho$ is called *h-circle* with center P and *h-radius* ρ.

11.9. Lemma. *Any h-circle is formed by a Euclidean circle which lies completely in h-plane.*

More precisely for any point P in the h-plane and $\rho \geqslant 0$ there is a $\hat\rho \geqslant 0$ and a point $\hat P$ such that

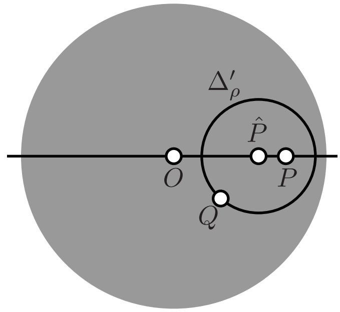
$$PQ_h = \rho \quad \Leftrightarrow \quad \hat{P}Q = \hat\rho.$$

Moreover, if O is the center of absolute then

1. *$\hat O = O$ for any ρ and*
2. *$\hat P \in (OP)$ for any $P \neq O$.*

Proof. According to Lemma 11.7, $OQ_h = \rho$ if and only if

$$OQ = \hat{\rho} = \frac{e^\rho - 1}{e^\rho + 1}.$$



Therefore the locus of points Q such that $OQ_h = \rho$ is formed by the Euclidean circle, denote it by Δ_ρ.

If $P \neq O$, applying Lemma 11.5 and the Main observation (11.6) we get a circle Γ perpendicular to the absolute such that P is the inversion of O in Γ.

Let Δ'_ρ be the inversion of Δ_ρ in Γ. Since the inversion in Γ preserves the h-distance, $PQ_h = \rho$ if and only if $Q \in \Delta'_\rho$.

According to Theorem 9.6, Δ'_ρ is a circle. Denote by $\hat{P}$ the center and by $\hat{\rho}$ the radius of Δ'_ρ.

Finally note that Δ'_ρ reflects to itself in (OP); i.e., the center $\hat{P}$ lies on (OP). □

The sketches of proofs

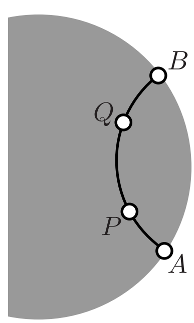
In this section we sketch the proofs of the statement 11.1–11.4. listed in the section one before last.

We will always assume that absolute is a unit circle centered at the point O.

Proof of 11.1; (a) and (b). Denote by O the center of absolute. Without loss of generality, we may assume that $Q = O$. If not apply Lemma 11.5, together with Main Observation (11.6).

Note that

$$\delta(O,P) = \frac{1 + OP}{1 - OP} \geqslant 1$$



and the equality holds only if $P = O$.

Therefore

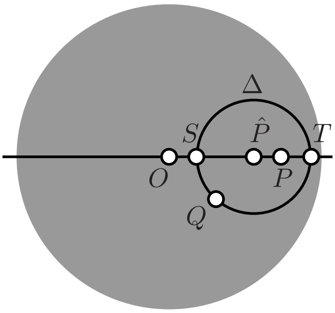
$$OP_h = \ln \delta(O,P) \geqslant 0.$$

and the equality holds if and only if $P = O$.

(c). Let A and B be ideal points of $(PQ)_h$ and A, P, Q, B appear on the cline containing $(PQ)_h$ in the same order.

Then

$$\begin{aligned}PQ_h &= \ln\frac{AQ\cdot BP}{QB\cdot PA} = \\ &= \ln\frac{BP\cdot AQ}{PA\cdot QB} = \\ &= QP_h.\end{aligned}$$



(d). Without loss of generality, we may assume that $RP_h \geqslant PQ_h$. Applying the main observation we may assume that $R = O$.

Denote by Δ the h-circle with center P and h-radius PQ_h. Let S and T be the points of intersection of (OP) and Δ.

Since $PQ_h \leqslant OP_h$, by Lemma 11.8 we can assume that the points O, S P and T appear on the h-line in the same order.

Let $\hat{P}$ be as in Lemma 11.9 for P and $\rho = PQ_h$. Note that $\hat{P}$ is the (Euclidean) midpoint of $[ST]$.

By the Euclidean triangle inequality

$$OT = O\hat{P} + \hat{P}Q \geqslant OQ.$$

Since the function $f(x) = \ln\frac{1+x}{1-x}$ is increasing for $x \in [0,1)$, the Lemma 11.7 implies that $OT_h \geqslant OQ_h$.

Finally applying Lemma 11.8 again, we get

$$OT_h = OP_h + PQ_h.$$

Therefore

❷ $$OQ_h \leqslant OP_h + PQ_h.$$ □

Proof of 11.2. Let ℓ be an h-line. Applying the main observation we can assume that ℓ contains the center of absolute. In this case ℓ is formed by intersection of diameter of absolute and the h-plane. Let A and B be the endpoints of the diameter.

Consider map $\iota\colon \ell \to \mathbb{R}$ defined as

$$\iota(X) = \ln\frac{AX}{XB}.$$

Note that $\iota\colon \ell \to \mathbb{R}$ is a bijection.

Further, if $X, Y \in \ell$ and the points A, X, Y and B appear on $[AB]$ in the same order then

$$|\iota(Y) - \iota(X)| = \left|\ln \frac{AY}{YB} - \ln \frac{AX}{XB}\right| = \left|\ln \frac{AY \cdot BX}{YB \cdot XB}\right| = XY_h;$$

i.e., any h-line is a line for h-metric.

Note that the equality in ❷ holds only if $Q = T$. In particular if Q lies on $(OP)_h$. Hence any line for h-distance is an h-line. □

Proof of 11.3. Axiom I follows from Corollary 9.14.

Let us prove Axiom II. Applying the main observation, we may assume that $Q = O$. In this case, for any point $X \neq O$ in h-plane, $[OX)_h$ is the intersection of $[OX)$ with h-plane. Hence all the statements in Axiom IIa and IIb follow.

In the proof of Axiom IIc, we can assume that Q is distinct from O. Denote by Z the inversion of Q in the absolute and by Γ the circle perpendicular to the absolute which is centered at Q'. According to Lemma 11.5, the point O is the inversion of Q in Γ; denote by A' and B' the inversions in Γ of the points A and B correspondingly. Note that the point A' is completely determined by the points Q and A, moreover the map $(Q, A) \mapsto A'$ is continuous at any pair of points (Q, A) such that $Q \neq O$. The same is true for the map $(Q, B) \mapsto B'$

According to the Main Observation

$$\measuredangle_h AQB \equiv -\measuredangle_h A'OB'.$$

Since $\measuredangle_h A'OB' = \measuredangle A'OB'$ and the maps $(Q, A) \mapsto A'$, $(Q, B) \mapsto B'$ are continuous, the Axiom IIc follows from the corresponding axiom of Euclidean plane.

Now let us show that Axiom III holds. Applying the main observation, we can assume that A and A' coincide with the center of absolute O. In this case

$$\measuredangle C'OB' = \measuredangle_h C'OB' = \pm\measuredangle_h COB = \pm\measuredangle COB.$$

Since

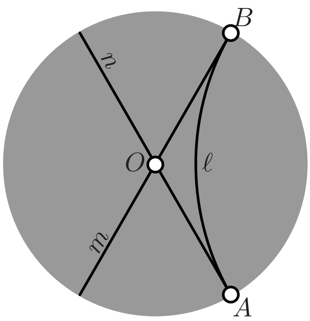
$$OB_h = OB'_h \quad \text{and} \quad OC_h = OC'_h,$$

Lemma 11.7 implies that the same holds for the Euclidean distances; i.e.,

$$OB = OB' \quad \text{and} \quad OC = OC'.$$

By SAS, there is a motion of Euclidean plane which sends O to itself, B to B' and C to C'

Note that the center of absolute is fixed by the corresponding motion. It follows that this motion gives also a motion of h-plane; in particular the h-triangles $\triangle_h OBC$ and $\triangle_h OB'C'$ are h-congruent.



Proof of 11.4. Finally we need to check that the Axiom IV_h holds.

Applying the main observation we can assume that $P = O$.

The remaining part of proof is left to the reader; it can be guessed from the picture □

Chapter 12

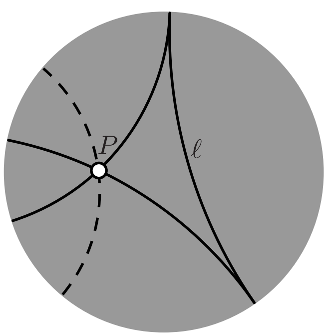
Geometry of h-plane

In this chapter we study the geometry of the plane described by Poincaré disc model. For briefness, this plane will be called *h-plane*. Note that we can work with this model directly from inside of Euclidean plane but we may also use the axioms of absolute geometry since according to the previous chapter they all hold in the h-plane.

Angle of parallelism

Let P be a point off an h-line ℓ. Drop a perpendicular $(PQ)_h$ from P to ℓ with foot point Q. Let φ be the least angle such that the h-line $(PZ)_h$ with $|\measuredangle_h QPZ| = \varphi$ does not intersect ℓ.

The angle φ is called *angle of parallelism* of P to ℓ. Clearly φ depends only on the distance $h = PQ_h$. Further $\varphi(h) \to \pi/2$ as $h \to 0$, and $\varphi(h) \to 0$ as $h \to \infty$. (In Euclidean geometry the angle of parallelism is identically equal to $\pi/2$.)



If ℓ, P and Z as above then the h-line $m = (PZ)_h$ is called *asymptotically parallel* to ℓ.[1] In other words, two h-lines are asymptotically parallel if they share one ideal point.

Given $P \notin \ell$ there are exactly two asymptotically parallel lines through P to ℓ; the remaining parallel lines t ℓ through P are called *ultra parallel*.

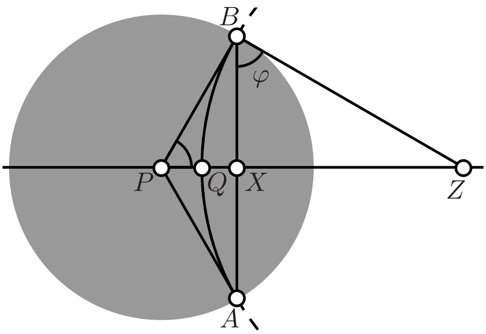
On the diagram, the two solid h-lines passing through P are asymptotically parallel to ℓ; the dotted

[1] In hyperbolic geometry the term *parallel lines* is often used for *asymptotically parallel lines*; we do not follow this convention.

h-line is ultra parallel to ℓ.

12.1. Proposition. *Let Q be the foot point of P on h-line ℓ. Denote by φ the angle of parallelism of P to ℓ and let $h = PQ_h$. Then*

$$h = \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}.$$



Proof. Applying a motion of h-plane if necessary, we may assume P is the center of absolute. Then the h-lines through P are formed by the intersections of Euclidean lines with the h-plane.

Let us denote by A and B the ideal points of ℓ. Without loss of generality we may assume that $\angle APB$ is positive. In this case

$$\varphi = \measuredangle QPB = \measuredangle APQ = \tfrac{1}{2}\cdot\measuredangle APB.$$

Let Z be the center of the circle Γ containing the h-line ℓ. Set X to be the point of intersection of the Euclidean segment $[AB]$ and (PQ).

Note that, $OX = \cos\varphi$ therefore by Lemma 11.7,

$$OX_h = \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}.$$

Note that both angles $\angle PBZ$ and $\angle BXZ$ are right. Therefore $\triangle ZBX \sim \triangle ZPB$, sine the $\angle PZB$ is shared. In particular

$$ZX\cdot XP = ZB^2;$$

i.e., X is the inversion of P in Γ.

The inversion in Γ is the reflection of h-plane through ℓ. Therefore

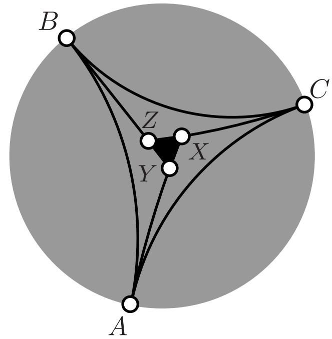
$$\begin{aligned} h = PQ_h = QX_h &= \\ &= \tfrac{1}{2}\cdot OX_h = \\ &= \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}. \end{aligned}$$

□

Inradius of h-triangle

12.2. Theorem. *Inradius of any h-triangle is less than $\frac{1}{2}\cdot \ln 3$.*

Proof. First note that any triangle in h-plane lies in an *ideal triangle*; i.e., a region bounded by three pairwise asymptotically parallel lines.



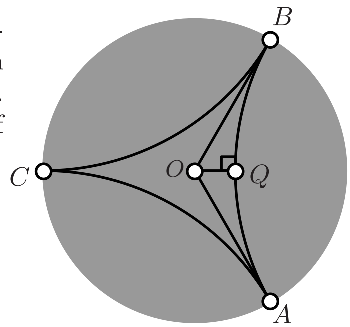
A proof can be seen in the picture. Consider arbitrary h-triangle $\triangle_h XYZ$. Denote by A, B and C the ideal points of the h-half-lines $[XY)_h$, $[YZ)_h$ and $[ZX)_h$.

It should be clear that inradius of the ideal triangle ABC is bigger than inradius of $\triangle_h XYZ$.

Applying an inverse if necessary, we can assume that h-incenter (O) of the ideal triangle is the center of absolute. Therefore, without loss of generality, we may assume

$$\measuredangle AOB = \measuredangle BOC = \measuredangle COA = \tfrac{2}{3}\cdot\pi.$$

It remains to find the inradius. Denote by Q the foot point of O on $(AB)_h$. Then OQ_h is the inradius. Note that the angle of parallelism of $(AB)_h$ at O is equal to $\frac{\pi}{3}$.



By Proposition 12.1,

$$\begin{aligned} OQ_h &= \tfrac{1}{2}\cdot \ln \frac{1+\cos\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \\ &= \tfrac{1}{2}\cdot \ln \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \\ &= \tfrac{1}{2}\cdot \ln 3. \end{aligned}$$

□

12.3. Exercise. *Let $ABCD$ be a quadrilateral in h-plane such that the h-angles at A, B and C are right and $AB_h = BC_h$. Find the optimal upper bound for AB_h.*

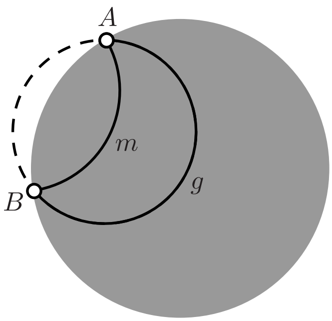
Circles, horocycles and equidistants

Note that according to Lemma 11.9, any h-circle is formed by a Euclidean circle which lies completely in the h-plane. Further any h-line is an intersection of the h-plane with the circle perpendicular to the absolute.

In this section we will describe the h-geometric meaning of the intersections of the other circles with the h-plane.

You will see that all these intersections formed by a *perfectly round shape* in the h-plane; i.e., h-geometrically all the points on an equidistant look the same.

One may think of these curves as about trajectories of a car which drives in the plane with fixed position of the wheel. In the Euclidean plane, this way you either run along a circles or along a line.



In hyperbolic plane the picture is different. If you turn wheel far right, you will run along a circle. If you turn it less, at certain position of wheel, you will never come back, the path will be different from the line. If you turn the wheel further a bit, you start to run along a path which stays on the same distant from an h-line.

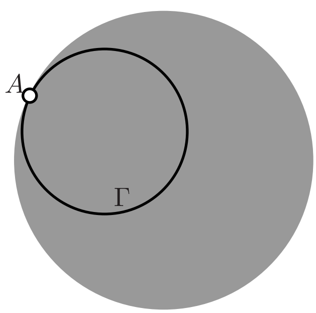
Equidistants of h-lines. Consider h-plane with absolute Ω. Assume a circle Γ intersects Ω in two distinct points A and B. Denote by g the intersection of Γ with h-plane. Let us draw an h-line m with the ideal points A and B.

12.4. Exercise. *Show that the h-line m is uniquely determined by its ideal points A and B.*

Consider any h-line ℓ perpendicular to m; let Δ be the circle containing ℓ.

Note that $\Delta \perp \Gamma$. Indeed, according to Corollary 9.12, m and Ω inverted to themselves in Δ. It follows that A is the inversion of B in Δ. Finally, by Corollary 9.13, we get that $\Delta \perp \Gamma$.

Therefore inversion in Δ sends both m and g to themselves. So if $P', P \in g$ are inversions of each other in Δ then they lie on the same distance from m. Clearly we have plenty of choice for ℓ, which can be used to move points along g arbitrary keeping the distance to m.



It follows that g is formed by the set of points which lie on fixed h-distance and the same h-side from m.

Such curve g is called *equidistant* to h-line m. In Euclidean geometry the equidistant from a line is a line; apparently in hyperbolic geometry the picture is different.

Horocycles. If the circle Γ touches the absolute from inside at one point A then

$h = \Gamma\backslash\{A\}$ lie in h-plane. This set is called *horocycle.* It also has perfectly round shape in the sense described above.

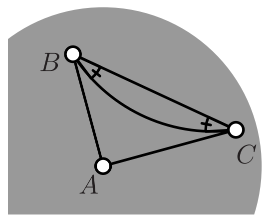
Horocycles are the boarder case between circles and equidistants to h-lines. An horocycle might be considered as a limit of circles which pass through fixed point which the centers running to infinity along a line. The same horocycle is a limit of equidistants which pass through fixed point to the h-lines running to infinity.

12.5. Exercise. *Find the leg of be a right h-triangle inscribed in a horocycle.*

Hyperbolic triangles

12.6. Theorem. *Any nondegenerate hyperbolic triangle has positive defect.*

Proof. Consider h-trinagle $\triangle_h ABC$. According to Theorem 10.8,



❶ $$\operatorname{defect}(\triangle_h ABC) \geqslant 0.$$

It remains to show that in the case of equality the triangle $\triangle_h ABC$ degenerates.

Without loss of generality, we may assume that A is the center of absolute; in this case $\measuredangle_h CAB = \measuredangle CAB$. Yet we may assume that

$$\measuredangle_h CAB,\ \measuredangle_h ABC,\ \measuredangle_h BCA,\ \measuredangle ABC,\ \measuredangle BCA \geqslant 0.$$

Let D be an arbitrary point in $[CB]_h$ distinct from B and C. From Proposition 8.9

$$\measuredangle ABC - \measuredangle_h ABC \equiv \pi - \measuredangle CDB \equiv \measuredangle BCA - \measuredangle_h BCA.$$

From Exercise 6.11, we get

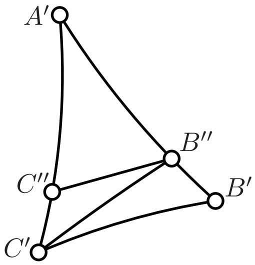
$$\operatorname{defect}(\triangle_h ABC) = 2\cdot(\pi - \measuredangle CDB).$$

Therefore if we have equality in ❶ then $\measuredangle CDB = \pi$. In particular the h-segment $[BC]_h$ coincides with Euclidean segment $[BC]$. The later can happen only if the h-line passes through the center of absolute; i.e., if $\triangle_h ABC$ degenerates. □

The following theorem states in particular that hyperbolic triangles are congruent if their corresponding angles are equal; in particular in hyperbolic geometry similar triangles have to be congruent.

12.7. AAA congruence condition. *Two nondegenerate triangles* $\triangle_h ABC$ *and* $\triangle_h A'B'C'$ *in the h-plane are congruent if* $\measuredangle_h ABC = \pm\measuredangle_h A'B'C'$, $\measuredangle_h BCA = \pm\measuredangle_h B'C'A'$ *and* $\measuredangle_h CAB = \pm\measuredangle_h C'A'B'$.

Proof. Note hat if $AB_h = A'B'_h$ then the theorem follows from ASA.



Assume contrary. Without loss of generality we may assume that $AB_h < A'B'_h$. Therefore we can choose the point $B'' \in [A'B']_h$ such that $A'B''_h = AB_h$.

Choose a point X so that $\measuredangle_h A'B''X = \measuredangle_h A'B'C'$. According to Exercise 10.4, $(B''X)_h \parallel (B'C')_h$.

By Pasch's theorem (3.10), $(B''X)_h$ intersects $[A'C']_h$. Denote by C'' the point of intersection.

According to ASA, $\triangle_h ABC \cong \triangle_h A'B''C''$; in particular

❷ $$\operatorname{defect}(\triangle_h ABC) = \operatorname{defect}(\triangle_h A'B''C'').$$

Applying Exercise 10.9 twice, we get

❸ $$\begin{aligned}\operatorname{defect}(\triangle_h A'B'C') &= \operatorname{defect}(\triangle_h A'B''C'')+\\ &\quad+\operatorname{defect}(\triangle_h B''C''C') + \operatorname{defect}(\triangle_h B''C'B').\end{aligned}$$

By Theorem 12.6, the defects has to be positive. Therefore

$$\operatorname{defect}(\triangle_h A'B'C') > \operatorname{defect}(\triangle_h ABC).$$

On the other hand,

$$\begin{aligned}\operatorname{defect}(\triangle_h A'B'C') &= |\measuredangle_h A'B'C'| + |\measuredangle_h B'C'A'| + |\measuredangle_h C'A'B'| =\\ &= |\measuredangle_h ABC| + |\measuredangle_h BCA| + |\measuredangle_h CAB| =\\ &= \operatorname{defect}(\triangle_h ABC),\end{aligned}$$

a contradiction. □

Conformal interpretation

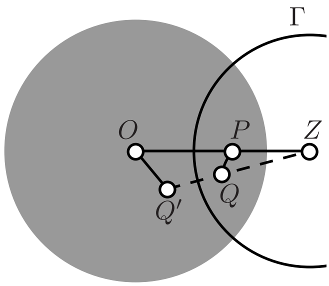
Let us give an other interpretation of the h-distance.

12.8. Lemma. *Consider h-plane with absolute formed by the unit circle centered at* O. *Fix a point* P *and let* Q *be an other point in the h-plane. Set* $x = PQ$ *and* $y = PQ_h$ *then*

$$\lim_{x\to 0}\frac{y}{x} = \frac{2}{1-OP^2}.$$

The above formula tells that the h-distance from P to a near by point Q is nearly proportional to the Euclidean distance with the coefficient $\frac{2}{1-OP^2}$. The value $\lambda(P) = \frac{2}{1-OP^2}$ is called *conformal factor* of h-metric.

One may think of conformal factor $\lambda(P)$ as the speed limit at the given point. In this case the h-distance is the the minimal time needed to travel from one point of h-plane to the other point.



Proof. If $P = O$, then according to Lemma 11.7

❹ $$\frac{y}{x} = \frac{\ln\frac{1+x}{1-x}}{x} \to 2$$

as $x \to 0$.

If $P \neq O$, denote by Z the inversion of P in the absolute. Denote by Γ the circle with center Z orthogonal to the absolute.

According to Main Observation 11.6 and Lemma 11.5 the inversion in Γ is a motion of h-plane which sends P to O. In particular, if we denote by Q' the inversion of Q in Γ then $OQ'_h = PQ_h$.

Set $x' = OQ'$ According to Lemma 9.2,

$$\frac{x'}{x} = \frac{OZ}{ZQ}.$$

Therefore

$$\frac{x'}{x} \to \frac{OZ}{ZP} = \frac{1}{1-OP^2}$$

as $x \to 0$.

Together with ❹, it implies

$$\frac{y}{x} = \frac{y}{x'} \cdot \frac{x'}{x} \to \frac{2}{1-OP^2}$$

as $x \to 0$. □

Here is an application of the lemma above.

12.9. Proposition. *The circumference of an h-circle of h-radius r is*

$$2 \cdot \pi \cdot \operatorname{sh} r,$$

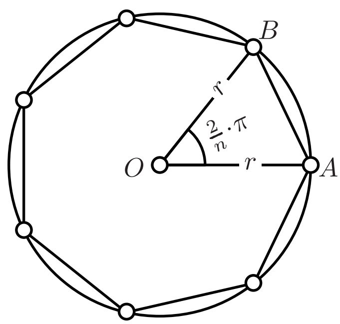
where $\operatorname{sh} r$ *denotes* hyperbolic sine *of r; i.e.,*

$$\operatorname{sh} r \stackrel{\text{def}}{=\!=} \frac{e^r - e^{-r}}{2}.$$

Before we proceed with the proof let us discuss the same problem in the Euclidean plane.

The circumference of the circle in the Euclidean plane can be defined as limit of perimeters of regular n-gons inscribed in the circle as $n \to \infty$.

Namely, let us fix $r > 0$. Given a positive integer n consider $\triangle AOB$ such that $\measuredangle AOB = \frac{2\cdot\pi}{n}$ and $OA = OB = r$. Set $x_n = AB$. Note that x_n is the side of regular n-gon inscribed in the circle of radius r. Therefore the perimeter of the n-gon is equal to $n\cdot x_n$.



The circumference of the circle with radius r might be defined as the limit of

❺ $$\lim_{n\to\infty} n\cdot x_n = 2\cdot\pi\cdot r.$$

(This limit can be taken as the definition of π.)

In the following proof we repeat the same construction in the h-plane.

Proof. Without loss of generality we can assume that the center O of the circle is the center of absolute.

By Lemma 11.7, the h-circle with h-radius r is formed by the Euclidean circle with center O and radius

$$a = \frac{e^r - 1}{e^r + 1}.$$

Denote by x_n and y_n the Euclidean and hyperbolic side lengths of the regular n-gon inscribed in the circle.

Note that $x_n \to 0$ as $n \to \infty$. By Lemma 12.8,

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \frac{2}{1 - a^2}.$$

Applying ❺, we get that the circumference of the h-circle can be found the following way

$$\begin{aligned}\lim_{n\to\infty} n\cdot y_n &= \frac{2}{1-a^2}\cdot \lim_{n\to\infty} n\cdot x_n = \\ &= \frac{4\cdot\pi\cdot a}{1-a^2} = \\ &= \frac{4\cdot\pi\cdot\left(\frac{e^r-1}{e^r+1}\right)}{1-\left(\frac{e^r-1}{e^r+1}\right)^2} = \\ &= 2\cdot\pi\cdot\frac{e^r-e^{-r}}{2} = \\ &= 2\cdot\pi\cdot \operatorname{sh} r.\end{aligned}$$

□

12.10. Exercise. *Denote by* $\mathrm{circum}_h(r)$ *the circumference of the h-circle of radius* r*. Show that*

$$\mathrm{circum}_h(r+1) > 2\cdot \mathrm{circum}_h(r)$$

for all $r > 0$*.*

Additional topics

Chapter 13

Spherical geometry

Spherical geometry is the geometry of the surface of the unit sphere. This type of geometry has practical applications in cartography, navigation and astronomy.

The spherical geometry is a close relative of Euclidean and hyperbolic geometries. Most of theorems of hyperbolic geometry have spherical analogs, but spherical geometry is easier to visualize.

We discuss few theorems in spherical geometry; the proofs are not completely rigorous.

Space and spheres

Let us repeat the construction of metric d_2 (page 12) in the space.

We will denote by $\mathbb{R}^3$ the set of all triples (x, y, z) of real numbers. Assume $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ are arbitrary points. Let us define the metric on $\mathbb{R}^3$ the following way

$$AB \stackrel{\text{def}}{=\!=} \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The obtained metric space is called *Euclidean space*.

Assume at least one of the real numbers a, b or c is distinct from zero. Then the subset of points $(x, y, z) \in \mathbb{R}^3$ described by equation

$$a{\cdot}x + b{\cdot}y + c{\cdot}z + d = 0$$

is called *plane*.

It is straightforward to show that any plane in Euclidean space is isometric to Euclidean plane. Further any three points on the space lie on one plane.

It makes possible to generalize many notions and results from Euclidean plane geometry to Euclidean space by applying plane geometry in the planes of the space.

Sphere in the space is an analog of circle in the plane. Formally, *sphere* with center O and radius r is the set of points in the space which lie on the distance r from O.

Let A and B be two points on the unit sphere centered at O. The *spherical distance* from A to B (briefly AB_s) is defined as $|\measuredangle AOB|$.

In the spherical geometry, the role of lines play the *great circles*; i.e., the intersection of the sphere with a plane passing through O.

Note that the great circles do not form lines in the sense of Definition 1.8. Also any two distinct great circles intersect at two antipodal points. In particular, the sphere does not satisfy the axioms of absolute plane.

Pythagorean theorem

Here is an analog of Pythagorean Theorems (6.8 and 14.5) in spherical geometry.

13.1. Theorem. *Let $\triangle_s ABC$ be a spherical triangle with right angle at C. Set $a = BC_s$, $b = CA_s$ and $c = AB_s$. Then*

$$\cos c = \cos a \cdot \cos b.$$

In the proof we will use the notion of scalar product which we are about to discuss.

Let A and B be two points in Euclidean space. Denote by $v_A = (x_A, y_A, z_A)$ and $v_B = (x_B, y_B, z_B)$ the position vectors of A and B correspondingly. The scalar product of two vectors v_A and v_B in $\mathbb{R}^3$ is defined as

$$\langle v_A, v_B\rangle \stackrel{\text{def}}{=\!=} x_A \cdot x_B + y_A \cdot y_B + z_A \cdot z_B. \quad ❶$$

Assume both vectors v_A and v_B are nonzero and φ is the angle measure between these two vectors. In this case the scalar product can be expressed the following way:

$$\langle v_A, v_B\rangle = |v_A| \cdot |v_B| \cdot \cos\varphi,$$

where

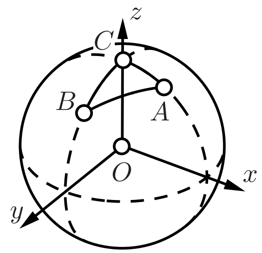
$$|v_A| = \sqrt{x_A^2 + y_A^2 + z_A^2}, \qquad |v_B| = \sqrt{x_B^2 + y_B^2 + z_B^2}.$$

Now, assume the points A and B lie on the unit sphere in $\mathbb{R}^3$ centered at the origin. In this case $|v_A| = |v_B| = 1$. By ❶ we get

❷ $$\cos AB_s = \langle v_A, v_B \rangle.$$

This is the key formula on which the following proof is build.

Proof. Since the angle at C is right, we can choose coordinates in $\mathbb{R}^3$ so that $v_C = (0,0,1)$, v_A lies in xz-plane, so $v_A = (x_A, 0, z_A)$ and v_B lies in yz-plane, so $v_B = (0, y_B, z_B)$.



Applying, ❷, we get

$$z_A = \langle v_C, v_A \rangle = \cos b,$$
$$z_B = \langle v_C, v_B \rangle = \cos a.$$

Applying, ❷ again, we get

$$\begin{aligned}\cos c &= \langle v_A, v_B \rangle = \\ &= x_A{\cdot}0 + 0{\cdot}y_B + z_A{\cdot}z_B = \\ &= \cos b{\cdot} \cos a.\end{aligned}$$

□

13.2. Exercise. *Show that if $\triangle_s ABC$ be a spherical triangle with right angle at C and $AC_s = BC_s = \frac{\pi}{4}$ then $AB_s = \frac{\pi}{3}$.*

Try to find two solutions, with and without using the spherical Pythagorean theorem.

Inversion of the space

Stereographic projection is special type of maps between sphere and the inversive plane. Poincare model of hyperbolic geometry is a direct analog of stereographic projection for spherical geometry.

One can also define inversion in the sphere the same way as we define inversion in the circle.

Formally, let Σ be the the sphere with center O and radius r. The *inversion* in Σ of a point P is the point $P' \in [OP)$ such that

$$OP{\cdot}OP' = r^2.$$

In this case, the sphere Σ will be called the *sphere of inversion* and its center is called *center of inversion.*

We also add ∞ to the space and assume that the center of inversion is mapped to ∞ and the other way around. The space $\mathbb{R}^3$ with the point ∞ will be called *inversive space.*

The inversion of the space has many properties of the inversion of the plane. Most important for us is the analogs of theorems 9.5, 9.6, 9.19 which can be summarized as follows.

13.3. Theorem. *The inversion in the sphere has the following properties:*

(a) Inversion maps sphere or plane into sphere or plane.
(b) Inversion maps circle or line into circle or line.
(c) Inversion preserves cross-ratio; i.e., if A', B', C' and D' be the inversions of the points A, B, C and D correspondingly then

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(d) Inversion maps arcs into arcs.
(e) Inversion preserves the absolute value of the angle measure between tangent half-lines to the arcs.

Instead of proof. We do not present the proofs here, but they are very similar to the corresponding proofs in plane geometry. If you want to do it yourself, prove the following lemma and use it together with the observation that *any circle in the space can be presented as an intersection of two spheres.*

13.4. Lemma. *Let Σ be a subset of Euclidean space which contains at least two points. Fix a point O in the space.*

Then Σ is a sphere if and only if for any plane Π passing through O, the intersection $\Pi\cap\Sigma$ is either empty set, one point set or a circle.

Stereographic projection

Consider the unit sphere Σ in Euclidean space centered at the origin $(0,0,0)$. This sphere can be described by equation $x^2+y^2+z^2=1$.

Denote by Π be the xy-plane; it is defined by the equation $z=0$. Clearly Π runs through the center of Σ.

Denote by $N=(0,0,1)$ the "North Pole" and by $S=(0,0,-1)$ be the "South Pole" of Σ; these are the points on the sphere which have extremal distances to Π. Denote by Ω the "equator" of Σ; it is the intersection $\Sigma\cap\Pi$.

For any point $P\ne S$ on Σ, consider the line (SP) in the space. This line intersects Π in exactly one point, say P'. We set in addition that $S'=\infty$.

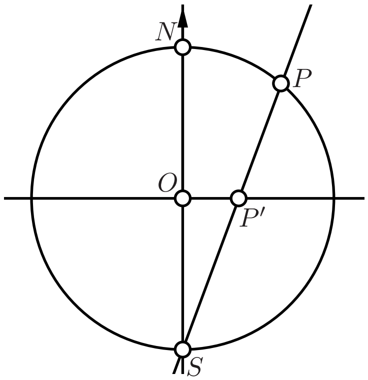
The map $P\mapsto P'$ is the *stereographic projection from Σ to Π from the South Pole.* The inverse of this map $P'\mapsto P$ is called *stereographic projection from Π to Σ from the South Pole.*

The same way one can define *stereographic projection from the North Pole*.

Note that $P = P'$ if and only if $P \in \Omega$.

Note that if Σ and Π as above. Then the stereographic projections $\Sigma \to \Pi$ and $\Pi \to \Sigma$ from S are the restrictions of the inversion in the sphere with center S and radius $\sqrt{2}$ to Σ and Π correspondingly.

From above and Theorem 13.3, it follows that the stereographic projection preserves the angles between arcs; more precisely *the absolute value of the angle measure* between arcs on the sphere.



The plane through P, O and S.

This makes it particularly useful in cartography. A map of a big region of earth can not be done in the same scale, but using stereographic projection, one can keep the angles between roads the same as on earth.

In the following exercises, we assume that Σ, Π, Ω, O, S and N are as above.

13.5. Exercise. *Show that the composition of stereographic projections from* Π *to* Σ *from* S *and from* Σ *to* Π *from* N *is the inversion of the plane* Π *in* Ω.

13.6. Exercise. *Show that image of great circle is a cline on the plane which intersects* Ω *at two opposite points.*

13.7. Exercise. *Let Fix a point* $P \in \Pi$ *and let* Q *be yet an other point in* Π. *Denote by* P' *and* Q' *their stereographic projections in* Σ. *Set* $x = PQ$ *and* $y = P'Q'_s$. *Show that*

$$\lim_{x\to 0} \frac{y}{x} = \frac{2}{1 + OP^2}.$$

Compare with Lemma 12.8.

Central projection

Let Σ be the unit sphere centered at the origin which will be denoted as O. Denote by Π^+ the plane described by equation $z = 1$. This plane is parallel to xy-plane and it pass through the North Pole $N = (0, 0, 1)$ of Σ.

Recall that north hemisphere of Σ, is the subset of points $(x, y, z) \in \in \Sigma$ such that $z > 0$. The north hemisphere will be denoted as Σ^+.

Given a point $P \in \Sigma^+$, consider half-line $[OP)$ and denote by P' the intersection of $[OP)$ and Π^+. Note that if $P = (x, y, z)$ then $P' = (\frac{x}{z}, \frac{y}{z}, 1)$. It follows that $P \mapsto P'$ is a bijection between Σ^+ and Π^+.

The described map $\Sigma^+ \to \Pi^+$ is called *central projection* of hemisphere Σ^+.

In spherical geometry, central projection is analogous to the Klein model of hyperbolic plane.

Note that the central projection sends intersections of great circles with Σ^+ to the lines in Π^+. The later follows since great circles are formed by intersection of Σ with planes passing through the origin and the lines in Π^+ are formed by intersection of Π^+ with these planes.

13.8. Exercise. *Assume that $\triangle_s NBC$ has right angle at C and N is the North Pole which lies completely in the north hemisphere. Let $\triangle NB'C'$ be the image of $\triangle_s NBC$ under central projection.*

Observe that $\triangle NB'C'$ has right angle at C'.

Use this observation and the standard Pythagorean for $\triangle NB'C'$ to prove spherical Pythagorean theorem for $\triangle_s NBC$.

13.9. Exercise. *Consider a nondegenerate spherical triangle $\triangle_s ABC$. Assume that Π^+ is parallel to the plane passing through A, B and C. Denote by A', B' and C' the central projections of A, B and C.*

(a) *Show that the midpoints of $[AB]$, $[BC]$ and $[CA]$ are central projections of the midpoints of $[AB]_s$, $[BC]_s$ correspondingly.*

(b) *Use part (a) to show that medians of spherical triangle intersect at one point.*

(c) *Compare to Exercise 14.4.*

Chapter 14

Klein model

Klein model is an other model of hyperbolic plane discovered by Beltrami. The Klein and Poincaré models are saying exactly the same thing but in two different languages. Some problems in hyperbolic geometry admit simpler proof using Klein model and others have simpler proof in Poincaré model. Therefore it worth to know both.

Special bijection of h-plane to itself

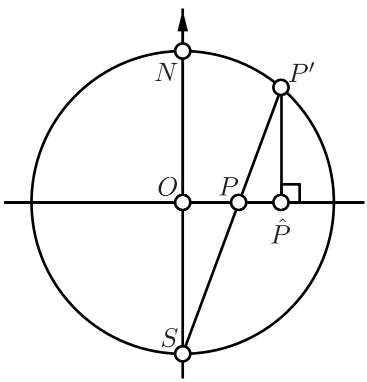
Consider the Poincaré disc model with absolute at the unit circle Ω centered at O. Choose a coordinate system (x, y) on the plane with origin at O, so the circle Ω is described by the equation $x^2 + y^2 = 1$.

Let us think of our plane Π as it lies in the Euclidean space as the xy-plane. Denote by Σ the unit sphere centered at O; it is described by the equation

$$x^2 + y^2 + z^2 = 1.$$

Set $S = (0, 0, -1)$ and $N = (0, 0, 1)$; these are the South and North Poles of Σ.

Consider stereographic projection $\Pi \to \Sigma$ from S; given point $P \in \Pi$ denote its image as P'. Note that the h-plane is mapped to the North Hemisphere; i.e., to the set of points (x, y, z) in Σ described by inequality $z > 0$.



The plane through P, O and S.

For a point $P' \in \Sigma$ consider its foot point $\hat{P}$ on Π; this is the closest point on Π from P'.

The composition $P \mapsto \hat{P}$ of these two maps is a bijection of h-plane to itself.

Note that $P = \hat{P}$ if and only if $P \in \Omega$ or $P = O$ or $P = \infty$.

14.1. Exercise. *Show that the map $P \mapsto \hat{P}$ described above can be described the following way: set $\hat{O} = O$ and for any other point point P take $\hat{P} \in [OP)$ such that*

$$O\hat{P} = \frac{2{\cdot}x}{1 - x^2},$$

where $x = OP$.

14.2. Lemma. *Let $(PQ)_h$ be an h-line with the ideal points A and B. Then $\hat{P}, \hat{Q} \in [AB]$.*

Moreover

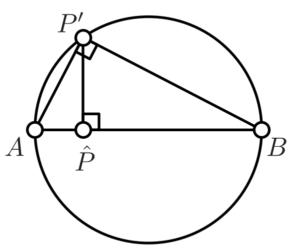
❶ $$\frac{A\hat{Q}{\cdot}B\hat{P}}{\hat{Q}B{\cdot}\hat{P}A} = \left(\frac{AQ{\cdot}BP}{QB{\cdot}PA}\right)^2.$$

In particular

$$PQ_h = \tfrac{1}{2}{\cdot}\left|\ln\frac{A\hat{Q}{\cdot}B\hat{P}}{\hat{Q}B{\cdot}\hat{P}A}\right|.$$

Proof. Consider the stereographic projection $\Pi \to \Sigma$ from the South Pole. Denote by P' and Q' the images of P and Q. According to Theorem 13.3(c),

❷ $$\frac{AQ{\cdot}BP}{QB{\cdot}PA} = \frac{AQ'{\cdot}BP'}{Q'B{\cdot}P'A}.$$



The plane Λ.

By Theorem 13.3(e), each cline in Π which is perpendicular to Ω is mapped to a circle in Σ which is still perpendicular to Ω. It follows that the stereographic projection sends $(PQ)_h$ to the intersection of the north hemisphere of Σ with a plane, say Λ, perpendicular to Π.

Consider the plane Λ. It contains points A, B, P', $\hat{P}$ and the circle $\Gamma = \Sigma \cap \Lambda$. (It also contains Q' and $\hat{Q}$ but we will not use these points for a while.)

Note that

- ⋄ $A, B, P' \in \Gamma$,
- ⋄ $[AB]$ is a diameter of Γ,
- ⋄ $(AB) = \Pi \cap \Sigma$,
- ⋄ $\hat{P} \in [AB]$
- ⋄ $(P'\hat{P}) \perp (AB)$.

Since $[AB]$ is the diameter, the angle $\angle APB$ is right. Hence $\triangle A\hat{P}P' \sim \triangle AP'B \sim \triangle P'\hat{P}B$. In particular

$$\frac{AP'}{BP'} = \frac{A\hat{P}}{P'\hat{P}} = \frac{P'\hat{P}}{B\hat{P}}.$$

Therefore

❸ $$\frac{A\hat{P}}{B\hat{P}} = \left(\frac{AP'}{BP'}\right)^2.$$

The same way we get

❹ $$\frac{A\hat{Q}}{B\hat{Q}} = \left(\frac{AQ'}{BQ'}\right)^2.$$

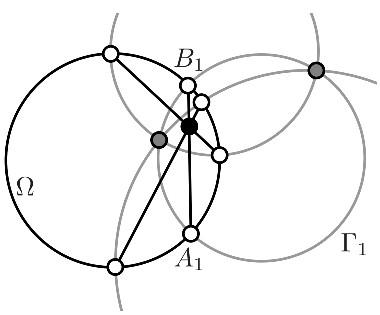
Finally note that ❷+❸+❹ imply ❶.

The last statement follows from ❶ and the definition of h-distance. Indeed,

$$\begin{aligned} PQ_h &\stackrel{\text{def}}{=\!=} \left|\ln \frac{AQ\cdot BP}{QB\cdot PA}\right| = \\ &= \left|\ln \left(\frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}\right)^{\frac{1}{2}}\right| = \\ &= \tfrac{1}{2}\cdot \left|\ln \frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}\right|. \end{aligned}$$

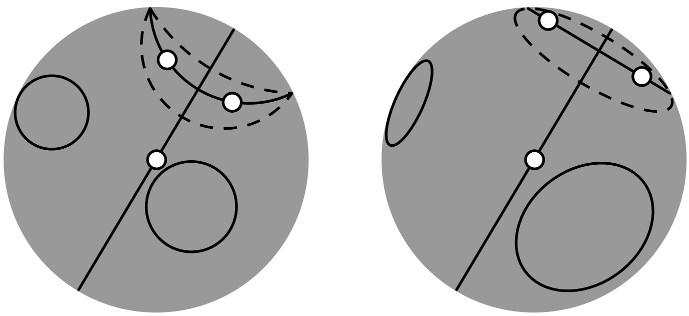
□

14.3. Exercise. *Let Γ_1, Γ_2 and Γ_3 be three circles perpendicular to the circle Ω. Let us denote by $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ the common chords of Ω and Γ_1, Γ_2, Γ_3 correspondingly. Show that the chords $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ intersect at one point inside Ω if and only if Γ_1, Γ_2 and Γ_3 intersect at two points.*



Klein model

The following picture illustrates the map $P \mapsto \hat{P}$ described in the previous section. If you take the picture on the left and apply the map $P \mapsto \hat{P}$, you get the picture on the right. The picture on the right gives a new way to look at the hyperbolic plane, which is called *Klein model*. One may think of the map $P \mapsto \hat{P}$ as about translation from one model to the other.

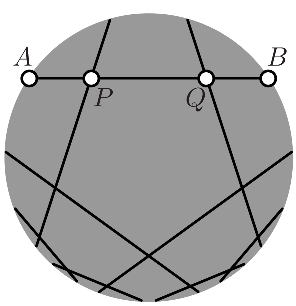


Poincaré model Klein model

In the Klein model things look different; some become simpler, other things become more complicated.

◇ The h-lines in the Klein model are formed by chords. More precisely, they are formed by the intersections of chords of the absolute wit the h-plane.

◇ The h-circles and equidistants in the Klein model are formed by ellipses and their intersections with the h-plane. It follows since the stereographic projection sends circles one the plane to the circles on the unit sphere and the orthogonal projection of circle back to plane is formed by ellipse[1].



◇ To find the h-distance between the points P and Q in the Klein model, you have to find the points of intersection, say A and B, of the Euclidean line (PQ) with the absolute; then, by Lemma 14.2,

$$PQ_h = \tfrac{1}{2}\cdot\left|\ln\frac{AQ\cdot BP}{QB\cdot PA}\right|.$$

[1]One may define ellipse as the projection of a circle which lies in the space to the plane.

◇ The angle measures in Klein model are very different from the Euclidean angles and it is hard to figure out by looking on the picture. For example all the intersecting h-lines on the picture above are perpendicular. There are two useful exceptions

  ◦ If O is the center of absolute then

$$\measuredangle_h AOB = \measuredangle AOB.$$

  ◦ If O is the center of absolute and $\measuredangle OAB = \pm\frac{\pi}{2}$ then

$$\measuredangle_h OAB = \measuredangle OAB = \pm\tfrac{\pi}{2}.$$

To find the angle measure in Klein model, you may apply a motion of h-plane which moves the vertex of the angle to the center of absolute; once it is done the hyperbolic and Euclidean angles have the same measure.

The following exercise is hyperbolic analog of Exercise 13.9. This is the first example of a statement which admits an easier proof using Klein model.

14.4. Exercise. *Let P and Q be the point in h-plane which lie on the same distance from the center of absolute. Observe that in Klein model, h-midpoint of $[PQ]_h$ coincides with the Euclidean midpoint of $[PQ]_h$.*

Conclude that if an h-triangle is inscribed in an h-circle then its medians intersect at one point.

Think how to prove the same for a general h-triangle.

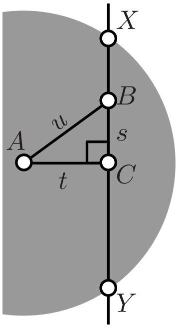
Hyperbolic Pythagorean theorem

14.5. Theorem. *Assume that $\triangle_h ACB$ is a triangle in h-plane with right angle at C. Set $a = BC_h$, $b = CB_h$ and $c = AB_h$. Then*

❺ $$\operatorname{ch} c = \operatorname{ch} a \cdot \operatorname{ch} b.$$

where ch *denotes* hyperbolic cosine*; i.e., the function defined the following way*

$$\operatorname{ch} x \stackrel{\text{def}}{=\!=} \tfrac{e^x + e^{-x}}{2}.$$



Proof. We will use Klein model of h-plane with a unit circle as the absolute.

We can assume that A is the center of absolute. Therefore both $\angle_h ACB$ and $\angle ACB$ are right.

Set $s = BC$, $t = CA$, $u = AB$. According to Euclidean Pythagorean theorem (6.8),

$$u^2 = s^2 + t^2.$$

Note that

$$b = \tfrac{1}{2} \cdot \ln \frac{1+t}{1-t};$$

therefore

$$\begin{aligned} \operatorname{ch} b &= \frac{\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} + \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}}}{2} = \\ &= \frac{1}{\sqrt{1-t^2}}. \end{aligned}$$

The same way we get

$$c = \tfrac{1}{2} \cdot \ln \frac{1+u}{1-u}$$

and

$$\begin{aligned} \operatorname{ch} c &= \frac{\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} + \left(\frac{1-u}{1+u}\right)^{\frac{1}{2}}}{2} = \\ &= \frac{1}{\sqrt{1-u^2}}. \end{aligned}$$

Let X and Y are the ideal points of $(BC)_h$. Applying the Pythagorean theorem (6.8) again, we get

$$CX^2 = CY^2 = 1 - t^2.$$

Therefore

$$a = \tfrac{1}{2} \cdot \ln \frac{\sqrt{1-t^2} + s}{\sqrt{1-t^2} - s}$$

and

$$\operatorname{ch} a = \frac{\left(\frac{\sqrt{1-t^2}+s}{\sqrt{1-t^2}-s}\right)^{\frac12} + \left(\frac{\sqrt{1-t^2}-s}{\sqrt{1-t^2}+s}\right)^{\frac12}}{2} =$$

$$= \frac{\sqrt{1-t^2}}{\sqrt{1-t^2-s^2}}$$

$$= \frac{\sqrt{1-t^2}}{\sqrt{1-u^2}}$$

Hence ❺ follows. □

14.6. Exercise. *Give a proof of Proposition 12.1 using Klein model.*

Bolyai's construction

Assume we need to construct a line asymptotically parallel to the given line through the given point. The initial configuration is given by three points, say P, A and B and we need to construct a line through P which is asymptotically parallel to $\ell = (AB)$.

Note that ideal points do not lie in the h-plane, so there is no way to use them in the construction.

The following construction was given by Bolyai. Unlike the other construction given earlier in the lectures, this construction works in absolute plane; i.e., it works in Euclidean and in hyperbolic plane as well. We assume that you know a compass-and-ruler construction of perpendicular line through the given point.

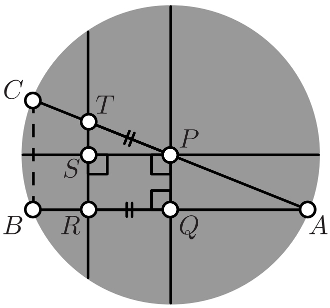
14.7. Bolyai's construction.

1. *Construct the line m through P which perpendicular to ℓ. Denote by Q the foot point of P on ℓ.*
2. *Construct the line n through P which perpendicular to m.*
3. *Draw the circle Γ_1 with center Q through P and mark by R a point of intersection of Γ_1 with ℓ.*
4. *Construct the line k through R which perpendicular to n.*
5. *Draw the circle Γ_2 with center P through Q and mark by T a point of intersection of Γ_2 with k.*
6. *The line PT is asymptotically parallel to ℓ.*

You can use this link to a java applet to perform the construction.

Note that in Euclidean plane Γ_2 is tangent to k, so the point T is uniquely defined. In hyperbolic plane the Γ_2 intersects k in two points, both of the corresponding lines are asymptotically parallel to ℓ, one from left and one from right.

To prove that Bolyai's construction gives the asymptotically parallel line in h-plane, it is sufficient to show the following.



14.8. Proposition. *Assume P, Q, R, S, T be points in h-plane such that*

- ◇ $S \in (RT)_h$,
- ◇ $(PQ)_h \perp (QR)_h$,
- ◇ $(PS)_h \perp (PQ)_h$,
- ◇ $(RT)_h \perp (PS)_h$ *and*
- ◇ $(PT)_h$ *and* $(QR)_h$ *are asymptotically parallel.*

Then $QR_h = PT_h$.

Proof. We will use the Klein's model. Without loss of generality, we may assume that P is the center of absolute. As it was noted on page 119, in this case the corresponding Euclidean lines are also perpendicular; i.e., $(PQ) \perp (QR)$, $(PS) \perp (PQ)$ and $(RT) \perp (PS)$.

Denote by A be the ideal point of $(QR)_h$ and $(PT)_h$. Denote by B and C the remaining ideal points of $(QR)_h$ and $(PT)_h$ correspondingly.

Note that the Euclidean lines (PQ), (TR) and (CB) are parallel. Therefore $\triangle AQP \sim \triangle ART \sim \triangle ABC$. In particular,

$$\frac{AC}{AB} = \frac{AT}{AR} = \frac{AP}{AQ}.$$

It follows that

$$\frac{AT}{AR} = \frac{AP}{AQ} = \frac{BR}{CT} = \frac{BQ}{CP}.$$

In particular

$$\frac{AT \cdot CP}{TC \cdot PA} = \frac{AR \cdot BQ}{RB \cdot QA};$$

hence $QR_h = PT_h$. □

Chapter 15

Complex coordinates

In this chapter we give an interpretation of inversive geometry using complex coordinates. The results of this chapter will not be used further in the lectures.

Complex numbers

Informally, a complex number is a number that can be put in the form

❶ $$z = x + i{\cdot}y,$$

where x and y are real numbers and $i^2 = -1$.

The set of complex numbers will be further denoted by $\mathbb{C}$. If x, y and z as in ❶, then x is called the real part and y the imaginary part of the complex number z. Briefly it is written as $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

On the more formal level, a complex number is a pair of real numbers (x, y) with addition and multiplication described below. The formula $x + i{\cdot}y$ is only convenient way to write the pair (x, y).

$$(x_1 + i{\cdot}y_1) + (x_2 + i{\cdot}y_2) \stackrel{\text{def}}{=\!=} (x_1 + x_2) + i{\cdot}(y_1 + y_2);$$

$$(x_1 + i{\cdot}y_1){\cdot}(x_2 + i{\cdot}y_2) \stackrel{\text{def}}{=\!=} (x_1{\cdot}x_2 - y_1{\cdot}y_2) + i{\cdot}(x_1{\cdot}y_2 + y_1{\cdot}x_2).$$

Complex coordinates

Recall that one can think of Euclidean plane as the set of all pairs of real numbers (x, y) equipped with the metric

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

where $A=(x_A,y_A)$ and $B=(x_B,y_B)$.

One can pack coordinates (x,y) of a point in the Euclidean plane, in one complex number $z=x+i{\cdot}y$. This way we get one-to-one correspondence between points of Euclidean plane and $\mathbb{C}$. Given a point $Z=(x,y)$, the complex number $z=x+i{\cdot}y$ is called *complex coordinate* of Z.

Note that if O, E and I are the points in the plane with complex coordinates 0, 1 and i then $\measuredangle EOI=\pm\frac{\pi}{2}$. Further we assume that $\measuredangle EOI=\frac{\pi}{2}$; if not, one has to change the direction of the y-coordinate.

Conjugation and absolute value

Let $z=x+i{\cdot}y$ and both x and y are real. Denote by Z the point in the plane with complex coordinate z.

If $y=0$, we say that z is a *real* and if $x=0$ we say that z is an *imaginary* complex number. The set of points with real and imaginary complex coordinates form lines in the plane, which are called *real* and *imaginary* lines which will be denoted as $\mathbb{R}$ and $i{\cdot}\mathbb{R}$.

The complex number $\bar z=x-iy$ is called *complex conjugate* of z.

Note that the point $\bar Z$ with complex coordinate $\bar z$ is the reflection of Z in the real line.

It is straightforward to check that

❷ $$x=\operatorname{Re}z=\frac{z+\bar z}{2},\quad y=\operatorname{Im}z=\frac{z-\bar z}{i{\cdot}2},\quad x^2+y^2=z{\cdot}\bar z.$$

The last formula in ❷ makes possible to express the quotient $\frac{w}{z}$ of two complex numbers w and $z=x+i{\cdot}y$:

$$\frac{w}{z}=\tfrac{1}{z\cdot\bar z}{\cdot}w{\cdot}\bar z=\tfrac{1}{x^2+y^2}{\cdot}w{\cdot}\bar z.$$

Note that

$$\overline{z+w}=\bar z+\bar w,\qquad \overline{z-w}=\bar z-\bar w,\qquad \overline{z{\cdot}w}=\bar z{\cdot}\bar w,\qquad \overline{z/w}=\bar z/\bar w;$$

i.e., all the algebraic operations *respect* conjugation.

The value $\sqrt{x^2+y^2}=\sqrt{z{\cdot}\bar z}$ is called *absolute value* of z and denoted by $|z|$.

Note that if Z and W are points in the Euclidean plane and z and w their complex coordinates then

$$ZW=|z-w|.$$

Euler's formula

Let α be a real number. The following identity is called *Euler's formula.*

❸ $$e^{i\cdot\alpha} = \cos\alpha + i\cdot\sin\alpha.$$

In particular, $e^{i\cdot\pi} = -1$ and $e^{i\cdot\frac{\pi}{2}} = i$.

Geometrically Euler's formula means the following. Assume that O and E are the point with complex coordinates 0 and 1 correspondingly. Assume $OZ = 1$ and $\measuredangle EOZ \equiv \alpha$ then $e^{i\cdot\alpha}$ is the complex coordinate of Z. In particular, the complex coordinate of any point on the unit circle centered at O can be uniquely expressed as $e^{i\cdot\alpha}$ for some $\alpha \in (-\pi, \pi]$.

A complex number z is called *unit* if $|z| = 1$. According to Euler's identity, in this case

$$z = e^{i\cdot\alpha} = \cos\alpha + i\cdot\sin\alpha$$

for some value $\alpha \in (-\pi, \pi]$.

Why should you think that ❸ is true? The proof of Euler's identity depends on the way you define exponent. If you never had to take exponent of imaginary number, you may take the right hand side in ❸ as the definition of the $e^{i\cdot\alpha}$.

In this case formally nothing has to be proved, but it is better to check that $e^{i\cdot\alpha}$ the satisfies familiar identities. For example

$$e^{i\cdot\alpha}\cdot e^{i\cdot\beta} = e^{i\cdot(\alpha+\beta)}.$$

Which can be proved using the following trigonometric formulas, which we assume to be known:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta\\ \sin(\alpha+\beta) &= \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta\end{aligned}$$

If you know power series for sine, cosine and exponent, the following might convince that ❸ is the right definition.

$$\begin{aligned}e^{i\cdot x} &= 1 + i\cdot x + \frac{(i\cdot x)^2}{2!} + \frac{(i\cdot x)^3}{3!} + \frac{(i\cdot x)^4}{4!} + \frac{(i\cdot x)^5}{5!} + \dots = \\ &= 1 + i\cdot x - \frac{x^2}{2!} - i\cdot\frac{x^3}{3!} + \frac{x^4}{4!} + i\cdot\frac{x^5}{5!} - \dots = \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = \\ &= \cos x + i\cdot\sin x.\end{aligned}$$

Argument and polar coordinates

As above, assume that O and E denote the points with complex coordinates 0 and 1 correspondingly.

Let Z be the point distinct form O. Set $\rho = OZ$ and $\vartheta = \measuredangle EOZ$. The pair (ρ, ϑ) is called *polar coordinates* of Z.

If z is the complex coordinate of Z then then $\rho = |z|$. The value ϑ is called argument of z (briefly, $\vartheta = \arg z$). In this case

$$z = \rho \cdot e^{i \cdot \vartheta} = \rho \cdot (\cos \vartheta + i \cdot \sin \vartheta).$$

Note that

$$\arg(z \cdot w) \equiv \arg z + \arg w \quad \text{and} \quad \arg \tfrac{z}{w} \equiv \arg z - \arg w$$

if $z, w \neq 0$. In particular, if Z, V, W be points with complex coordinates z, v and w correspondingly then

❹ $$\measuredangle VZW = \arg\left(\frac{w-z}{v-z}\right) \equiv \arg(w-z) - \arg(v-z)$$

once the left hand side is defined.

15.1. Exercise. *Use the formula* ❹ *to show that in any triangle* $\triangle ZVW$

$$\measuredangle ZVW + \measuredangle VWZ + \measuredangle WZV \equiv \pi.$$

15.2. Exercise. *Assume that points* V, W *and* Z *have complex coordinates* v, w *and* $v \cdot w$ *correspondingly and the point* O *and* E *as above. Sow that*

$$\triangle OEV \sim \triangle OWZ.$$

The following Theorem is a reformulation of Theorem 8.5 which use complex coordinates.

15.3. Theorem. *Let* $UVWZ$ *be a quadrilateral and* u, v, w *and* z *be the complex coordinates of its vertices. Then* $UVWZ$ *is inscribed if and only if the number*

$$\frac{(v-u)\cdot(w-z)}{(v-w)\cdot(z-u)}$$

is real.

The value $\frac{(v-u)\cdot(w-z)}{(v-w)\cdot(z-u)}$ will be called *complex cross-ratio*, it will be discussed in more details below.

15.4. Exercise. *Observe that the complex number* $z \neq 0$ *is real if and only if* $\arg z = 0$ *or* π*; in other words,* $2 \cdot \arg z \equiv 0$.

Use this observation to show that Theorem 15.3 is indeed a reformulation of Theorem 8.5.

Möbius transformations

15.5. Exercise. *Watch video "Möbius Transformations Revealed" by Douglas Arnold and Jonathan Rogness. (It is 3 minutes long and available on YouTube.)*

The complex plane $\mathbb{C}$ extended by one ideal number ∞ is called extended complex plane. It is denoted by $\hat{\mathbb{C}}$, so $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Möbius transformation of $\hat{\mathbb{C}}$ is a function of one complex variable z which can be written as

$$f(z) = \frac{a{\cdot}z + b}{c{\cdot}z + d},$$

where the coefficients a, b, c, d are complex numbers satisfying $a{\cdot}d - b{\cdot}c \neq 0$. (If $a{\cdot}d - b{\cdot}c = 0$ the function defined above is a constant and is not considered to be a Möbius transformation.)

In case $c \neq 0$, we assume that

$$f(-d/c) = \infty \quad \text{and} \quad f(\infty) = a/c;$$

and if $c = 0$ we assume

$$f(\infty) = \infty.$$

Elementary transformations

The following three types of Möbius transformations are called *elementary.*

1. $z \mapsto z + w$,
2. $z \mapsto w{\cdot}z$ for $w \neq 0$,
3. $z \mapsto \frac{1}{z}$.

The geometric interpretations. As before we will denote by O the point with complex coordinate 0.

The first map $z \mapsto z + w$, corresponds to so called *parallel translation* of Euclidean plane, its geometric meaning should be evident.

The second map is called *rotational homothety* with center at O. I.e., the point O maps to itself and any other point Z maps to a point Z' such that $OZ' = |w|{\cdot}OZ$ and $\measuredangle ZOZ' = \arg w$.

The third map can be described as a composition of inversion in the unit circle centered at O and the reflection in $\mathbb{R}$ (any order). Indeed, $\arg z \equiv -\arg \frac{1}{z}$ therefore

$$\arg z = \arg(1/\bar{z});$$

i.e., if the points Z and Z' have complex coordinates z and $1/\bar{z}$ then $Z' \in [OZ)$. Clearly $OZ = |z|$ and $OZ' = |1/\bar{z}| = \frac{1}{|z|}$. Therefore Z' is inversion of Z in the unit circle centered at O. Finally the reflection of Z' in $\mathbb{R}$, has complex coordinate $\frac{1}{z} = \overline{(1/\bar{z})}$.

15.6. Proposition. *A map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation if and only if it can be expressed as a composition of elementary Möbius transformation.*

Proof; (⇒). Consider, the Möbius transformation

$$f(z) = \frac{a\cdot z + b}{c\cdot z + d}.$$

It is straightforward to check that

❺ $$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z),$$

where

- ◇ $f_1(z) = z + \frac{d}{c}$,
- ◇ $f_2(z) = \frac{1}{z}$,
- ◇ $f_3(z) = -\frac{a\cdot d - b\cdot c}{c^2}\cdot z$,
- ◇ $f_4(z) = z + \frac{a}{c}$

if $c \neq 0$ and

- ◇ $f_1(z) = \frac{a}{d}\cdot z$,
- ◇ $f_2(z) = z + \frac{b}{d}$,
- ◇ $f_3(z) = f_4(z) = z$

if $c = 0$.

(⇐). We need to show that composing elementary transformations, we can only get Möbius transformations. Note that it is sufficient to check that composition of a Möbius transformations

$$f(z) = \frac{a\cdot z + b}{c\cdot z + d}.$$

with any elementary transformation is a Möbius transformations.

The later is done by means of direct calculations.

$$\frac{a\cdot(z+w)+b}{c\cdot(z+w)+d} = \frac{a\cdot z + (b + a\cdot w)}{c\cdot z + (d + c\cdot w)}$$

$$\frac{a\cdot(w\cdot z)+b}{c\cdot(w\cdot z)+d} = \frac{(a\cdot w)\cdot z + b}{(c\cdot w)\cdot z + d}$$

$$\frac{a\cdot\frac{1}{z}+b}{c\cdot\frac{1}{z}+d} = \frac{b\cdot z + a}{d\cdot z + c}$$

□

15.7. Corollary. *The image of cline under Möbius transformation is a cline.*

Proof. By Proposition 15.6, it is sufficient to check that each elementary transformation sends cline to cline.

For the first and second elementary transformation the later is evident.

As it was noted above, the map $z \mapsto \frac{1}{z}$ is a composition of inversion and reflection. By Theorem 9.9, inversion sends cline to cline. Hence the result follows. □

15.8. Exercise. *Show that inverse of Möbius transformation is a Möbius transformation.*

15.9. Exercise. *Given distinct values $z_0, z_1, z_\infty \in \hat{\mathbb{C}}$, construct a Möbius transformation f such that $f(z_0) = 0$, $f(z_1) = 1$ and $f(z_\infty) = \infty$. Show that such transformation is unique.*

Complex cross-ratio

Given four distinct complex numbers u, v, w, z, the complex number

$$\frac{(u-w)\cdot(v-z)}{(v-w)\cdot(u-z)}$$

is called *complex cross-ratio*; it will be denoted as $(u, v; w, z)$.

If one of the numbers u, v, w, z, is ∞, then the complex cross-ratio has to be defined by taking the appropriate limit; in other words, we assume that $\frac{\infty}{\infty} = 1$. For example,

$$(u, v; w, \infty) = \frac{(u-w)}{(v-w)}.$$

Assume that U, V, W and Z be the points with complex coordinates u, v, w and z correspondingly. Note that

$$\begin{aligned}\frac{UW\cdot VZ}{VW\cdot UZ} &= |(u, v; w, z)|,\\ \measuredangle WUZ + \measuredangle ZVW &= \arg\frac{u-w}{u-z} + \arg\frac{v-z}{v-w} \equiv\\ &\equiv \arg(u, v; w, z).\end{aligned}$$

It makes possible to reformulate Theorem 9.5 using the complex coordinates the following way.

15.10. Theorem. *Let $UWVZ$ and $U'W'V'Z'$ be two quadrilaterals such that the points U', W', V' and Z' are inversions of U, W, V, and Z correspondingly. Assume u, w, v, z, u', w', v' and z' be the complex coordinates of U, W, V, Z, U', W', V' and Z' correspondingly.*

Then

$$(u', v'; w', z') = \overline{(u, v; w, z)}.$$

The following Exercise is a generalization of the Theorem above. It admits a short and simple solution which use Proposition 15.6.

15.11. Exercise. *Show that complex cross-ratios are invariant under Möbius transformations. That is, if a Möbius transformation maps four distinct complex numbers u, v, w, z to complex numbers u', v', w', z' respectively, then*

$$(u', v'; w', z') = (u, v; w, z).$$

Index

$(u, v; w, z)$, 129
$\cong$, 15
∞, 69
$\|$, 45
$\perp$, 37
$\sim$, 46
d_1, 12
d_2, 12
d_∞, 12

absolute, 88
absolute plane, 79
absolute value, 124
acute
  acute angle, 37
  acute triangle, 48
altitude, 53
angle, 14
  acute angle, 37
  angle of parallelism, 97
  negative angle, 27
  obtuse angle, 37
  positive angle, 27
  right angle, 37
  straight angle, 24
  vertical angles, 25
angle measure, 22
  hyperbolic angle measure, 89
angle-side-angle congruence condition, 33
arc, 65
area, 86
ASA congruence condition, 33
asymptotically parallel lines, 97

base of isosceles triangle, 34
between, 24
bijection, 13
bisector, 41
  external bisector, 41

center, 18, 43
centroid, 54
chord, 43
circle, 18, 43
circle arc, 64
cline, 70
complex conjugate, 124
conformal factor, 103
congruent triangles, 15
cross-ratio, 68
  complex cross-ratio, 126, 129
curvature, 86

defect of triangle, 83
diagonal
  diagonals of quadrilateral, 51
diameter, 43
direct motio, 40
discrete metric, 11
distance, 11
distance-preserving map, 13

elementary transformation, 127
endpoint, 65
equidistant, 100

equivalence relation, 47
Euclidean metric, 12
Euclidean plane, 22
Euclidean space, 109
Euler's formula, 125

foot point, 38

geometric construction, 18
great circle, 110

h-angle measure, 89
h-circle, 92
h-half-line, 88
h-line, 88
h-plane, 88
h-radius, 92
h-segment, 88
half-plane, 29
horocycle, 101
hyperbolic angle measure, 89
hyperbolic cosine, 119
hyperbolic plane, 88
hypotenuse, 48

ideal point, 88
imaginary complex number, 124
imaginary line, 124
incenter, 56
incircle, 56
indirect motion, 40
inradius, 56
inscribed triangle, 62
intersecting lines, 45
inverse, 13
inversion, 67
  center of inversion, 67, 111
  circle of inversion, 67
  inversion in a sphere, 111
  sphere of inversion, 111
inversive plane, 70
inversive space, 111
isometry, 13
isosceles triangle, 34

Klein model, 118

leg, 48
line, 14

Möbius transformation, 127
  elementary transformation, 127
Manhattan metric, 12
maximum metric, 12
metric, 11
metric space, 11
motion, 13

neutral plane, 79

obtuse
  obtuse angle, 37
orthic triangle, 57
orthocenter, 54

parallel lines, 45
  ultra parallel lines, 97
parallel translation, 127
parallelogram, 51
perpendicular, 37
perpendicular bisector, 37
perpendicular circles, 72
pint at infinity, 69
plan
  hyperbolic plane, 88
plane
  absolute plane, 79
  Euclidean plane, 22
  h-plane, 88
  inversive plane, 70
  neutral plane, 79
  plane in the space, 109
Poincaré disk model, 88
point, 11
  ideal point, 88
polar coordinates, 126

quadrilateral, 51
  inscribed quadrilateral, 63

nondegenerate quadrilateral, 51

radius, 18, 43
real complex number, 124
real line, 12, 124
reflection, 39
rotational homothety, 127

SAA condition, 81
SAS condition, 33
secant line, 43
side
 side of quadrilateral, 51
 side of the triangle, 30
side-angle-angle congruence condition, 81
side-angle-side condition, 33
side-side-side congruence condition, 34
similar triangles, 46
sphere, 110
spherical distance, 110
stereographic projection, 112

tangent circles, 43
tangent half-line, 65
tangent line, 43
triangle, 15
 congruent triangles, 15
 degenerate triangle, 25
 ideal triangle, 99
 orthic triangle, 57
 right triangle, 48
 similar triangles, 46

unit complex number, 125

vertex of the angle, 14
vertical angles, 25

Used resources

[1] *Euclid's Elements*, a web version by David Joyce available at `aleph0.clarku.edu/~djoyce/java/elements/toc.html`

[2] Eugenio Beltrami, *Teoria fondamentale degli spazii di curvatura costante,* Annali. di Mat., ser II, 2 (1868), 232–255

[3] Birkhoff, George David, A Set of postulates for plane geometry, based on scale and protractors, *Annals of Mathematics* 33 (1932), 329–345.

[4] Marvin J. Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History,* 4th ed., New York: W. H. Freeman, 2007.

[5] Lambert, Johann Heinrich, *Theorie der Parallellinien,* F. Engel, P. Stäckel (Eds.) (1786) Leipzig.

[6] Legendre, Adrien-Marie, *Eléments de géométrie*, 1794.

[7] Лобачевский, Никлай Иванович, О началах геометрии, *Казанский вестник,* вып. 25–28 (1829–1830 гг.).

[8] Moise, Edwin. *Elementary Geometry From An Advanced Standpoint,* 3rd ed. Boston: Addison-Wesley. 1990

[9] Saccheri, Giovanni Girolamo, *Euclides ab omni nævo vindicatus*, 1733

[10] Шарыгин, Игорь Фёдорович, *Геометрия 7–9*, М.: Дрофа, 1997.

[11] Kiselev, A. P., *Kiselev's Geometry. Book I. Planimetry,* Adapted from Russian by Alexander Givental.