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Topologie

Organised by
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Abstract. This conference is one of the few occasions where researchers from many different areas in algebraic and geometric topology are able to meet and exchange ideas. Accordingly, the program covered a wide range of new developments in such fields as geometric group theory, rigidity of group actions, knot theory, and stable and unstable homotopy theory. More specifically, we discussed progress on problems such as the Farrell-Jones conjecture, the Levine conjecture in grope cobordism of knots and Rosenberg's conjecture about homotopy invariance of negative algebraic K-theory, to mention just a few subjects with a name attached. One of the highlights was a series of four talks on the solution of Arf-Kervaire invariant problem by Mike Hill and Doug Ravenel, reporting on their joint work with Mike Hopkins.

Mathematics Subject Classification (2000): 55-xx, 57-xx.

Introduction by the Organisers

This conference was the first topology conferences in Oberwolfach organized by Thomas Schick, Peter Teichner, Nathalie Wahl and Michael Weiss. About 50 mathematicians participated, working in many different areas of algebraic and geometric topology.

The 18 regular talks of the conference covered a wide range of areas such as 3-manifolds and knot theory, geometric group theory, algebraic K- and L-theory, and homotopy theory. One of the goals of the conference is to foster interaction between such different areas and the passage of methods from one to the other. Four of these talks were devoted to the solution of the Kervaire invariant question by Mike Hopkins, Mike Hill and Doug Ravenel, allowing an in-depth discussion of the new ideas necessary for this breakthrough on a decades-old problem in homotopy theory.

In addition, to give the many young and very young participants the opportunity to present themselves and their work to a broader audience, a "gong show" was organized where eight participants gave an overview on their research efforts and results. Here, **Alexander Kahle** from Göttingen reported on joint work with Alessandro Valentino concerning T-duality and differential K-theory. In this work, topological T-duality (studied for example by Bunke-Schick or Mathai-Rosenberg) is enriched with geometric information using differential K-theory, and indeed a T-duality isomorphism for differential K-theory is established for pairs which are T-duals of each other. **Arturo Prat-Waldron** from Berkeley discussed Thom classes for field theories. The space of certain low dimensional field theories is used as a geometric model for associated generalized cohomology theories. The goal now is to find a geometric description of Thom classes in this model. **Ulrich Pennig** from Münster reported on twisted K-theory and obstructions to positive scalar curvature. In his work, he gives a new geometric model for twisted K-theory, proves an index theorem in the spirit of Kasparov's in this context and uses this to extend the enlargeability-obstruction to positive scalar curvature to non-spin manifolds. **Georgios Raptis** from Osnabrück described K-theory of derivators. He gives an example of two differential graded algebras that have the same derivator K-theory but non-isomorphic Waldhausen K-theory. He also proves that Maltsiniotis' comparison and localization conjectures for derivator K-theory cannot be simultaneously true. **Justin Noel** from Strasbourg studied the complex orientations preserving power operations. This is used in particular to classify which complex oriented cohomology theories can be given an H-infinity ring structure compatible with the standard E-infinity ring structure on MU. **Lennart Meier** from Bonn gave a new proof, from a stacky point of view, of Bousfield's classification of modules over real K-theory KO. The crucial ingredient is to find all KO-modules M such that $M \wedge_{KO} KU$ is KU-free. This approach should eventually lead to a generalization to modules over topological modular forms TMF. **Wolfgang Steimle** from Münster described obstructions to stably fibering manifolds. In particular, he introduced and calculated family versions of Whitehead torsion, using higher algebraic K-theory of spaces. **David Ayala** from Copenhagen described a simple and combinatorial E_n operad which is built out of finite posets indexing a stratification of configuration spaces of points in an n-disk, posets which recently became important for modeling weak n-categories. A version of Dunn's additivity theorem is a formal consequence of the set up.

We now report on some of the highlights of the regular talks, whose abstracts form the main part of this report.

Arthur Bartels talked about joint work with Wolfgang Lück and Tom Farrell, proving the Farrell-Jones conjecture and algebraic K-theory conjecture for cocompact lattices in connected Lie groups. One of the most spectacular consequences is the Borel conjecture on topological rigidity of aspherical manifolds whose fundamental group is a cocompact lattice as above. The proof uses controlled topology and new constructions of transfer homomorphisms. In other talks on geometric group theory, Michelle Bucher-Karlsson described the explicit calculation of the Gromov norm of the universal Euler class in the cohomology of classifying space for oriented vector bundles, and Stefan Friedl gave a survey on the group theoretic properties of fundamental groups of three manifolds, culminating in his joint result with Matthias Aschenbrenner that all such groups are for every prime number p virtually residually p.

Andreas Thom presented his joint work with Guillermo Cortiñas that negative algebraic K-theory of the algebra $C(X)$ is a homotopy invariant of the space X. This finally implements a strategy proposed several decades ago by Jonathan Rosenberg. Among many other tools, algebraic geometry —including Hironaka's resolution of singularities— is crucially used.

Cameron Gordon gave (in joint work with Danny Calegari) a complete classification of all knots in closed 3-manifolds of small rational genus. Jim Conant presented a proof of the Levine conjecture and applications to grope cobordism (this is joint work with Peter Teichner).

An curious application of homotopy theory to the theory of finite groups was described by Bob Oliver. He presented a purely algebraic result about the p-subgroup structure in finite groups of Lie type, one which was reduced to a statement about their classifying spaces and then proved using the homotopy theory of p-local finite groups. No algebraic proof of this result is available up to now. Jesper Grodal used homotopy theory to study the group of self-homotopy equivalences of a finite CW-complex X and found explicit bounds on the order of groups with a faithful action up to homotopy on X.

Bernhard Hanke described a new construction of families of manifolds with positive scalar curvature, jointly carried out with Boris Botvinnik, Thomas Schick and Mark Walsh, which is based on a family version of the Gromov-Lawson surgery method and which provides the first examples of non-trivial elements of the moduli space of metrics of positive scalar curvature.

Other talks addressed a construction of a delooping of the 2-category of von Neumann algebras (with bimodules and Connes fusion as 1-morphism and 2-morphism) via conformal nets, the complete classification of characteristic cohomology classes of Morita-Miller-Mumford type for bundles of closed manifolds of arbitrary dimension, stable moduli spaces of highly connected high-dimensional manifolds, and algebraic models of equivariant stable homotopy theories.

The famous Oberwolfach atmosphere made this meeting another wonderful success, and all thanks go to the institute for creating this atmosphere and making the conference possible.

Workshop: Topologie

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Abstracts

The Farrell-Jones Conjecture for cocompact lattices

Arthur Bartels
(joint work with Wolfgang Lück, Tom Farrell)

Via surgery theory the classification of high-dimensional manifolds depends on the understanding of the algebraic K-theory and L-theory of integral group rings. The work of Farrell-Jones on the Borel conjecture has lead to what is now known as the Farrell-Jones conjecture:
Let G be a group and R be a ring (with involution). Then the assembly maps

$$\begin{aligned} \alpha^K_{VCyc} \quad &: \quad H^G_*(E_{VCyc}G; \mathbf{K}_R) \to K_*(R[G]) \\ \alpha^L_{VCyc} \quad &: \quad H^G_*(E_{VCyc}G; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_*^{\langle -\infty \rangle}(R[G]) \end{aligned}$$

are isomorphisms.

In the original formulation only integral group rings were considered. This conjecture implies many other conjectures, most notable the Borel conjecture in dimension ≥ 5. If the conjecture is known, then it reduces in some sense the computation of $K_*(R[G])$ to the computations of the groups $K_*(R[V])$ where the groups V are virtually cyclic subgroups of G. Similarly, it reduces in some sense the computation of $L_*^{\langle -\infty \rangle}(R[G])$ to the computations of the groups $L_*^{\langle -\infty \rangle}(R[V])$ where the groups V are virtually cyclic subgroups of G.

Theorem 1. *Let G be a cocompact lattice in a virtually connected Lie group. Then*

(1) α^L_{VCyc} *is an isomorphism,*
(2) α^K_{VCyc} *is an isomorphism for* $* < 1$ *and surjective for* $* = 1$.

This does not quite prove the full Farrell-Jones conjecture, but it suffices for many important applications.

By an argument of Farrell-Jones to prove the above theorem it suffices to prove the assertion for $CAT(0)$-groups and for polycyclic groups. The methods used to prove the result for these groups are formally quite similar. In both cases controlled topology is used to give a more geometric description of the assembly maps as forget control maps, a transfer argument and contracting maps are used. However, the construction of the transfer and the contracting maps are quite different in these two cases.

In the case of $CAT(0)$-groups the fiber for the transfer is a contractible space and construction of the transfer uses the singular chain complex of this contractible space. The contracting map depends in this case on dynamic properties of the group. More precisely, an analog of the geodesic flow of non-positively curved manifolds is used. Ultimately this is a refinement of a method developed and used by Farrell-Jones.

In the case of polycyclic groups group homomorphisms $f\colon G \to F$ to finite quotients are used. The fiber for the transfer is then the finite discrete space

$$\coprod_{H \in \mathcal{H}} G/f^{-1}(H)$$

where $\mathcal{H}$ is the family of hyper-elementary subgroups of F. The construction of the transfer uses Frobenius induction and depends on work of Swan (for K-theory) and Dress (in for L-theory). The construction of the contracting map depends on the correct choice of f. For example it is important here to choose f such that $f^{-1}(H)$ will have large index in G for all $H \in \mathcal{H}$. Ultimately this is a refinement of a method developed and used by Farrell-Hsiang.

From Heegaard Floer homology to embedded contact homology via open book decompositions

Paolo Ghiggini

(joint work with Vincent Colin, Ko Honda)

1. Introduction

Heegaard Floer homology, embedded contact homology and monopole Floer homology are three Floer theories which produce TQFT-like invariants for smooth 3- and 4-manifolds. In their simplest form (the so-called "hat" version), they all associate finite dimensional vector spaces over $\mathbb{Z}/2\mathbb{Z}$ to any closed, oriented and connected 3-manifold.

These theories have different origins: monopole Floer homology is a 3-dimensional version of Seiberg–Witten theory, and was defined by Kronheimer and Mrowka [KM]. Embedded contact homology is a variant of symplectic field theory, and was defined by Hutchings and Taubes [Hu, HT1, HT2]. Finally Heegaard Floer homology is a Lagrangian intersection Floer homology in a symmetric product of a Heegaard surface, and was defined by Ozsvàth and Szábo [OSz1, OSz2].

Despite their differences, these three theories show similar formal properties and are conjecturally equivalent. A large portion of the isomorphism between monopole Floer homology and embedded contact homology has been recently established by Taubes [Ta]. The goal of this report is to announce the following isomorphism:

Theorem 1. *Let M be a closed, connected, oriented 3-manifold. There is an isomorphism*

$$\Phi\colon \widehat{HF}(-M) \to \widehat{ECH}(M).$$

where $\widehat{HF}$ and $\widehat{ECH}$ denote the hat-versions of Heegaard Floer homology and of embedded, and $-M$ is the manifold M with the opposite orientation.

We refer to the original papers for the definitions, to [Li] for the cylindrical reformulation of Heegaard Floer homology which is used in the definition of Φ, and to [CGHH] for the definition of the hat-version of embedded contact homology.

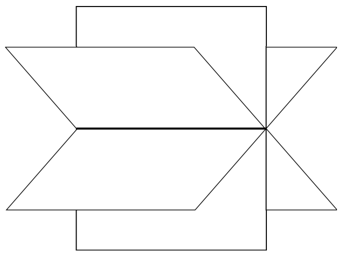
A different proof of a similar result has been announced recently by Kutluhan, Lee, and Taubes: [KLT1, KLT2, KLT3].

2. Open book decompositions

The link between the geometry underlying Heegaard Floer homology and and the geometry underlying embedded contact homology is provided by open book decompositions.

Definition. *An* open book decomposition *of M is a pair (B, π) where:*

- *$B \subset M$ is a compact 1-dimensional submanifold (i. e. a link)*
- *$\pi \colon M \setminus B \to S^1$ is a locally trivial fibration*
- *near every component of B the fibration looks like:*



Theorem 2. *(Alexander) Every* 3*-manifold admits an open book decomposition.*

B is called the *binding*, and $S_\theta = \overline{\pi^{-1}(\theta)}$ a *page*. We can identify the complement of a neighbourhood of B with the suspension of a diffeomorphism $\phi\colon S \to S$ where S is a surface with boundary which is homeomorphic to S_θ. We can assume without loss of generality that ∂S is connected, and ϕ is the identity on ∂S and a small negative rotation in a neighbourhood of ∂S. The suspension of ϕ will be denoted by N_ϕ.

An open book decomposition (B, π) gives rise to a Heegaard splitting $M = H_1 \cup_\Sigma H_2$ where:

$$H_1 = \overline{\pi^{-1}([0,\pi])}, \quad H_2 = \overline{\pi^{-1}([\pi, 2\pi])}, \quad \Sigma = -\overline{\pi^{-1}(0)} \cup \overline{\pi^{-1}(\pi)}.$$

An open book decomposition gives rise also to a contact structure.

Definition. *(Giroux) A contact form α on M is* supported *by an open book decomposition (B, π) if*

- *every connected component of B is a periodic orbit of the Reeb vector field*
- *the Reeb vector field is positively transverse to the pages.*

Theorem 3. *(Thurston–Winkelnkemper,* [TW]*) Every open book decomposition of a* 3*-manifold supports a contact form.*

The converse is also true, by the fundamental work of Giroux [Gi].

3. Modification of $\widehat{HF}$

On S we choose a collection of disjoint arcs $\mathbf{a} = (a_1, \dots, a_{2g})$ which cut S into a disc, and consider also the collection of arcs $\phi(\mathbf{a}) = (\phi(a_1), \dots, \phi(a_{2g}))$. Then $(S, \mathbf{a}, \phi(a_{2g})$ is half of a Heegaard diagram (Σ, α, β) for the Heegaard splitting induced by the open book decomposition.

We define $\widetilde{CF}(S, \underline{a}, \phi(\underline{a}))$ as the subcomplex of the Heegaard Floer complex defined from the Heegaard diagram (Σ, α, β). We denote by $\overline{x}_i, \overline{x}_i'$ the intersection points in $a_i \cap \phi(a_i) \cap \partial S$, and finally we define $\widehat{CF}(S, \underline{a}, \phi(\underline{a}))$ as the quotient of $\widetilde{CF}(S, \underline{a}, \phi(\underline{a}))$ obtained by identifying the intersection point $\overline{x}_i$ with $\overline{x}_i'$ for all i. It is not hard to see that $\widehat{CF}(S, \underline{a}, \phi(\underline{a}))$ is a chain complex, and that its homology is isomorphic to $\widehat{HF}(-M)$.

4. Modification of $\widehat{ECH}$

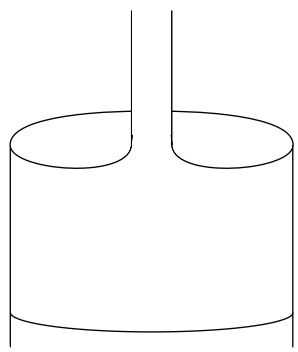
In [CGH] we proved that we can perform the Thurston-Winkelnkemper construction to (B, π) with some extra care so that the following are true:

- the Reeb vector field of α has an elliptic orbit γ_e and a hyperbolic orbit γ_h on ∂N_ϕ, and
- the subgroup $\widetilde{ECC}(M, \alpha)$ of $ECC(M, \alpha)$ generated by orbit sets containing only Reeb orbits in N_ϕ is a subcomplex.

Moreover the quotient of $\widetilde{ECC}(M, \alpha)$ obtained by declaring equivalent all orbit sets which differ only by the multiplicity of γ_e is a chain complex and its homology is isomorphic to $\widehat{ECH}(M)$.

5. Construction of the map

Let B be the surface with a positive strip-like end and a negative cylindrical end obtained by smoothing the corners of $(\mathbb{R} \times S^1) \setminus ([0, +\infty) \times [0, \pi])$.



Let $\pi_B \colon W_B \to B$ be a bundle with fibre S and monodromy ϕ. On W_B we put a symplectic form Ω_B which makes it into a symplectic fibration, and a compatible almost complex structure J_B which makes the projection $\pi_B \colon W_B \to B$ J_B-holomorphic. On $\pi_B^{-1}(\partial B)$ we put a Lagrangian submanifold Λ which coincides with $(n, +\infty) \times \{\pi\} \times \underline{a}$ on $\pi_B^{-1}((n, +\infty) \times \{\pi\})$ and with $(n, +\infty) \times \{0\} \times h(\underline{a})$ on $\pi_B^{-1}((n, +\infty) \times \{0\})$.

$(W_B, \Omega_B, J_B, \Lambda)$ provide a cobordism between the geometric setup of the definition of $\widetilde{CF}(\Sigma, \underline{a}, h(\underline{a}))$ and the geometric setup for the definition of $\widetilde{ECC}(M, \alpha)$. A chain map $\widetilde{\Phi}\colon \widetilde{CF}(\Sigma, \underline{a}, h(\underline{a})) \to \widetilde{ECC}(M, \alpha)$ is defined by counting J_B-holomorphic multi-sections of W_B with boundary on Λ and connecting HF generators at the strip-like end with ECH generators at the cylindrical end. The map $\widetilde{\Phi}$ passes to the quotient and defines a map $\Phi\colon \widehat{HF}(-M) \to \widehat{ECH}(M)$.

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A solution to the Arf-Kervaire invariant problem

Mike Hill and Doug Ravenel

(joint work with Mike Hopkins)

For more information on this topic, including links to our preprint and detailed notes for our talks, we refer the reader to the second author's website

`http://www.math.rochester.edu/u/faculty/doug/kervaire.html`

We gave a series of four lectures, the first by the second author and the rest by the first author. The latter was followed by a 2 hour question and answer session.

The problem in question is nearly 50 years old and began with Kervaire's paper [Ker60] of 1960 in which he defined a $\mathbf{Z}/2$-valued invariant $\phi(M)$ on certain manifolds M of dimension $4m+2$. He showed that for $m = 2$ and M aclosed smooth manifolds, it must vanish. He also constructed a topological 10-manifold M on which it is nontrivial. This was one of the earliest examples of a nonsmoothable manifold. Milnor's paper on exotic 7-spheres [Mil56] had appeared four years earlier. In their subsequent joint work [KM63] they gave a complete classification of exotic spheres in dimensions ≥ 5 in terms of the stable homotopy groups of spheres, modulo a question about manifolds which they left unanswered:

> For which m is there a smooth framed manifold of dimension $4m+2$ with nontrivial Kervaire invariant?

Such manifolds were known to exist for $m = 0$, 1 and 3, and Kervaire had shown there are none for $m = 2$. A pivotal step in answering the question was the following result of Browder [Bro69] published in 1969.

Browder's Theorem. *The Kervaire invariant $\phi(M)$ of a smooth framed manifold M of dimension $4m+2$ is trivial unless $m = 2^{j-1} - 1$ for some $j > 0$. In that case such an M with $\phi(M) \neq 0$ exists if and only if the element h_j^2 in the Adams spectral sequence is a permanent cycle.*

The Adams spectral sequence referred to in the theorem was first introduced in [Ada58], and we refer the reader to [Rav04] for more information. The relation between framed manifolds and stable homotopy groups of spheres had been established decades earlier by Pontryagin.

This result raised the stakes considerably and brought the problem into the realm of stable homotopy thoery. The name θ_j was given to the hypothetical element in the stable homotopy group $\pi_{2^{j+1}-2}(S^0)$ representing the permanent cycle h_j^2. It was known to exist for $j = 1$, 2 and 3. In the next few years its existence was established for $j = 4$ ([BMT70] and [Jon78]) and $j = 5$ [BJM84]. It was widely believed that such framed manifolds existed for *all* values of j. In the ensuing decade there were many unsuccessful (and unpublished) attempts to construct them. *We now know that they were trying to prove the wrong theorem.*

In [Mah67] Mahowald described a beautiful pattern in the unstable homotopy groups of spheres based on the assumption that the θ_j exist for all j. It was so compelling that the possibility that they did not all exist was later called the DOOMSDAY HYPOTHESIS. After 1985 the problem faded into the background because it was thought to be inaccessible. In early 2009 Snaith published a book [Sna09] on it "to stem the tide of oblivion."

Soon after we announced the following.

Main Theorem. *The element $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ (representing h_j^2 in the Adams spectral sequence and corresponding to a framed manifold in the same dimension with nontrivial Kervaire invariant) does not exist for $j \geq 7$.*

Our proof relies heavily on equivariant stable homotopy theory and complex cobordism theory. Neither was available in the 1970s. Here is our strategy.

We construct a nonconnective ring spectrum Ω with the following properties:

(i) **Detection Theorem.** *If θ_j exists, its composition with the unit map $S^0 \to \Omega$ is nontrivial.*

(ii) **Periodicity Theorem.** *$\pi_k\Omega$ depends only on k modulo 256.*

(iii) **Gap Theorem.** $\pi_{-2}\Omega = 0$.

Note that (ii) and (iii) imply that $\pi_{254}\Omega = 0$, and 254 is the dimension of θ_7. But (i) says that if θ_7 exists it has nontrivial image in this group, so it cannot exist. The argument for larger j is similar.

Our spectrum Ω is the fixed point set of a C_8-equivariant spectrum $\tilde{\Omega}$, i.e., $\Omega = \tilde{\Omega}^{C_8}$. We will describe $\tilde{\Omega}$ below. It also has a homotopy fixed point $\tilde{\Omega}^{hC_8}$. We show that it has properties (i) and (ii), while the actual fixed point set satisfies (iii). Thus we need a fourth result,

(iv) **Fixed Point Theorem.** *The map $\tilde{\Omega}^{C_8} \to \tilde{\Omega}^{hC_8}$ is an equivalence.*

The starting point for constructing $\tilde{\Omega}$ is the observation, originally due to Landweber [Lan68], that the complex cobordism spectrum MU has a C_2-equivariant structure defined in terms of complex conjugation. Recall that MU is defined in terms of Thom spaces of certain complex vector bundles over complex Grassmannians. Complex conjugation acts on everything in sight and commutes with the relevant structure maps. The resulting equivariant spectrum is known as real cobordism theory and is denoted by $MU_{\mathbf{R}}$.

Next there is a formal construction which we call the norm for inducing up from an H-equivariant spectrum X to form a G-equivariant spectrum $N_H^G X$ for any finite group G containing H. The underlying spectrum (meaning the one we get by forgetting the equivariant structure) of $N_H^G X$ is the $|G/H|$-fold smash power of X. G then acts by permuting the factors, each of which is invariant under H. The case of interest to us is $H = C_2$, $X = MU_{\mathbf{R}}$ and $G = C_8$. The underlying spectrum of $N_H^G MU_{\mathbf{R}}$ is $MU^{(4)}$, the 4-fold smash power of MU.

Let V be a real representation of G and let S^V be its one point compactification. For a G-equivariant space or spectrum X we denote the group of equivariant maps from S^V to X by $\pi_V^G X$. In this way a G-equivariant spectrum X has homotopy groups indexed by $RO(G)$, the real representation ring of G. These are denoted collectively by $\pi_\star^G X$.

We can now describe our C_8-equivariant spectrum $\tilde{\Omega}$. We choose a certain element $D \in \pi_{19\rho}^{C_8} MU_{\mathbf{R}}^{(4)}$, where ρ denotes the real regular representation of C_8. There are many choices of D that would lead to Periodicity (possible with periods other than 256) and Gap Theorems. Ours is the simplest one that also gives the Detection Threorem. Since $MU_{\mathbf{R}}^{(4)}$ is a ring spectrum, we get a map

$$MU_{\mathbf{R}}^{(4)} \xrightarrow{D} \Sigma^{-19\rho} MU_{\mathbf{R}}^{(4)}.$$

This can be iterated, and we define $\tilde{\Omega}$ to be the resulting telescope,

$$\tilde{\Omega} = D^{-1} MU_{\mathbf{R}}^{(4)}.$$

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Knots with small rational genus

Cameron McA. Gordon
(joint work with Danny Calegari)

If K is a rationally null-homologous knot in a 3-manifold M then there is a compact orientable surface S in the exterior of K whose boundary represents $p[K]$ in $H_1(N(K))$ for some $p > 0$. We define $\|K\|$, the *rational genus* of K, to be the infimum of $-\chi^-(S)/2p$ over all S and p. (Here $\chi^-(S)$ is the sum of $\chi^-(S_0) = \min\{\chi(S_0), 0\}$ over the components S_0 of S, as in the definition of the Thurston norm.) If M is a homology sphere then $\|K\|$ is essentially the genus of K. By doing surgery on knots in S^3 one can produce knots in 3-manifolds with arbitrarily small rational genus. We show that such knots are related to the geometry of the ambient manifold in a specific way. More precisely we show that there is a positive constant C (1/50 will do) such that if K is a knot in a 3-manifold M with $\|K\| < C$ then (M, K) belongs to one of a small number of classes; for example, M is hyperbolic and K is isotopic to a core of a *Margulis tube* (i.e. a geodesic of length ≤ 0.162 with tube radius ≥ 0.531), M is Seifert fibered and K is a fiber, K lies in a JSJ torus in M, etc. Conversely we show that there are pairs (M, K) in each of these classes with $\|K\|$ arbitrarily small.

Fundamental groups of 3-manifolds

Stefan Friedl

(joint work with Matthias Aschenbrenner)

Our goal is to study properties of fundamental groups of closed 3-dimensional manifolds. (Here and throughout the paper all manifolds are assumed to be oriented.) This class of groups sits between the class of fundamental groups of surfaces, which for the most part are well understood, and the class of fundamental groups of 4-manifolds which are very badly understood for the simple reason that any finitely presented group can appear as the fundamental group of a closed 4-manifold.

In the following we say that a closed 3-manifold N is *prime* if it does not admit a non-trivial connected sum decomposition. A classic theorem due to Kneser states that any closed 3-manifold is the connected sum of finitely many prime 3-manifolds. Put differently, the fundamental group of any closed 3-manifold is the free product of fundamental groups of prime 3-manifolds. For most intents and purposes we can thus restrict ourselves to the study of fundamental groups of prime 3-manifolds.

By Perelman's proof of the Thurston Geometrization Conjecture we know that prime closed 3-manifolds fall into three categories:

(1) hyperbolic 3-manifolds,
(2) Seifert fibered 3-manifolds (loosely speaking these are 3-manifolds which are the total space of a singular S^1-bundle over a surface), and
(3) 3-manifolds which are obtained from gluing hyperbolic 3-manifolds and Seifert fibered spaces along incompressible tori.

It is well-known that fundamental groups of hyperbolic 3-manifolds are subgroups of $\mathrm{SL}(2, \mathbb{C})$ and that fundamental groups of Seifert fibered 3-manifolds are linear over $\mathbb{Z}$, i.e. subgroups of $\mathrm{SL}(n, \mathbb{Z})$ for some sufficiently large n. The following question naturally arises.

Question. *Let N be a prime 3-manifold. Is $\pi_1(N)$ linear, i.e. is $\pi_1(N)$ a subgroup of $GL(n, \mathbb{C})$ for some n?*

This question remains wide open. In support of an affirmative answer we will in the following verify that 3-manifold groups satisfy all the properties of finitely generated linear groups which we are aware of.

We first need to introduce the following definitions.

(1) A group π satisfies the *Tits alternative* if π is either virtually solvable or if it contains a non-abelian free group.

(2) Given a ring R we say that π has *virtual infinite R-Betti number* (which we denote by $vb_1(\pi; R) = \infty$) if given any $k \in \mathbb{N}$ there exists a finite index subgroup $\pi' \subset \pi$ such that $b_1(\pi'; R) \geq k$.
(3) A group π is *residually finite* if given any non-trivial $g \in \pi$ there exists a homomorphism $\alpha : \pi \to G$ to a finite group G such that $\alpha(g) \neq e$.
(4) Given a prime p we say that a group π is *residually p* if given any non-trivial $g \in \pi$ there exists a homomorphism $\alpha : \pi \to G$ to a p-group G (i.e. a group of p-power order) such that $\alpha(g) \neq e$.
(5) Given a property $\mathcal{P}$ of groups we say that π is *virtually $\mathcal{P}$* if π admits a finite index subgroup which satisfies $\mathcal{P}$.

Note that a group which is virtually residually p for some prime p is also residually finite. We can now list the most interesting properties of linear groups which we are aware of.

Theorem 1. *Let π be a finitely generated linear group, then the following hold:*
(1) *π satisfies the Tits alternative,*
(2) *the group π is either solvable or for all primes p the group π satisfies $vb_1(\pi; \mathbb{F}_p) = \infty$,*
(3) *π is residually finite,*
(4) *for almost all primes p the group π is virtually residually p.*

The first property was discovered by Tits [9], the second property is a consequence of the Lubotzky alternative (see [5, Corollary 16.4.18] and [4, Theorem 1.3]) and the third property was proved by Mal'cev [7]. The last property admits a short and elegant proof which we provide below. We refer to [10, Theorem 4.7] for full details.

Proof of Theorem 1 (4). Let π be a finitely generated subgroup of $\mathrm{GL}(n, \mathbb{C})$. Since π is finitely generated there exists a finitely generated subring R of $\mathbb{C}$ such that $\pi \subset \mathrm{GL}(n, R)$. It is well–known that for almost all primes p there exists a maximal ideal $\mathfrak{m}$ of R with $\mathrm{char}(R/\mathfrak{m}) = p$ (see [5, p. 376f]).

Now let p be a prime for which there exists a maximal ideal $\mathfrak{m}$ of R with $\mathrm{char}(R/\mathfrak{m}) = p$. We will show that π is virtually residually p. Before we continue note that $R/\mathfrak{m}^k$ is a finite ring for any $k \geq 1$ and that $\bigcap_{k=1}^{\infty} \mathfrak{m}^k = \{0\}$ by the Krull Intersection Theorem. For $k \geq 1$ we let

$$\pi_k = \ker\left(\pi \to \mathrm{GL}(n, R) \to \mathrm{GL}(n, R/\mathfrak{m}^k)\right).$$

Each π_k is a normal subgroup of π, of finite index, and clearly $\pi_{k+1} \subset \pi_k$ for every $k \geq 1$. Moreover $\bigcap_{k=1}^{\infty} \pi_k = \{1\}$ since $\bigcap_{k=1}^{\infty} \mathfrak{m}^k = \{0\}$.

We claim that π_1 is residually p. We will prove this by showing that π_1/π_k is a p-group for any k. This in turn follows from showing that any non–trivial element in π_k/π_{k+1} has order p. In order to show this pick $A \in \pi_k$. By definition we can write

$$A = \mathrm{id} + C \qquad \text{for some } n \times n\text{-matrix } C \text{ with entries in } \mathfrak{m}^k.$$

From $p \in \mathfrak{m}$ and $k \geq 1$ we get that

$$\begin{array}{rcl} A^p = (\mathrm{id} + C)^p & = & \mathrm{id} + pC + \frac{p(p-1)}{2} C^2 + \cdots + C^p \\ & = & \mathrm{id} + (\text{some } n \times n\text{-matrix with entries in } \mathfrak{m}^{k+1}). \end{array}$$

Hence $A^p \in \pi_{k+1}$. □

The following theorem gives evidence to the conjecture that 3-manifold groups are linear.

Theorem 2. *Let N be a prime 3-manifold. We write $\pi = \pi_1(N)$. Then the following hold:*

(1) *π satisfies the Tits alternative,*
(2) *the group π is either solvable or for all primes p the group π satisfies $vb_1(\pi; \mathbb{F}_p) = \infty$,*
(3) *π is residually finite,*
(4) *for almost all primes p the group π is virtually residually p.*

By the above discussion we only have to prove Theorem 2 for 3-manifolds N which admit an incompressible torus. Note that N is Haken and Property (1) now follows from [2, Corollary 4.10]. Since N admits an incompressible torus it follows from work of Luecke [6] that $\pi = \pi_1(N)$ is either virtually solvable or $vb_1(N; \mathbb{Z}) = \infty$. In particular in the latter case we have $vb_1(N; \mathbb{F}_p) = \infty$ for any prime p.

Hempel [3] (see also [8]) showed that $\pi = \pi_1(N)$ is residually finite. The proof of Hempel has two key ingredients:

(1) subgroups carried by JSJ tori are maximal abelian, separable subgroups of the cobounding JSJ components,
(2) HNN extensions and amalgamated products of finite groups are residually finite.

In [1] we showed that for almost all primes p the group $\pi = \pi_1(N)$ is virtually residually p. A first standard argument shows that since the fundamental groups of all JSJ components of N are virtually residually p one can find a finite cover N' of N such that the fundamental groups of all JSJ components of N' are residually p. The key technical difficulty in extending Hempel's argument at that stage is the fact that amalgamated products of p-groups are not necessarily residually p. Overcoming this problem requires a very careful study of the filtrations by p-power index subgroups of the JSJ components of N'.

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The norm of the Euler class

Michelle Bucher-Karlsson
(joint work with Nicolas Monod)

Norms of cohomology classes were introduced by Gromov and have important applications since they give *a priori* bounds for characteristic numbers. These in turn give rise to Milnor-Wood inequialities. Unfortunately, these norms turn out to be complicated to compute explicitly, and the only norms known to this day are those of the Kähler class of Hermitian symmetric spaces in degree 2 (Domic-Toledo, Clerc-Ørsted), of the Euler class in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and of the volume form in hyperbolic n-space (Gromov, Thurston), though the latter norm is only explicit in low dimension. We compute the norm of the universal Euler class of flat oriented vector bundles in degree n; this is new for $n > 2$.

Monoids of moduli spaces of manifolds, II

Oscar Randal-Williams
(joint work with Søren Galatius)

In my talk I announced new results on the homology of stable diffeomorphism groups of highly-connected manifolds. In order to motivate these results, we consider first the diffeomorphism groups of compact 0-manifolds: these are the symmetric groups and have classifying spaces $B\Sigma_n$. Adding an extra point to a 0-manifold induces maps $B\Sigma_n \to B\Sigma_{n+1}$, and theorems of Barratt–Priddy [1] and Quillen [7] provide a homology equivalence

$$\mathrm{hocolim}_{n\to\infty} B\Sigma_n \longrightarrow \Omega_0^\infty \mathbf{S}$$

from the direct limit to a component of the infinite loop space corresponding to the sphere spectrum.

Moving up two dimensions, let us write $B\mathrm{Diff}_\partial^+(\Sigma_{g,1})$ for the classifying space of the group of diffeomorphisms of a connected, oriented surface of genus g with a single boundary component, where the diffeomorphisms are required to restrict to the identity on the boundary. Gluing on a torus with two boundary components along a single boundary gives a map $B\mathrm{Diff}^+_\partial(\Sigma_{g,1}) \to B\mathrm{Diff}^+_\partial(\Sigma_{g+1,1})$. The theorem of Madsen and Weiss [5] provides a homology equivalence

$$\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}^+_\partial(\Sigma_{g,1}) \longrightarrow \Omega^\infty_0\mathbf{MTSO}(2)$$

from the direct limit to a component of the infinite loop space corresponding to the negative of the tautological bundle over $BSO(2)$, which was used to compute the stable rational homology of Riemann's moduli space of curves, and hence prove the Mumford conjecture.

Our results give an extension of these theorems to all higher even dimensions, except dimension 4. In order to state them, let us write

$$\theta : BO(2n)[n+1,\infty) \longrightarrow BO(2n) \qquad \bar\theta : BO(2n)[n,\infty) \longrightarrow BO(2n)$$

for the n- and $(n-1)$-connected covers of $BSO(2n)$, and let $\mathbf{MT}\theta$ and $\mathbf{MT}\bar\theta$ be the Thom spectra of the virtual bundles $-\theta^*\gamma_{2n}$ and $-\bar\theta^*\gamma_{2n}$ respectively, where $\gamma_{2n} \to BO(2n)$ is the universal bundle.

Theorem A. *Let $W_g := \#^g S^n \times S^n$, and $n \neq 2$. The map*

$$\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}(W_g, D^{2n}) \longrightarrow \Omega^\infty_0\mathbf{MT}\theta$$

is a homology equivalence.

Theorem B. *Suppose W is a $(n-1)$-connected $2n$-manifold such that $\pi_n(W) \to \pi_n(BO)$ is surjective. Let $\overline{W}_g := W\#W_g$, and $n \neq 2$. The map*

$$\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}(\overline{W}_g, D^{2n}) \longrightarrow \Omega^\infty_0\mathbf{MT}\bar\theta$$

is a homology equivalence.

In both of these statements, we may replace the homotopy colimit by the classifying space of a diffeomorphism group with compact support, and so compute the homology of $B\mathrm{Diff}_c(W_\infty)$ and $B\mathrm{Diff}_c(\overline{W}_\infty)$ respectively.

These theorems follow from a more technical statement about *cobordism categories*. Let $\theta : \mathbf{X} \to BO(d)$ be a map and $\mathcal{C}_\theta$ be the associated cobordism category. Roughly speaking, it's morphisms are d-dimensional cobordisms in $[0,t]\times\mathbb{R}^\infty$ equipped with a θ-structure on their tangent bundles, and objects are closed $(d-1)$-manifolds in $\mathbb{R}^\infty$ with a θ-structure on their once-stabilised tangent bundles. For technical reasons it is convenient to also assume that the line $[0,t]\times\{0\}$ is contained in every cobordism, and the origin is contained in every object. A full and detailed definition appears in [3] (based on the definition in [2]) where the category is called $\mathcal{C}^\bullet_\theta$.

We first filter the category $\mathcal{C}_\theta$ by subcategories $\mathcal{C}^\kappa_\theta$ containing all objects, but only those morphisms W such that $(W, \partial_{out}W)$ is κ-connected. We further filter each $\mathcal{C}^\kappa_\theta$ by the full subcategories $\mathcal{C}^{\kappa,\ell}_\theta$ on the objects which are ℓ-connected: that is, $M \subset \mathbb{R}^\infty$ containing the origin is an object of $\mathcal{C}^{\kappa,\ell}_\theta$ precisely if $\pi_{\leq\ell}(M,0) = 0$.

Theorem C. *For $d \neq 4$, the inclusion*

$$B\mathcal{C}_\theta^{\kappa,\ell} \longrightarrow B\mathcal{C}_\theta \simeq \Omega^{\infty-1}\mathbf{MT}\theta$$

is a weak homotopy equivalence as long as

(1) $2\kappa \leq d-1$,
(2) $\ell \leq \kappa$,
(3) $\ell + \kappa \leq d-2$
(4) $\mathbf{X}$ *is ℓ-connected.*

Furthermore, we require the technical assumption that $\theta : \mathbf{X} \to BO(d)$ is the homotopy pullback of a map to $BO(d+1)$.

In the case $d = 2n \neq 4$, the objects of $\mathcal{C}_\theta^{n-1,n-1}$ are $(n-1)$-connected $(2n-1)$-manifolds and hence homotopy spheres, but we can say more. Let us restrict ourselves to the case where $\theta : \mathbf{X} \to BO(2n)$ is either the n- or $(n-1)$-connected cover of $BO(2n)$, and choose an object of $\mathcal{C}_\theta^{n-1,n-1}$ diffeomorphic to the standard sphere. Let $\mathcal{E}$ denote the monoid of endomorphisms of S^{2n-1} in the category $\mathcal{C}_\theta^{n-1,n-1}$. We prove that in this case the inclusion

$$B\mathcal{E} \longrightarrow B\mathcal{C}_\theta^{n-1,n-1}$$

is a weak homotopy equivalence onto the path component it hits, which along with Theorem C identifies the group-completion

$$\Omega B\mathcal{E} \simeq \Omega^\infty \mathbf{MT}\theta.$$

It is not difficult to verify that the monoid $\mathcal{E}$ is homotopy-commutative, and so the group-completion theorem [6] may be applied to it. Theorems A and B follow from two observations about the monoid $\mathcal{E}$: firstly that the submonoid $\mathcal{E}'$ consisting of those path components represented by manifolds W such that $\pi_d(W) \to \pi_d(BO)$ is surjective is a cofinal submonoid (and hence has the same group completion), and secondly that the discrete monoid $\pi_0(\mathcal{E}')$ may be group-completed by inverting multiplication by $S^d \times S^d$, which follows by a theorem of Kreck [4].

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Algebraic independence of generalized Morita-Miller-Mumford classes

Johannes Ebert

Let M be a closed oriented smooth n-manifold, $\mathrm{Diff}^+(M)$ be the topological group of orientation-preserving diffeomorphisms and $f : E \to B$ be a fibre bundle with fibre M and structure group $\mathrm{Diff}^+(M)$. Let $T_v E \to E$ be the vertical tangent bundle; it is an n-dimensional oriented real vector bundle. Given $c \in H^{k+n}(BSO(n);\mathbb{Q})$, we can define the *generalized Miller-Morita-Mumford classes* of E using the Gysin homomorphism:

$$\kappa_E(c) := f_!(c(T_v E)) \in H^k(B;\mathbb{Q}).$$

In the case $n = 2$, one obtains the characteristic classes of surface bundles that were defined by Miller [5], Morita [6] and Mumford [7]. Miller and Morita have shown that these classes are algebraically independent in a sense that is explained below. The goal of this project is to extend the result to the higher-dimensional case. To state our theorem, we need some notation.

Let $\sigma^{-n}H^*(BSO(n))$ be the graded vector space whose degree k part is 0 for $k \leq 0$ and $H^{n+k}(BSO(n);\mathbb{Q})$ for $k > 0$. Then $c \mapsto \kappa_E(c)$ is a linear map $\sigma^{-n}H^*(BSO(n)) \to H^*(B)$ of graded vector spaces.

Let $\mathcal{R}_n$ be a set of representatives for the oriented diffeomorphism classes of oriented closed n-manifolds (connected or non-connected). Put

$$\mathcal{B}_n := \coprod_{M \in \mathcal{R}_n} B\,\mathrm{Diff}^+(M). \tag{1}$$

There are tautological manifold bundles on these spaces and therefore we get a linear map

$$\kappa^n : \sigma^{-n}H^*(BSO(n)) \to H^*(\mathcal{B}_n). \tag{2}$$

Let $\Lambda\sigma^{-n}H^*(BSO(n))$ denote the free graded-commutative algebra generated by $\sigma^{-n}H^*(BSO(n))$; there is an algebra map

$$\Lambda\kappa^n : \Lambda\sigma^{-n}H^*(BSO(n)) \to H^*(\mathcal{B}_n). \tag{3}$$

Here is our main result:

Theorem 2. *[J. Ebert,* [1]*]*

(1) *If n is even, then $\Lambda\kappa^n : \Lambda\sigma^{-n}H^*(BSO(n)) \to H^*(\mathcal{B}_n)$ is injective.*
(2) *If n is odd, then the kernel of $\Lambda\kappa^n : \Lambda\sigma^{-n}H^*(BSO(n)) \to H^*(\mathcal{B}_n)$ is the ideal that is generated by the components of the Hirzebruch $\mathcal{L}$-class.*

The case $n = 2$ of Theorem 2 is due to Miller and Morita. This is also an immediate consequence of the Madsen-Weiss theorem [4]. The fact that the kernel of $\Lambda\kappa^{2m+1}$ contains the Hirzebruch has been proven in [2] using a classical index-theoretic argument. The proof of Theorem 2 goes in the following steps:

(1) A formal argument using Barratt-Priddy-Quillen-Segal and Nakaoka stability reduces the problem to the injectivity of the linear map κ^n.
(2) Multiples of the Euler class can be easily detected on sphere bundles; this reduces the problem to the detection of polynomials in the Pontrjagin classes.
(3) By taking products of manifold bundles, one shows the following conditional statement for the even-dimensional case: Let n be even, suppose that Theorem 2 has been shown for all even dimensions less than n and suppose finally that $\kappa_n(\mathrm{ph}_k) \neq 0$, where $\mathrm{ph}_k \in H^{4k}(BSO(n); \mathbb{Q})$ is the component of the Pontrjagin character. Then Theorem 2 holds for dimension n. The Miller-Morita theorem serves as an induction beginning.
(4) A similar inductive statement is true for odd dimensions, but we have to assume $n \geq 7$ (the 3-dimensional case of Theorem 2 is empty).
(5) One shows that the universal $\mathbb{CP}^{2r}$-bundle on $BSU(r+1)$ detects the classes ph_k. Moreover, the universal $\mathbb{CP}^2$-bundle detects all classes except the Hirzebruch $\mathcal{L}$-class.
(6) As a device to increase the dimension of a bundle, we use the following pullback construction. Given an M-bundle $E \to X$, we form the pullback via the evaluation map $\mathbb{S}^1 \times LX \to X$ from the free loop space and project to LX to obtain an $\mathbb{S}^1 \times M$-bundle on LX. The MMM-classes of this new bundle relate to the MMM-classes of E via the transgression $H^*(X) \to H^{*-1}(LX)$ which is injective as long as X is simply-connected and has the rational cohomology as a product of Eilenberg-MacLane space. This argument is used to tie the several loose ends of the proof.

The map $\Lambda\kappa^n$ is closely related to the universal Madsen-Tillmann-Weiss map [3] $\alpha_n : \mathcal{B}_n \to \Omega^\infty \mathrm{MTSO}(n)$ into the infinite loop space of a well-known Thom spectrum. There is a classical isomorphism

$$H^*(\Omega^\infty \mathrm{MTSO}(n); \mathbb{Q}) \cong \Lambda \sigma^{-n} H^*(BSO(n); \mathbb{Q}),$$

under which the maps α_n^* and $\Lambda\kappa_n$ correspond to each other. Thus our theorem answers the question whether the map $\mathcal{B}_n \to \Omega^\infty \mathrm{MTSO}(n)$ is injective in rational cohomology for dimensions different from 2.

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Symmetry of spaces and subgroups of $B\mathrm{aut}(X)$

Jesper Grodal
(joint work with Bill Dwyer)

The goal of this talk is to report on some joint work in progress [2] on understanding the subgroup structure of the topological monoid of self-homotopy equivalences $\mathrm{aut}(X)$ of a finite simply connected CW–complex X. Here, by a subgroup we mean a map $BG \to B\mathrm{aut}(X)$ that satisfies one of several equivalent conditions that justifies calling it a "monomorphism". Subgroups in this sense correspond to faithful group actions on X, up to homotopy.

Perhaps the first indication that $B\mathrm{aut}(X)$ should have interesting group theoretic properties was discovered by Sullivan and Wilkerson, who showed that $\pi_0(\mathrm{aut}(X))$ is an arithmetic group [3]. It is natural to speculate in which ways this can be extended to a space-level statement.

The type of naïve questions we are interested in are the following:

(1) How many subgroups are there? (e.g., finite or infinite?)
(2) How "large" can a subgroup be?
(3) What does $B\mathrm{aut}(X)$ look like cohomologically?

Concerning (1), it was observed some time ago by J. Smith that the set of conjugacy classes of subgroups of, say, order p can be infinite, e.g., for $X = S^3 \vee S^3 \vee S^5$, a departure from what happens for arithmetic groups. However, it follows from the work of Grodal–Smith, that for $X = S^n$ the set of conjugacy classes of subgroups isomorphic to a fixed finite group G is indeed finite. Here we show that for any simply connected finite CW–complex the set of conjugacy classes of subgroups which correspond to free actions is also finite, and examine other cases where the same conclusion holds.

Concerning (2), one can define a homotopical version of the classical notion of the p–rank of symmetry as follows:

$$h\text{-}rk_p(X) = \max\{r \,|\, \exists \text{ mono } f\colon\ B(\mathbb{Z}/p)^r \to B\mathrm{aut}(X)\}$$

The homotopy p–rank of symmetry $h\text{-}rk_p(X)$ provides an upper bound for the free p–rank of symmetry, as well as the corresponding S^1–rank of symmetry. Earlier work of Grodal–Smith implies that $h\text{-}rk_2(S^n) = n+1$ and $h\text{-}rk_p(S^{2n-1}) = h\text{-}rk_p(S^{2n}) = n$, p odd, realized by the standard reflections in the coordinates.

One basic question one may ask is whether the rank is always finite. We answer this in the affirmative:

Theorem 4 (Dwyer–Grodal)**.** *For any finite simply connected CW–complex X, $h\text{-}rk_p(X) < \infty$.*

Via Lannes' theory the statement implies that the transcendence degree of $H^*(B\mathrm{aut}_1(X); \mathbb{F}_p)$ is finite, where the subscript denotes the identity component, providing information regarding (3).

The current proof produces bounds on $h\text{-}rk_p(X)$ which are homotopic in nature, in particular they depend on information about the homotopy type of Postnikov sections of $B\mathrm{aut}(X)$. It seems reasonable to expect better and more algebraic bounds. This could provide analogs for faithful actions of various results and conjectures of Browder, Carlsson, and others on the free rank of symmetry of products of spheres and more general finite complexes [1].

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Algebraic models for rational equivariant stable homotopy theories

Brooke Shipley

(joint work with John Greenlees)

For a compact Lie group G of rank r, Greenlees has conjectured that there is an abelian category $A(G)$ of injective dimension r such that the homotopy category of rational G-equivariant cohomology theories is modeled by the derived category of $A(G)$. This conjecture holds for finite groups since rational G-equivariant cohomology theories are just graded rational Mackey functors (which are all injective). This conjecture also holds for $SO(2), O(2)$ and $SO(3)$ by work of Greenlees [Gr1, Gr2, Gr3]. In this talk we discussed joint work with Greenlees on this conjecture for G a torus. Specifically we outlined this new, simplified, proof for $SO(2)$ and indicate the changes needed for higher rank tori.

Theorem 3. [GS2] *For any torus, T^n, there is an abelian category $\mathcal{A}(T^n)$ of sheaves over the space of closed subgroups of T^n and a Quillen equivalence of model categories:*

$$\textit{Rational } T^n\textit{-spectra} \simeq_Q \textit{Rational } dg\mathcal{A}(T^n)$$

Furthermore, $\mathcal{A}(T^n)$ has injective dimension n (the rank.)

In the case of $T = SO(2)$ this Quillen equivalence appears in [Sh1] based on the work in [Gr2]. In [Gr4], Greenlees uses this work to construct $SO(2)$-equivariant elliptic cohomlogy. Our generalization to higher dimensional tori leads to the possibility of constructions of T^g-equivariant cohomology theories associated to complex curves of genus g; see also [Gr5].

One can also specialize to families versions of Theorem 3 for rational G-spectra with fixed points concentrated in a given family, $\mathcal{F}$. For example, free T^n-spectra are modeled by differential graded torsion modules over H^*BT^n; this gives a new proof for tori of the results in [GS1].

This work uses the work on algebraicization of rational stable homotopy theories from [Sh2], and intrinsic formality results for (diagrams of) rational polynomial rings. We develop several other general techniques as well which may have other applications. The next three paragraphs list these.

First, stably *any* Quillen adjunction inducing an equivalence on certain cells induces a Quillen equivalence on the cellularizations (or Bousfield colocalizations).

Proposition A. [GS2] *Let M and N be right proper, stable, cellular model categories with $F : M \to N$ a Quillen adjunction with right adjoint U. Let Q be a cofibrant replacement functor in M and R a fibrant replacement functor in N. Let A be a compact object in M and B a compact object in N.*

- *If $QA \to URFQA$ is a weak equivalence in M, then F and U induce a Quillen equivalence*
$$A\text{-cell-}M \simeq_Q FQA\text{-cell-}N$$
- *If $FQURB \to RB$ is a weak equivalence in N, then F and U induce a Quillen equivalence*
$$URB\text{-cell-}M \simeq_Q B\text{-cell-}N$$

Second, we often use diagram categories of modules over diagrams of rings. Up to colocalization and Quillen equivalence, we show that omitted entries in the diagram can be reconstructed by extension of scalars or pullback. These are bascially generalized Hasse square type statements, such as for R the homotopy pull-back of a diagram of rings, R-modules can be modeled by a localization of the category of modules over the diagram.

Third, we consider a Quillen adjunction induced by fixed points between G-equivariant modules over a ring G-spectrum A and G/K-equivariant modules over the fixed points A^K. We show that this adjunction induces equivalences on certain cells for the particular rings under consideration and then use Proposition A to deduce associated Quillen equivalences.

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Local TQFT versus local CFT: results and speculations

André Henriques

Conformal nets form the objects of a symmetric monoidal category. Applying Jacob Lurie's theorem to a fully dualizable objects of that category, one gets TQFTs

$$\mathrm{Bord}_3^{\mathrm{framed}} \to \mathrm{CNets}.$$

But our geometric understanding of those TQFTs is still quite incomplete. Our best attempt to recreate those TQFTs geometrically seems to produce CFTs... which is also quite interesting.

Homotopy invariance in algebraic K-theory

Andreas Thom
(joint work with Guillermo Cortiñas)

In this talk I presented joint work with Guillermo Cortiñas [1]. The talk was centered arround a conjecture which arose from the work of Johnathan Rosenberg (see [2]) about the negative algebraic K-theory of algebras of continuous functions on compact Hausdorff spaces. Negative algebraic K-theory is a natural invariant of a ring, which is defined in terms of projective modules over the ring itself and certain polynomial or Laurent extensions over it. Rosenberg conjectured that the assignment

$$X \mapsto K_{-n}(C(X))$$

is homotopy invariant as a functor on the category of compact Hausdorff topological spaces. Here, $C(X)$ denotes the ring of complex-valued continuous functions on X. Building on work Eric Friedlander and Marc Walker [3], we managed to prove this conjecture, see [1].

As a key tool we use a technique of *algebraic approximation* which considers the algebra $C(X)$ as a union of its finitely generated subalgebras. More precisely, for every finite set $F \subset C(X)$, we consider the algebra $A_F = \mathbb{C}[f \in F] \subset C(X)$. Since these subalgebras are reduced, they are algebras of regular functions on affine algebraic varieties V_F. This allows to introduce algebraic techniques such as desingularization. Hironaka desingularization gives a (more of less) canonical smooth quasi-projective variety $\tilde{V}_F$ and a proper algebraic morphism $f_F \colon \tilde{V}_F \to V_F$. In order to illustrate how this can be useful, let us consider the special case

$C(\beta\mathbb{N}) = \ell^\infty(\mathbb{N})$. Then, the inclusion $A_F \subset \ell^\infty(\mathbb{N})$ is dual (in the sense of Gel'fand) to a map $\mathbb{N} \to V_F$ with pre-compact image in the euclidean topology. Since f_F is proper, such maps can be lifted through the desingularization $\tilde{V}_F$. This finally leads to the conclusion that the negative algebraic K-theory of $\ell^\infty(\mathbb{N})$ vanishes since this is the case of smooth quasi-projective varieties. Similar conclusions holds for group rings $\ell^\infty(\mathbb{N})[\Gamma]$ and certain monoid rings, being a consequence of the Farrell-Jones Isomorphism Conjecture and results of Richard Swan. Indeed, the argument reveals that the Farrell-Jones Isomorphism Conjecture with coefficients in rings like $C(X)$ or $\ell^\infty(\mathbb{N})$ can give interesting results about the algebraic structure of complex group rings themselves, since non-existence of exotic classes in $K_0(\ell^\infty(\mathbb{N})[\mathbb{Z}^n])$ can be interpreted as a *algebraic compactness* statement.

There are various applications of those results; I want to just mention one. Let R be a topological ring and let X be a compact Hausdorff topological space. A R-quasi-bundle is a continuous map $\pi\colon E \to X$ together with a continuous action $R \times E \to E$ which endows each fiber $E_x := \pi^{-1}(x)$ with the structure of an R-module. It is straighforward to define products and similar constructions for R-quasi-bundle. A R-quasi-bundle $\pi\colon E \to X$ is said to be complemented if there exists another R-quasi-bundle $\pi'\colon E' \to X$ such that $\pi \times \pi'\colon E \times_X E' \to X$ is homeomorphic (over X and respecting the action of the ring R) to the R-quasi-bundle $X \times R^n$ for some $n \in \mathbb{N}$. It is well-known that if R is a Banach algebra or more generally a topological algebra with an open group of invertible elements, then complemented bundles are locally trivial. Our main result says that this is still true for the ring $R := \mathbb{C}[\mathbb{Z}^n \times \mathbb{N}^m]$ equipped with the fine locally convex topology. In fact, we can show that complemented R-quasi-bundles for $R = \mathbb{C}[\mathbb{Z}^n \times \mathbb{N}^m]$ are locally free, which gives a parametrized generalization of the resolution of Serre's Conjecture by Daniel Quillen and Andrei Suslin.

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An application of homotopy theory to comparing finite groups of Lie type

Bob Oliver

(joint work with Carles Broto, Jesper Møller)

Fix a prime p. Two finite groups G and H will be called *p-locally equivalent*, denoted here $G \underset{p}{\sim} H$, if there are Sylow p-subgroups $S \in \mathrm{Syl}_p(G)$ and $T \in \mathrm{Syl}_p(H)$, and an isomorphism $\varphi\colon S \xrightarrow{\cong} T$ which preserves all conjugacy relations between elements and subgroups of S and T. More precisely, for $P, Q \leq S$ and $\alpha \in \mathrm{Iso}(P,Q)$, one requires that α is conjugation by some element of G if and only if $\varphi\alpha\varphi^{-1} \in \mathrm{Iso}(\varphi(P), \varphi(Q))$ is conjugation by some element of H.

It is not hard to see that there are many cases of p-local equivalences among finite groups of Lie type (such as linear, symplectic, or orthogonal groups over finite fields). For example, if q and q' are two prime powers such that $v_2(q-1) = v_2(q'-1) \geq 2$ (where $v_2(-)$ is the 2-adic valuation), and $n \geq 2$, then $GL_n(q) \underset{2}{\sim} GL_n(q')$, $SL_n(q) \underset{2}{\sim} SL_n(q')$, $PSL_n(q) \underset{2}{\sim} PSL_n(q')$, etc. Constructing an isomorphism between Sylow 2-subgroups is straightforward in each case, and one can show using character theory that those isomorphisms preserve conjugacy relations.

In joint work with Carles Broto and Jesper Møller [BMO], we proved, among other results, the following very general theorem about such p-local equivalences between finite groups of Lie type.

Theorem 1. *Fix a connected, reductive, integral group scheme $\mathbb{G}$ and a prime p. For any pair of prime powers q and q' such that $p \nmid qq'$, if $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$, then $\mathbb{G}(q) \underset{p}{\sim} \mathbb{G}(q')$.*

Theorem 1 thus applies not only to $\mathbb{G} = (P)GL_n$, $(P)SL_n$, and $(P)Sp_{2n}$, but also to the exceptional groups such as G_2, F_4, etc.

The following is another theorem of the same type. It compares conjugacy in unitary and linear groups, and is a special case of a more general theorem in [BMO].

Theorem 2. *Fix $n \geq 2$ and a prime p. For any pair of prime powers q and q' such that $p \nmid qq'$, if $\overline{\langle -q \rangle} = \overline{\langle q' \rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$, then $U(q) \underset{p}{\sim} GL(q')$, $SU(q) \underset{p}{\sim} SL(q')$, and $PSU(q) \underset{p}{\sim} PSL(q')$.*

Our proof of Theorems 1 and 2 is homotopy theoretic. In [MP], Martino and Priddy proved that if the p-completed classifying spaces $BG_p^\wedge$ and $BH_p^\wedge$ are homotopy equivalent, then G and H are p-locally equivalent. They also conjectured the converse, a result which has now been proven [O1, O2], but only by using the classification theorem of finite simple groups.

Thus, to prove Theorem 1, it suffices to show that $B\mathbb{G}(q)_p^\wedge \simeq B\mathbb{G}(q')_p^\wedge$. The starting point when doing this is a theorem of Friedlander [Fr], which proves that the space $B\mathbb{G}(q)_p^\wedge$ is equivalent to the "homotopy fixed space" $(B\mathbb{G}(\mathbb{C})_p^\wedge)^{h\psi^q}$ of the action of some "unstable Adams operation" ψ^q on $B\mathbb{G}(\mathbb{C})_p^\wedge$. This map ψ^q is characterized by requiring that its restriction to BT (where $T \leq \mathbb{G}(\mathbb{C})$ is a maximal torus) is induced by the endomorphism $(t \mapsto t^q)$ of T. By a theorem of Jackowski, McClure, and Oliver [JMO], this condition determines ψ^q uniquely up to homotopy.

These results are then combined with the following theorem, proven in [BMO, Theorem 2.4]:

Theorem 3. *Fix a prime p. Let X be a connected, p-complete space such that*

- $H^*(X; \mathbb{F}_p)$ *is noetherian, and*
- $\mathrm{Out}(X)$ *is detected on* $\lim_n H^*(X; \mathbb{Z}/p^n)$.

Let α and β be self homotopy equivalences of X which generate the same closed subgroup of $\mathrm{Out}(X)$ *under the p-adic topology. Then $X^{h\alpha} \simeq X^{h\beta}$.*

Here, for any space X, $\mathrm{Out}(X)$ is the group of homotopy classes of self homotopy equivalences of X. The "p-adic topology" on $\mathrm{Out}(X)$ is that determined by its filtration by subgroups acting via the identity on $H^*(X; \mathbb{Z}/p^n)$, for $n \to \infty$.

Theorems 1 and 2 are not surprising to group theorists. But currently, no other proof seems to be known of these purely algebraic results.

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Homotopy groups of the moduli space of metrics of positive scalar curvature

Bernhard Hanke

(joint work with Boris Botvinnik, Thomas Schick and Mark Walsh)

Let M be a connected closed smooth manifold admitting a Riemannian metric of positive scalar curvature. We are interested in topological properties of $\mathrm{Riem}^+(\mathrm{M})$, the space of positive scalar curvature metrics on M with the C^∞-topology, and the associated moduli space $\mathcal{M}^+(M) = \mathrm{Riem}^+(\mathrm{M})/\mathrm{Diff}(\mathrm{M})$. The knowledge about global properties of these spaces is sparse. For example it has been an open problem to decide whether $\pi_k(\mathrm{Riem}^+(\mathrm{M}))$ for $k > 1$ or $\pi_k(\mathcal{M}^+(M))$ for $k > 0$ can be non-zero.

The following result identifies higher degree non-zero homotopy groups of moduli spaces of positive scalar curvature metrics.

Theorem 5 ([1])**.** *For any $d > 0$ there exists a connected closed smooth manifold M admitting a metric of positive scalar curvature with*

$$\pi_{4q}(\mathcal{M}^+(M)) \neq 0$$

for $0 < q \leq d$.

We shall sketch a proof of this fact.

Because the action of $\mathrm{Diff}(M)$ on Riem(M), the space of Riemannian metrics on M, is not free in general, we first restrict attention to a certain subgroup of $\mathrm{Diff}(M)$. Let $x_0 \in M$ be a basepoint. Then we denote by $\mathrm{Diff}_{x_0}(M) \subset \mathrm{Diff}(M)$ the subgroup of diffeomorphisms fixing x_0 and inducing the identity on the tangent space $T_{x_0}M$.

We think of $\mathrm{Diff}_{x_0}(M)$ as diffeomorphisms fixing an observer at x_0. Note that $\mathrm{Diff}_{x_0}(M)$ acts freely on Riem(M), since M is connected. The action admitting local slices [2] and Riem(M) being contractible we obtain a $\mathrm{Diff}_{x_0}(M)$-principal fibration

$$\mathrm{Diff}_{x_0}(M) \hookrightarrow \mathrm{Riem(M)} \to \mathcal{M}_{\mathrm{x_0}}(\mathrm{M})$$

where the *observer moduli space* of Riemannian metrics

$$\mathcal{M}_{x_0}(M) = \mathrm{Riem(M)}/\mathrm{Diff}_{\mathrm{x_0}}(\mathrm{M})$$

is homotopy equivalent to $B\mathrm{Diff}_{x_0}(M)$.

Note that a map $f : S^k \to B\mathrm{Diff}_{x_0}(M)$ classifying a bundle $M \hookrightarrow E \to S^k$ gives rise to a commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{Riem(M)} \times_{\mathrm{Diff}_{\mathrm{x_0}}(\mathrm{M})} \mathrm{M} \arrow[d] \\ S^k \arrow[r, "f"] & B\mathrm{Diff}_{x_0}(M) \end{tikzcd}$$

and hence f is given by a smooth family of Riemannian metrics on the fibres of E.

The proof of Theorem 5 comprises the following steps.

a) Construct maps $f : S^{4q} \to B\mathrm{Diff}_{x_0}(M)$ representing non-zero elements in $\pi_{4q}(B\mathrm{Diff}_{x_0}(M))$.
b) Construct smooth families of positive scalar curvature metrics on the fibres of the resulting bundles $E \to S^{4q}$.
c) Show that in some cases the resulting non-zero elements in $\pi_{4q}(\mathcal{M}^+_{x_0}(M))$ are not in the kernel of the canonical map $\pi_{4q}(\mathcal{M}^+_{x_0}(M)) \to \pi_{4q}(\mathcal{M}^+(M))$ forgetting the base point.

The first two steps are first carried out for the special manifold $M = S^n$.

Step a) is now based on the following classical result.

Theorem 6 ([3]). *Let $0 < k \ll n$. Then*

$$\pi_k(B\mathrm{Diff}_{x_0}(S^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} \ \textit{if } n \textit{ is odd and } k = 4q \\ 0 \ \textit{else.} \end{cases}$$

Here the shorthand notation $k \ll n$ means that for fixed k there is an $N \in \mathbb{N}$ so that the statement is true for all $n \geq N$.

Concerning step b) we state

Theorem 7 ([1]). *The canonical map*

$$\pi_k(\mathcal{M}^+_{x_0}(S^n)) \otimes \mathbb{Q} \to \pi_k(\mathcal{M}_{x_0}(S^n)) \otimes \mathbb{Q}$$

is an epimorphism for $0 < k \ll n$. In particular, the groups $\pi_k(\mathcal{M}^+_{x_0}(S^n)) \otimes \mathbb{Q}$ are non-zero for odd n and $0 < k = 4q \ll n$.

As remarked before this amounts to constructing smooth families of positive scalar curvature metrics on the bundles $S^n \hookrightarrow E \to S^{4q}$ corresponding to the nontrivial elements in $\pi_{4q}(B\mathrm{Diff}_{x_0}(S^n)) \otimes \mathbb{Q}$ from Theorem 6.

Here we make use of an explicit description of these bundles due to Hatcher. Each of these bundles can be constructed by doubling a smooth fibre bundle

$$D^n \hookrightarrow D \to S^{4q}$$

along the boundary. There are fibrewise Morse functions $D \to [0,1]$ on these bundles constructed in [5] which are self-indexing, i.e. larger criticial values correspond to larger (fibrewise) indices of the corresponding critical sets, and which have critical sets with indices smaller than or equal to $n-3$.

These fibrewise Morse functions serve two purposes.

On the one hand, they are used in [5, Theorem 5.13] to show by explicit calculations that certain higher Franz-Reidemeister torsion invariants of the resulting S^n-bundles $E \to S^{4q}$ are non-zero. Therefore these bundles correspond to non-zero generators in Theorem 6. On the other hand they can be used to apply a fibrewise version of the surgery technique due to Gromov-Lawson [6] and Gajer [4] to obtain fibrewise metrics of positive scalar curvature on the bundles $D \to S^{4q}$ and $E \to S^{4q}$. The extension of these surgery techniques to a fibred situation is the technical heart of our argument, cf. also [9].

For an arbitrary manifold M^n of odd dimension and equipped with a metric of positive scalar curvature, steps a) and b) are carried out by taking a fibrewise connected sum of the Hatcher bundle $E \to S^{4q}$ with a trivial bundle $S^{4q} \times M$ along the maximum of the fibrewise Morse function on E constructed earlier. The resulting element in $\pi_{4q}(B\mathrm{Diff}_{x_0}(M)) \otimes \mathbb{Q}$ is still non-zero and can be detected by a higher Franz-Reidemeister torsion invariant. Furthermore, the resulting M-bundle over S^{4q} can be equipped with a smooth family of positive scalar curvature metrics, since these metrics can be extended over connected sums.

Step c) is based on the construction of closed orientiable odd dimensional manifolds M which carry positive scalar curvature metrics, but only trivial S^1-actions. For these manifolds the isotropy groups of the action of $\mathrm{Diff}(M)$ on Riem(M) must be finite (they are compact Lie groups by [8]) so that the argument needed for step c) can be completed by a Leray spectral sequence argument. The construction of the manifolds M is somewhat difficult, because the well known Atiyah-Hirzebruch obstruction to nontrivial S^1-actions on smooth manifolds given in terms of the $\hat{A}$-genus applies only to even dimensional spin manifolds and in particular also obstructs positive scalar curvature metrics by the Lichnerowicz-Weitzenböck argument.

In our paper we make use of a refinement of the Atiyah-Hirzebruch obstruction in terms of higher $\hat{A}$-genera, which does not only apply to spin manifolds, but to orientable manifolds with finite second and fourth homotopy groups. This recent result is due to Herrera-Herrera [7].

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