3-MANIFOLDS AFTER PERELMAN

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Abstract. We summarize some of the main results on 3–manifolds after Perelman's proof of the Geometrization Conjecture.

Introduction

In 2003 Perelman [Pe02, Pe03a, Pe03b] (see also [MT07], [CZ06a, CZ06b], [KL08], [FM10] and [BBBMP10]) proved the Geometrization Theorem, which can be formulated as follows:

Geometrization Theorem. *Let N be an orientable, compact, irreducible 3–manifold with empty or toroidal boundary. Then one of the following three cases occurs:*

1. *N is Seifert fibered, i.e. finitely covered by an S^1–bundle over a surface,*
2. *N is hyperbolic, i.e. N admits a complete metric of constant curvature -1,*
3. *N admits incompressible disjoint tori $T_1, \dots, T_l$ such that each component of N cut along $T_1 \cup \dots \cup T_l$ is Seifert fibered or hyperbolic.*

Remark. This theorem was first conjectured by Thurston [Th79, Th82a, Th82b] in the late 1970's. Thurston had also provided a proof in the case that the 3–manifold is Haken, i.e. for irreducible 3–manifolds which admit an incompressible surface.

Seifert fibered 3–manifolds are fully classified (see e.g. [Sei33, **?**, He76]), so it remains to study hyperbolic 3–manifolds, and then to understand which results on hyperbolic 3–manifolds extend to the general case.

Thurston [Th82b, Questions 15 to 18] asked the following 'challenge questions' regarding hyperbolic 3–manifolds:

- (T1) Is every hyperbolic 3–manifold virtually Haken, i.e. does every hyperbolic 3–manifold admit a finite cover which is Haken?
- (T2) Does every hyperbolic 3–manifold admit a finite cover with positive first Betti number?

(T3) Is every hyperbolic 3–manifold finitely covered by a fibered 3–manifold? [1]

(T4) Is the fundamental group of a hyperbolic 3–manifold subgroup separable? Recall that a group π is called *subgroup separable* if given any finitely generated subgroup $A \subset \pi$ and any $g \notin A$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group such that $\alpha(g) \notin \alpha(A)$.

The goal of this talk is to report on the progress towards answering these questions by Agol, Calegari-Gabai, Kahn-Markovic and Wise.

Note though that a complete understanding of Seifert fibered 3–manifolds and hyperbolic 3–manifolds does not necessarily lead immediately to a good understanding of 3–manifolds with a non–trivial JSJ decomposition. For example it is still not known fundamental groups of such 3–manifolds are linear.

Remark. This short survey is mostly 'zu Guttenberged' from the forthcoming survey paper [AFW11] on 3–manifold groups. This survey is only meant as a guide to the literature, all statements should in particular be taken with a grain of salt. Also note that Wise's results have not been fully verified yet.

2004: The tameness theorem of Agol and Calegari–Gabai

Agol [Ag07] and Calegari–Gabai [CG06] (see also [Ca08, Corollary 8.1] and [Bow10] for further details) proved independently in 2004 the following theorem, which was first conjectured by Marden in 1974:

Tameness Theorem. *Let N be a hyperbolic 3–manifold with finitely generated fundamental group. Then N is topologically tame, i.e. N is homeomorphic to the interior of a compact 3–manifold.*

The tameness theorem together with Canary's covering theorem (see [Ca94, Section 4] and [Ca96]) implies the following dichotomy:

Dichotomy Theorem. *Let N be a hyperbolic 3-manifold and let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup of infinite index. Then either*

(1) Γ *is a virtual surface fiber group, i.e. there exists a finite cover $\tilde{N} \to N$ and a fiber surface $\Sigma \subset \tilde{N}$ of a fibration $\tilde{N} \to S^1$ such that $\Gamma = \pi_1(\Sigma)$, or*

(2) Γ *is geometrically finite.*

[1] Thurston comments the question with 'This dubious-sounding question seems to have a definite chance for a positive answer.'

We will not give the definition of a geometrically finite subgroup (see [KAG86, p. 10] for details). Instead below we will rephrase the dichotomy in various alternative ways.

In order to state one possible reformulation we need the notion of the commensurator of a subgroup Γ of a group π, which is defined as

$$\mathrm{Comm}_\pi(\Gamma) := \{g \in \pi \,|\, \Gamma \cap g\Gamma g^{-1} \text{ has finite index in } \Gamma\}.$$

Then the above dichotomy can be phrased as follows: If $\Gamma \subset \pi_1(N)$ is a finitely generated subgroup of infinite index of the fundamental group of a hyperbolic 3–manifold, then either

(1) $\mathrm{Comm}_\pi(\Gamma)$ is a finite index subgroup of π, or
(2) $\mathrm{Comm}_\pi(\Gamma)$ is a finite index supergroup of Γ.

We refer to [Ca08, Theorem 8.7] for a proof.

Loosely speaking this version says that a finitely generated subgroup of the fundamental group of a hyperbolic 3–manifold is either 'almost normal' or 'very non-normal'. Another way of phrasing this dichotomy is in terms of the 'width' of a subgroups, which is a different measure of 'normality' respectively 'non-normality' of a subgroup. We refer to [GMRS98], [AGM09] and [Wi11a, Definition 12.7] for details.

In order to give one more formulation of the dichotomy we will need a few more definitions:

Definition. (1) Let X be a geodesic metric space. A subspace Y is said to be *quasi-convex* if there exists $\kappa \geq 0$ such that any geodesic in X with endpoints in Y is contained within the κ-neighbourhood of Y.

(2) Let π be a group with a fixed generating set S. A subgroup $H \subseteq \pi$ is said to be *quasi-convex* if it is a quasi-convex subspace of $\mathrm{Cay}_S(\pi)$, the Cayley graph of π with respect to the generating set S. In general quasi-convexity depends on the choice of generating set S. However, if π is word-hyperbolic, then the quasi-convexity of a subgroup H does not depend on the choice of generating set.

Let N be a hyperbolic 3–manifold. A subgroup of $\pi_1(N)$ is geometrically finite if and only if it is quasiconvex (see for example [Hr10, Corollary 1.3] for a reference and see [KS96, Theorem 2]). We thus obtain the following reformulation of the above dichotomy theorem:

Dichotomy Theorem. *Let N be a hyperbolic 3-manifold and let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup of infinite index. Then one of the following occurs:*

(1) Γ *is a virtual surface fiber group, or*

(2) Γ *is a quasi–convex subgroup of* π.

2007: The virtual fibering theorem of Agol

Let G be a finite graph with vertex set V, then it gives rise to a group presentation as follows:

$$A_\Gamma = \langle \{g_v\}_{v\in V} \mid [g_u, g_v] = e \text{ if } u \text{ and } v \text{ are connected by an edge}\rangle.$$

Any group which is isomorphic to such a group is called a *right angled Artin group (RAAG)*.

Note that right angled Artin groups are commensurable with the perhaps more familiar right angled Coxeter groups (see [DJ00]), which correspond to reflections in orthogonal hyperplanes.

In 2007 Agol proved the following theorem:

Theorem. *Let N be an irreducible 3–manifold such that $\pi_1(N)$ is virtually a subgroup of a RAAG. Let $\phi \in H^1(N;\mathbb{Q})$ be a non-fibered non–trivial class, then there exists a finite cover $p\colon N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Q})$ lies on the boundary of a fibered cone of the Thurston norm ball of N'.*

Note that the pull–back of a non–fibered class can not be fibered, the theorem thus says, loosely speaking, that provided that $\pi_1(N)$ is virtually a subgroup of a RAAG any non–fibered class can be made 'as fibered as possible' in a finite cover.

This theorem gave the first general criterion for virtual fiberedness. The condition that $\pi_1(N)$ is virtually a subgroup of a RAAG is a priori rather stringent. In fact at the time of the writing of [Ag08] only few hyperbolic 3–manifold groups were known to have this property, e.g. arithmetic hyperbolic groups defined by a quadratic form.

2009: The surface subgroup theorem of Kahn–Markovic

If N is closed a 3–manifold which is virtually Haken, then $\pi_1(N)$ contains in particular a surface subgroup. Kahn and Markovic [KM09] proved the following result, which by the above can be viewed as a major step towards a resolution of the virtual Haken conjecture:

Theorem. *Let N be a closed hyperbolic 3-manifold, then N admits a π_1–injective immersion $\iota\colon \Sigma \to N$ of a connected surface such that $\iota_*(\pi_1(\Sigma))$ is quasi-Fuchsian[2] surface.*

In fact, in a sense, which can be made precise, [KM09] provides 'lots of surface subgroups'.

[2]See [KAG86, p. 4] for the definition of a *quasi-Fuchsian surface group.*

2009: The 'virtually special' theorem of Wise

The statement of the theorem. We refer to [HW08] for details regarding the following definitions:

Definition.
(1) An *n-cube* is a copy of $[-1,1]^n$ and a 0-cube is a single point.
(2) A *cube complex* is a cell complex formed from cubes, such that the attaching map of each cube is combinatorial in the sense that it sends cubes homeomorphically to cubes by a map modelled on a combinatorial isometry of n-cubes.
(3) The *link* of a 0-cube v is the complex whose 0-simplices correspond to ends of 1-cubes adjacent to v, and these 0-simplices are joined up by n–simplices for each corner of an $(n+1)$–cube adjacent to v.
(4) A *flag complex* is a simplicial complex with the property that any finite pairwise adjacent collection of vertices spans a simplex.
(5) A cube complex C is *non–positively curved* if link(v) is a flag complex for each 0-cube $v \in C^0$.
(6) There is a natural notion of immersed hyperplanes in cube complexes, a cube complex is called *special* if certain 'pathologies' do not arise from the immersed hyperplanes.

Definition. A group π is *(compact) special* if π is the fundamental group of a non-positively curved special (compact) cube complex X.

The following theorem of Haglund and Wise [HW08] gives a purely group theoretic reformulation of the property of being virtually (compact) special.

Theorem. (Haglund–Wise) *A group π is virtually (compact) special if and only if π admits a subgroup of finite index which is a (quasiconvex) subgroup of a RAAG.*

The connection between being special and being a subgroup of a RAAG comes through the 'nice hyperplanes' in special cube complexes (which necessarily meet 'orthogonally') and the orthogonal hyperplanes in the definition of a right angled Coxeter groups, which are in turn commensurable with right angled Artin groups.

The following theorem was proved by Wise [Wi09, Wi11a, Wi11b].

Theorem. (Wise) *Let π be a word hyperbolic group which admits a quasiconvex hierarchy, then π is virtually compact special.*

We refer to [Wi09, Definition 1.1] for the definition of a quasiconvex hierarchy, but loosely speaking it means that π can be obtain from the trivial group through iterated HNN extensions and amalgamated products along quasiconvex subgroups.

Let N be a closed, hyperbolic 3-manifold which contains a geometrically finite surface. Thurston showed that N admits in fact a hierarchy of geometrically finite surfaces (see [Ca94, Theorem 2.1]). As we mentioned before, a subgroup of $\pi_1(N)$ is geometrically finite if and only if it is quasiconvex. We thus obtain the following result:

Theorem. (Wise) *Let N be a closed hyperbolic 3–manifold which contains a geometrically finite surface, then $\pi_1(N)$ is virtually compact special.*

Note that by the dichotomy theorem an incompressible surface is either geometrically finite or it lifts to a fiber in a finite cover (in fact this special case had been proved already by Thurston and Bonahon, see [Bon86]). It follows in particular that a closed Haken hyperbolic 3–manifold either admits a geometrically finite surface, or it is virtually fibered. Furthermore, a standard Thurston norm argument shows that any closed hyperbolic 3–manifold N with $b_1(N) \geq 2$ admits a geometrically finite surface.

Note that fundamental groups of hyperbolic 3–manifolds with non–trivial boundary are *not* word hyperbolic. So Wise's theorem can not be applied directly. Nonetheless, Wise [Wi09, Wi11a, Wi11b] also proved the following theorem.

Theorem. (Wise) *Let N be a hyperbolic 3–manifold with non–trivial boundary, then $\pi_1(N)$ is virtually compact special.*

Consequences of Wise's theorem. A non–trivial group which is virtually a subgroup of a RAAG group admits a finite index subgroup with positive first Betti number (see e.g. [Ag08]). The combination of the results of Agol and Wise, and the discussion in the previous section, therefore implies the following theorem

Theorem. *Let N be a hyperbolic Haken 3–manifold, then N is virtually fibered.*

It also follows from the discussion in the previous section that the following theorem holds:

Theorem. *Let N be a hyperbolic 3–manifold. Let $\phi \in H^1(N;\mathbb{Q})$ be a non-fibered non–trivial class, then there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Q})$ lies on the boundary of a fibered cone of the Thurston norm ball of N'.*

Right angled Artin groups can be viewed as a common generalization of free groups and free abelian groups. In particular many properties of free groups and free abelian groups also hold for (subgroups of) right angled Artin groups. We thus obtain the following theorem:

Theorem. *Let N be a 3–manifold such that $\pi = \pi_1(N)$ is virtually special, then the following hold:*

(1) *if N is neither spherical nor virtually a torus bundle over S^1, then $vb_1(N) = \infty$, i.e. N admits finite covers with arbitrarily large first Betti numbers,*
(2) *π admits a finite index subgroup which is residually torsion–free nilpotent,*
(3) *π admits a finite index subgroup which is residually p for any prime p,*
(4) *π admits a finite index subgroup which is biorderable,*
(5) *π is linear over $\mathbb{Z}$, i.e. $\pi \subset \mathrm{GL}(n, \mathbb{Z})$ for some $n \in \mathbb{N}$.*

The first statement follows from [Ag08], the second statement is shown in [DK92]. The third statement is a consequence of the second statement (see [Gru57, Theorem 2.1]) and the fourth statement is a consequence of the third statement (see [Rh73]). Finally recall that a RAAG is commensurable with a right angled Coxeter group, which in turn is easily seen to be linear over $\mathbb{Z}$ (see [HsW99] for details).

We will now see that groups which are virtually *compact* special (or equivalently, groups which are virtually a quasi–convex subgroup of a RAAG) are even better behaved. The reason is the following theorem of Haglund [Ha08, Theorem F]:

Theorem. (Haglund) *Let Γ be a quasiconvex subgroup of a RAAG* [3] *A, then Γ is a virtual retract of A, i.e. there exists a finite index subgroup A' of A which contains Γ and a homomorphism $\varphi\colon A' \to \Gamma$ such that $\varphi(g) = g$ for all $g \in \Gamma$.*

Note that the conclusion of the theorem trivially holds for all finitely generated subgroups of abelian groups and it is a classical theorem that it also holds for finitely generated subgroups of free groups. Haglund's result is therefore a generalization of these two classical results.

Recall that a group π is called *conjugacy separable* if for any two non-conjugate elements $g, h \in \pi$ there exists an epimorphism $\alpha\colon \pi \to G$ onto a finite group G such that $\alpha(g)$ and $\alpha(h)$ are not conjugate. Minasyan [Min09] showed that finite index subgroups of RAAGs are conjugacy separable. Using the fact that retracts of conjugacy separable groups are again conjugacy separable one can now easily prove the following theorem:

[3]Here we mean 'quasiconvex' with respect to a canonical generating set of a RAAG, as in the definition of a RAAG.

Theorem. (Minasyan) *Let N be a 3–manifold such that $\pi = \pi_1(N)$ is virtually compact special, then π is conjugacy separable.*

Let N be a hyperbolic 3–manifold. Recall that by the dichotomy theorem a finitely generated subgroup $\Gamma \subset \pi = \pi_1(N)$ of infinite index is either a virtual surface fiber group, or it is a quasi–convex subgroup of π. Using Haglund's theorem and using the philosophy [4] that if $\Gamma \subset \pi$ is quasi–convex and if $\pi \subset A$ is quasi–convex, then $\Gamma \subset A$ should be quasi–convex one can prove the following result:

Theorem. *Let N be a hyperbolic 3–manifold which either admits a geometrically finite surface or has non–trivial boundary. Let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup which is not a virtual surface fiber group. Then Γ is a virtual retract of $\pi_1(N)$.*

Note that a virtual surface fiber group of a hyperbolic 3–manifold N can not be a virtual retract of $\pi_1(N)$. We thus obtain the following reformulation of the dichotomy theorem: If N is a hyperbolic 3–manifold and if Γ is a finitely generated subgroup of $\pi_1(N)$, then

(1) Γ is either a virtual surface fiber group, or
(2) Γ is a virtual retract of $\pi_1(N)$.

Note that an elementary argument shows that virtual surface fiber groups are separable in $\pi_1(N)$. Furthermore, a virtual retract of a group π is also separable in π (see e.g. [Ha08, Section 3.4]), we thus obtain the following result:

Theorem. *Let N be a hyperbolic 3–manifold which either admits a geometrically finite surface or has non–trivial boundary. Then $\pi_1(N)$ is subgroup separable.*

Possible future directions

3–manifolds with a non–trivial JSJ decomposition. It is a natural question to ask which of the results in the previous section extend to 3–manifolds with a non–trivial JSJ decomposition. First note that there exist graph manifolds which are not virtually fibered (see e.g. [Ne96]), furthermore the fundamental groups of certain torus bundles are not virtually residually torsion–free nilpotent. In particular in both cases the fundamental group can not be virtually special.

On the other hand the following conjecture seems rather reasonable:

Conjecture 1. *Let N be an irreducible 3–manifold which supports a non-positively curved metric, then $\pi_1(N)$ is virtually special.*

[4] This statement does not hold in general, but one can apply this philosophy to prove the subsequent theorem.

For graph manifolds this conjecture was proved by Liu [Liu11] (see also the work of Przytycki and Wise [PW11]). Also note that Leeb [Leb95] showed that an irreducible 3–manifold which is not a graph manifold supports in fact a non-positively curved metric. Put differently, it remains to show if N is an irreducible 3–manifold which contains at least one hyperbolic piece in its JSJ decomposition, then $\pi_1(N)$ is virtually special.

Recall that a solution to the conjecture would show that 3–manifolds which support a non-positively curved metric are virtually fibered, that their fundamental groups are virtually residually torsion–free nilpotent and that their fundamental groups are linear over $\mathbb{Z}$.

It is less clear though whether fundamental groups of 3–manifolds which support a non-positively curved metric are also virtually *compact* special. Also note that if N is a non–hyperbolic 3–manifold such that if $\pi_1(N)$ is virtually compact special, then this does not imply that $\pi_1(N)$ is subgroup separable. For instance, the link group exhibited in [NW01, Theorem 1.3] equals the right-angled Artin group defined by the graph with four vertices that is homeomorphic to the interval, but the group is known not to be subgroup separable.

Finally note that oddly enough, the only class of 3–manifolds for which so far no clear picture is emerging is the class of graph manifolds with a non–trivial JSJ decomposition which do not support a metric of non–positive curvature.

The virtual Haken conjecture. The discussion in the previous sections shows that the aforementioned questions of Thurston are now reduced to the following two questions:

(1) Is every closed hyperbolic 3–manifold virtually Haken?
(2) Does every fibered closed hyperbolic 3–manifold admit a finite cover which contains a geometrically finite surface?

One possible approach might be the following. Kahn and Markovic [KM09] provide a wealth of geometrically finite [5] surface groups in a closed hyperbolic 3-manifold–so many that one can apply Sageev's cubulation construction (see [Sa95]). Using this observation, Bergeron and Wise proved that every closed hyperbolic 3-manifold group is also the fundamental group of a compact non-positively curved cube complex [BW09]. A solution to the following conjecture of Wise (see [Wi11a, Conjecture 19.5]) would therefore resolve both of the above questions:

[5] Here we use that quasi-Fuchsian surface groups are geometrically finite (see e.g. [Oh02, Lemma 4.66]).

Conjecture 2. *Let π be a word-hyperbolic group which is also the fundamental group of a compact non-positively curved cube complex. Then π is virtually compact special.*

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lecture notes for the NSF-CBMS conference '3-Manifolds, Artin Groups and Cubical Geometry' from August 1-5, 2011 held at the CUNY Graduate Center

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