Newton-Okounkov Bodies of Partial Flag Varieties

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Motivation

Goal: Associate combinatorial objects (e. g. polytopes) to algebraic/geometric objects and recover information about them.
Toy example: toric varieties.
Possible uses: toric degenerations, mirror symmetry, rep. theory ...

What if there is no canonical combinatorial object?
$\rightsquigarrow$ Newton-Okounkov bodies!

Definition

A convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is called **reflexive** if $\mathcal{P}$ and $\mathcal{P}^* := \{y \in \mathbb{R}^d \mid \langle x, y \rangle \geq -1 \text{ for all } x \in \mathcal{P}\}$ are lattice polytopes.

When is the NO-Body a reflexive polytope? (based on Rusinko '08)
Q1: When is the NO-Body a lattice polytope?
Q2: When is the dual of the NO-Body a lattice polytope?

G	simple complex algebraic group
$T \subset B \subset P \subset G$	a Borel and a parabolic subgroup
$\Phi_P^+ \subset \Phi_P$	(positive) roots with respect to T, B and P
$\Lambda_P^+ \subset \Lambda_P$	dominant integral weights with respect to P
$\mathcal{L}_\lambda$	ample line bundle over G/P

$N_P := |\Phi_P^+|$, $R_\lambda := \bigoplus_{n \geq 0} H^0(G/P, \mathcal{L}_{n\lambda})$

Definition

Fix a total ordering on $\mathbb{Z}^{N_P}$. A map $v : R_\lambda \setminus \{0\} \to \mathbb{Z}^{N_P}$ is called a **valuation** if for all $c \in \mathbb{C}^\times$, $f, g \in R_\lambda \setminus \{0\}$

- $v(cf) = v(f)$,
- $v(fg) = v(f) + v(g)$ and
- $v(f+g) \geq \min\{v(f), v(g)\}$ (if $f + g \neq 0$).

We say that v has **full rank** if $\dim \langle \mathrm{Im}\, v \rangle_{\mathbb{R}} = N_P$.

Definition of NO-Bodies

Definition

Given a valuation v we define the graded semigroup

$$\Gamma(\lambda) := \{0\} \cup \bigcup_{n>0} \left\{(n, v(f)) \,\middle|\, f \in H^0(G/P, \mathcal{L}_{n\lambda}) \setminus \{0\}\right\},$$

the closed convex cone $C(\lambda) = \overline{\operatorname{cone} \Gamma(\lambda)} \subseteq \mathbb{R} \times \mathbb{R}^{N_P}$ and finally the **Newton-Okounkov body**

$$\{1\} \times \Delta(\lambda) := C(\lambda) \cap \{x_0 = 1\}.$$

Lemma

If v has full rank and $\Gamma(\lambda)$ is finitely generated and saturated, $\Delta(\lambda)$ is a rational convex polytope of dimension N_P with exactly $\dim H^0(G/P, \mathcal{L}_\lambda)$ many lattice points and $\Delta(n\lambda) = n\Delta(\lambda)$ for all $n \in \mathbb{N}$.

A Partial Solution

Theorem

Let $\lambda \in \Lambda_P^+$ be P-regular, v a full-rank valuation with $\Gamma(\lambda)$ finitely generated and saturated. The following are equivalent.

- *$\Delta(\lambda)$ contains exactly one interior lattice point p_λ.*
- *$\mathcal{L}_\lambda$ is the anticanonical line bundle over G/P.*

In this case $(\Delta(\lambda) - p_\lambda)^*$ is a lattice polytope.

Corollary

Under the same assumptions $\Delta(\lambda)$ is reflexive if and only if it is a lattice polytope and $\mathcal{L}_\lambda$ is the anticanonical line bundle over G/P.

Corollary

The Gelfand-Tsetlin polytope (in type A_n), the Feigin-Fourier-Littelmann-Vinberg polytope (in types A_n and C_n) and the Gornitskii polytope (in type G_2) for λ are reflexive (after translation by p_λ) if and only if λ is the anticanonical weight over G/P.

$\mathcal{P} \subset \mathbb{R}^d$ rational convex polytope.

$$L_{\mathcal{P}}(n) := \#(n\mathcal{P} \cap \mathbb{Z}^d).$$

Theorem (Ehrhart-Macdonald Reciprocity '71)

$L_{\mathcal{P}}$ is a quasi-polynomial of degree d and

$$L_{\operatorname{int}\mathcal{P}}(n) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-n)$$

for all $n \in \mathbb{N}$.

Theorem (Hibi '92)

Suppose $\mathcal{P}$ is full-dimensional with $0 \in \operatorname{int}\mathcal{P}$ and $L_{\mathcal{P}}$ is a polynomial. Then $\mathcal{P}^*$ is a lattice polytope if

$$L_{\mathcal{P}}(-n-1) = (-1)^d L_{\mathcal{P}}(n)$$

for all $n \in \mathbb{N}$.

We know that

$$\begin{aligned} L_{\Delta(\lambda)}(n) &= \#(\Delta(n\lambda) \cap \mathbb{Z}^{N_P}) \\ &= \dim H^0(G/P, \mathcal{L}_{n\lambda}) \\ &= \dim V(n\lambda) \\ &= \prod_{\beta \in \Phi_P^+} \frac{\langle n\lambda + \rho, \beta^\vee \rangle}{\langle \rho, \beta^\vee \rangle}. \end{aligned}$$

(For $P = B$: $L_{\Delta(2\rho)}(n) = (2n+1)^N$)

Now calculate $1 = L_{\mathrm{int}\,\Delta(\lambda)}(1)$ if and only if $\mathcal{L}_\lambda$ is the anticanonical line bundle and in that case $L_{\Delta(\lambda)}(-n-1) = (-1)^{N_P} L_{\Delta(\lambda)}(n)$.

Then Hibi's Theorem completes the proof. $\square$

$\rightsquigarrow$ **Q1':** When is $\Delta(\lambda^{\mathrm{ac}})$ a lattice polytope?

What about String Polytopes?

For every reduced decomposition $\underline{w_0}$ of the longest word of the Weyl group we get a string polytope $Q_{\underline{w_0}}(\lambda)$ [Littelmann '98, Berenstein-Zelevinsky '01, Alexeev-Brion '04, Kaveh '15].

Theorem (Rusinko '08)

Let $G = SL_n$. Then $(Q_{\underline{w_0}}(2\rho) - p_{2\rho})^*$ is a lattice polytope for every reduced decomposition $\underline{w_0}$.

Problem: String polytopes are not always lattice polytopes!

Example

Let G be of type B_2 and $\underline{w_0} = s_2 s_1 s_2 s_1$. Then $Q_{\underline{w_0}}(\omega_2)$ has one half-integral vertex. So $Q_{\underline{w_0}}(4\omega_2)$ is a lattice polytope.

Example

Let G be of type G_2 and $\underline{w_0} = s_1 s_2 s_1 s_2 s_1 s_2$. Then $Q_{\underline{w_0}}(2\rho)$ is not a lattice polytope.

Two Conjectures

Conjecture A

Let G be of type $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n$ or D_n, let $\lambda \in \Lambda^+$ and let $\underline{w_0}$ be the "standard" reduced decomposition. Then $Q_{\underline{w_0}}(\lambda)$ is a lattice polytope if and only if one of the following conditions hold.

(i) *G is of type A_n,*

(ii) *G is of type B_n and $\langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$,*

(iii) *G is of type C_n or*

(iv) *G is of type D_n and $\langle \lambda, \alpha_{n-1}^\vee \rangle + \langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$.*

Conjecture B

Let G be of type $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n$ or D_n, let G/P be a partial flag variety and let $\underline{w_0}$ be the "standard" reduced decomposition. Let $\lambda \in \Lambda_P^+$. Then $Q_{\underline{w_0}}(\lambda)$ is reflexive (after unique translation) if and only if λ is the weight of the anticanonical line bundle over G/P.

And for non-standard $\underline{w_0}$?

Example

Let $G = SL_6$ and $\underline{w_0} = s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_1$. Then $Q_{\underline{w_0}}(\omega_3)$ has one half-integral vertex. So for $G/P = Gr(3,6)$ the "anticanonical" string polytope $Q_{\underline{w_0}}(6\omega_3)$ will still be a lattice polytope.

Example

Let $G = SL_7$ and $\underline{w_0} = s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_4 s_3 s_2 s_1$. Then $Q_{\underline{w_0}}(\omega_3)$ has half-integral vertices. So for $G/P = Gr(3,7)$ the "anticanonical" string polytope $Q_{\underline{w_0}}(7\omega_3)$ will not be a lattice polytope!

Thank you for your attention!