METABELIAN SL($n, \mathbb{C}$) REPRESENTATIONS OF KNOT GROUPS
II: FIXED POINTS AND DEFORMATIONS

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Abstract. Given a knot $K$ in an integral homology sphere $\Sigma$ with exterior $N_K$, there is a natural action of the cyclic group $\mathbb{Z}/n$ on the space of SL($n, \mathbb{C}$) representations of the knot group $\pi_1(N_K)$, and this induces an action on the SL($n, \mathbb{C}$) character variety $X_n(N_K)$. We identify the fixed points of this action in $X^*_n(N_K)$ with characters of irreducible metabelian representations.

We then show that for any irreducible metabelian representation $\alpha$, we have $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad, \alpha}) \geq n - 1$. If equality holds, then we prove that the character $\xi_\alpha$ is a smooth point in $X_n(N_K)$ and that there exists an $(n - 1)$-dimensional family of characters of irreducible SL($n, \mathbb{C}$) representations of $\pi_1(N_K)$ near $\xi_\alpha$. We relate the cohomological condition that $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad, \alpha}) = n - 1$ to the vanishing of $H_1(\hat{\Sigma}_\varphi)$, where $\hat{\Sigma}_\varphi$ denotes the associated metabelian branched cover of $\Sigma$ branched along $K$.

1. Introduction

Suppose $K$ is a knot. Throughout this paper we will always understand this to mean that $K$ is an oriented simple closed curve in an integral homology 3-sphere $\Sigma$. We write $N_K = \Sigma^3 \setminus \tau(K)$, where $\tau(K)$ denotes an open tubular neighborhood of $K$. In a previous paper ([BF08]) the authors classified the metabelian SL($n, \mathbb{C}$) representations from the knot group $\pi_1(N_K)$. In this paper we continue our study of metabelian representations of knot groups.

In order to state our results, we need to introduce some terminology.

Given a topological space $M$, let $R_n(M)$ be the space of SL($n, \mathbb{C}$) representations of $\pi_1(M)$ and $X_n(M)$ the associated character variety. We use $\xi_\alpha$ to denote the character of the representation $\alpha$: $\pi_1(M) \to$ SL($n, \mathbb{C}$), and we recall the elementary but important fact that two irreducible representations determine the same character if and only if they are conjugate (see [LM85, Corollary 1.33]).

Now suppose $K$ is a knot. There is an action of the group $\mathbb{Z}/n$ on the representation variety $R_n(N_K)$ given by twisting by the $n$–th roots of unity $\omega^k = e^{2\pi ik/n} \in U(1)$.

Date: September 20, 2009.


Key words and phrases. metabelian representation, knot group, character variety, Zariski tangent space, group action, fixed point, deformation.

The first named author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.
(This is a special case of the more general twisting operation described in [LM85, Ch. 5].) More precisely, we write \( \mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle \) and set \( (\sigma \cdot \alpha)(g) = \omega^\varepsilon(g)\alpha(g) \) for each \( g \in \pi_1(N_K) \), where \( \varepsilon : \pi_1(N_K) \to H_1(N_K) = \mathbb{Z} \) is determined by the given orientation of the knot.

This constructs an action of \( \mathbb{Z}/n \) on \( R_\alpha(N_K) \) which, in turn, determines a corresponding action on the character variety \( X_n(N_K) \). Our first result identifies the fixed points of \( \mathbb{Z}/n \) in \( X_n^*(N_K) \), the irreducible characters, as those associated to metabelian representations.

**Theorem 1.** The character \( \xi_\alpha \) of an irreducible representation \( \alpha : \pi_1(M) \to SL(n, \mathbb{C}) \) is fixed under the \( \mathbb{Z}/n \) action if and only if \( \alpha \) is metabelian.

Theorem 12 goes a little further in that it gives a characterization of the entire fixed point set \( X_n(N_K)^{\mathbb{Z}/n} \) in terms of characters \( \xi_\alpha \) of the metabelian representations \( \alpha = \alpha_{(n,\chi)} \) described in Subsection 2.3. When \( n = 2 \), it turns out that every metabelian representation is dihedral and in this case Theorem 1 was first proved by F. Nagasato and Y. Yamaguchi (cf. [NY08, Proposition 4.8]).

Our second result concerns deformations of irreducible metabelian \( SL(n, \mathbb{C}) \) representations. In order to formulate it, we need to introduce some additional notation. To any representation \( \alpha : \pi_1(M) \to SL(n, \mathbb{C}) \) we can associate the twisted homology and cohomology groups with coefficients in the Lie algebra \( sl(n, \mathbb{C}) \) endowed with the adjoint action of \( \pi_1(M) \) given by \( g \cdot A := \alpha(g)A\alpha(g)^{-1} \) for \( g \in \pi_1(M) \) and \( A \in sl(n, \mathbb{C}) \). Throughout the paper, we use \( sl(n, \mathbb{C})_{ad,\alpha} \) when referring to \( sl(n, \mathbb{C}) \) as a \( \pi_1(M) \)-module via the adjoint action.

In Proposition 15 we show that with respect to the natural symplectic structure on \( H^1(\partial N_K; sl(n, \mathbb{C})_{ad,\alpha}) \), the image of \( H^1(N_K; sl(n, \mathbb{C})_{ad,\alpha}) \) in \( H^1(\partial N_K; sl(n, \mathbb{C})_{ad,\alpha}) \) is Lagrangian. (This is a special case of a general phenomenon elucidated nicely by A. Sikora in [Si09].) Because \( \dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad,\alpha}) = 2n - 2 \), and since the image of \( H^1(N_K; sl(n, \mathbb{C})_{ad,\alpha}) \) must be half–dimensional, we see that \( \dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad,\alpha}) \geq n - 1 \).

In the case that equality holds, i.e. if \( \dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad,\alpha}) = n - 1 \), Lemma 16 says that \( \alpha \) has finite image (in particular \( \alpha \) is conjugate to a unitary representation) and the following theorem says that \( \alpha \) can be deformed within a smooth \((n - 1)\)-dimensional family of conjugacy classes of irreducible \( SL(n, \mathbb{C}) \) representations.

**Theorem 2.** Suppose \( \alpha : \pi_1(N_K) \to SL(n, \mathbb{C}) \) is an irreducible metabelian representation such that \( \dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad,\alpha}) = n - 1 \). Then the following hold:

(i) The character \( \xi_\alpha \) is a smooth point in the character variety \( X_n(N_K) \) and there exists a smooth complex \((n - 1)\)-dimensional family of characters of irreducible \( SL(n, \mathbb{C}) \) representations near \( \xi_\alpha \in X_n(N_K) \).

(ii) If \( \alpha \) is a unitary representation, then \( \xi_\alpha \) is a smooth point in \( X_{SU(n)}(N_K) \) and there exists a smooth real \((n - 1)\)-dimensional family of characters of irreducible \( SU(n) \) representations near \( \xi_\alpha \in X_{SU(n)}(N_K) \).
Finally both of the deformation families can be chosen so that \( \xi_\alpha \) is the only metabelian character.

The key step in Theorem 2 is presented in Theorem 17, which is proved by developing an \( \text{SL}(n, \mathbb{C}) \) version of the deformation argument given originally by M. Heusener, J. Porti, and E. Suárez Peiró in [HPS01] for \( \text{SL}(2, \mathbb{C}) \). Recall that Lemma 16 also implies \( \alpha \) is conjugate to a unitary representation, and once we choose a unitary representative we have that \( \dim \mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad \alpha}) = \dim \mathbb{C} \text{H}^1(N_K; sl(n, \mathbb{C})_{ad \alpha}) = n - 1 \). An application of Theorem 19 now shows that \( \alpha \) can be deformed within a real \((n - 1)\)-dimensional family of conjugacy classes of irreducible \( SU(n) \) representations.

It would be interesting to study those metabelian representations \( \alpha \) for which Theorem 2 applies. For example, the results of [Na07, Bu90] show that, if \( K \) is a 2-bridge knot, then every irreducible metabelian representation \( \alpha: \pi_1(N_K) \to \text{SL}(2, \mathbb{C}) \) satisfies \( \text{H}^1(N_K; sl(2, \mathbb{C})_{ad \alpha}) = \mathbb{C} \). One could attempt direct computations of \( \text{H}^1(N_K; sl(n, \mathbb{C})_{ad \alpha}) \) for irreducible metabelian representations of other knots and for \( n > 2 \). We do not take up this problem here, but instead content ourselves with the next result, which provides an alternative criterion for the condition \( \dim \mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad \alpha}) = n - 1 \) of Theorem 2 to hold.

Suppose that \( n \) is chosen so that \( b_1(L_n) = 0 \) (e.g. \( n \) is a prime power) and let \( \alpha: \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) be an irreducible metabelian representation. Suppose that \( \varphi: \pi_1(N_K) \to \mathbb{Z}/n \ltimes H \) is a group homomorphism with \( H \) finite and abelian such that \( \alpha \) factors through \( \varphi \). For example we could take \( \varphi \) to be the natural surjection \( \pi_1(N_K) \to \mathbb{Z}/n \ltimes \text{H}_1(L_n) \). The following theorem has been proved in the case \( \mathbb{Z}/n \ltimes H \) is a dihedral group by Boileau and Boyer (cf. [BB07, Lemma A.2]).

**Theorem 3.** Suppose that \( n \) is such that \( b_1(L_n) = 0 \). Let \( \alpha: \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) be an irreducible metabelian representation and \( \varphi: \pi_1(N_K) \to \mathbb{Z}/n \ltimes H \) be a group homomorphism with \( H \) finite and abelian such that \( \alpha \) factors through \( \varphi \). Denote by \( \tilde{N}_\varphi \to N_K \) the covering map corresponding to \( \varphi \). Then the following hold:

(i) \( b_1(\tilde{N}_\varphi) \geq |H| \) and if \( b_1(\tilde{N}_\varphi) = |H| \), then \( \dim \mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad \alpha}) = n - 1 \).

(ii) The cover \( \tilde{N}_\varphi \to N_K \) extends to a cover \( \tilde{\Sigma}_\varphi \to \Sigma \) branched over \( K \).

(iii) If \( b_1(\tilde{\Sigma}_\varphi) = 0 \), then \( \dim \mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad \alpha}) = n - 1 \).

**Acknowledgments.** The authors would like to thank Steven Boyer, Christopher Herald, Michael Heusener, Paul Kirk, Charles Livingston, Andrew Nicas and Adam Sikora for generously sharing their knowledge, wisdom, and insight. We would also like to thank Fumikazu Nagasato and Yoshikazu Yamaguchi for communicating the results of their paper to us.
2. The classification of metabelian representations of knot groups

In this section we recall some results from [BF08] regarding the classification of metabelian representations of knot groups (see also [Fr04]), and we provide some general results on the twisted homology and cohomology groups.

2.1. Preliminaries. Given a group \( \pi \), we shall write \( \pi^{(n)} \) for the \( n \)-th term of the derived series of \( \pi \). These subgroups are defined inductively by setting \( \pi^{(0)} = \pi \) and \( \pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}] \). The group \( \pi \) is called metabelian if \( \pi^{(2)} = \{e\} \).

Suppose \( V \) is a finite dimensional vector space over \( \mathbb{C} \). A representation \( \varrho : \pi \to \text{Aut}(V) \) is called metabelian if \( \varrho \) factors through \( \pi/\pi^{(2)} \). The representation \( \varrho \) is called reducible if there exists a proper subspace \( U \subset V \) invariant under \( \varrho(\gamma) \) for all \( \gamma \in \pi \). Otherwise \( \varrho \) is called irreducible or simple. If \( \varrho \) is the direct sum of simple representations, then \( \varrho \) is called semisimple.

Two representations \( \varrho_1 : \pi \to \text{Aut}(V) \) and \( \varrho_2 : \pi \to \text{Aut}(W) \) are called isomorphic if there exists an isomorphism \( \phi : V \to W \) such that \( \phi^{-1} \circ \varrho_1(g) \circ \phi = \varrho_2(g) \) for all \( g \in \pi \).

2.2. Metabelian quotients of knot groups. Let \( K \subset \Sigma^3 \) be a knot in an integral homology 3-sphere. In the following we denote by \( \tilde{N}_K \) the infinite cyclic cover of \( N_K \) corresponding to the abelianization \( \pi_1(N_K) \to H_1(N_K) \cong \mathbb{Z} \). Therefore \( \pi_1(\tilde{N}_K) = \pi_1(N_K)^{(1)} \) and

\[
H_1(N_K; \mathbb{Z}[t^{\pm 1}]) = H_1(\tilde{N}_K) \cong \pi_1(N_K)^{(1)}/\pi_1(N_K)^{(2)}.
\]

The \( \mathbb{Z}[t^{\pm 1}] \)-module structure is given on the right hand side by \( t^n \cdot g := \mu^{-n} g \mu^n \), where \( \mu \) is a meridian of \( K \).

For a knot \( K \), we set \( \pi := \pi_1(N_K) \) and consider the short exact sequence

\[
1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1.
\]

Since \( \pi/\pi^{(1)} = H_1(N_K) \cong \mathbb{Z} \), this sequence splits and we get isomorphisms

\[
\pi/\pi^{(1)} \cong \pi/\pi^{(1)} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \times H_1(\tilde{N}_K; \mathbb{Z}[t^{\pm 1}])
\]

where the semidirect products are taken with respect to the \( \mathbb{Z} \) actions defined by letting \( n \in \mathbb{Z} \) act by conjugation by \( \mu^n \) on \( \pi^{(1)}/\pi^{(2)} \) and by multiplication by \( t^n \) on \( H_1(\tilde{N}_K; \mathbb{Z}[t^{\pm 1}]) \). This demonstrates the following lemma.

**Lemma 4.** Given a knot \( K \), the set of metabelian representations of \( \pi_1(N_K) \) can be canonically identified with the set of representations of \( \mathbb{Z} \times H_1(\tilde{N}_K; \mathbb{Z}[t^{\pm 1}]) \).
2.3. Irreducible metabelian \( SL(n, \mathbb{C}) \) representations of knot groups. Let \( K \) be a knot. We write \( H = H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \). In order to classify the irreducible metabelian \( SL(n, \mathbb{C}) \) representations of \( \pi_1(N_K) \) it suffices by Lemma 4 to classify the irreducible \( SL(n, \mathbb{C}) \) representations of \( \mathbb{Z} \ltimes H \).

Throughout this paper we will frequently make use of the following well–known facts:

1. \( H \) is a finitely generated \( \mathbb{Z}[t^{\pm 1}] \)–module such that multiplication by \( t - 1 \) is an isomorphism.

2. Given \( n \in \mathbb{N} \) there exists a natural identification \( H/(t^n - 1) \cong H_1(L_n) \), where \( L_n \) denotes the \( n \)–fold cover of \( \Sigma^3 \) branched over \( K \).

Let \( \chi : H \rightarrow \mathbb{C}^* \) be a character which factors through \( H/(t^n - 1) \) and \( z \in U(1) \). Then it follows from [BF08, Section 3] that, for \( (j, h) \in \mathbb{Z} \ltimes H \), setting

\[
\alpha_{(n,\chi,z)}(j, h) = \begin{pmatrix} 0 & \cdots & z \\ z & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \cdots & 0 \\ 0 & \chi(th) & \cdots & 0 \\ \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & \chi(t^{n-1}h) \end{pmatrix} 
\]

defines a \( GL(n, \mathbb{C}) \) representation. Note that \( \alpha_{(n,\chi,z)} \) factors through \( \mathbb{Z} \ltimes H/(t^n - 1) \).

We are mostly interested in the special values for \( z \):

1. If \( z \in \mathbb{C} \) such that \( z^n = (-1)^{n+1} \), then we write \( \alpha_{(n,\chi)} = \alpha_{(n,\chi,z)} \). Note that \( \alpha_{(n,\chi)} \) defines an \( SL(n, \mathbb{C}) \)–representation and that the isomorphism type of this representation is independent of the choice of \( z \).

2. If \( z = 1 \), then we write \( \beta_{(n,\chi)} = \alpha_{(n,\chi,1)} \).

In our notation we will normally not distinguish between metabelian representations of \( \pi_1(N_K) \) and representations of \( \mathbb{Z} \ltimes H \).

In the following we say that a character \( \chi : H \rightarrow \mathbb{C}^* \) has order \( n \) if it factors through \( H/(t^n - 1) \) and but not through \( H/(t^\ell - 1) \) for any \( \ell < n \). Given a character \( \chi : H \rightarrow \mathbb{C}^* \), let \( t^i\chi \) be the character defined by \( (t^i\chi)(h) = \chi(t^ih) \). Any character \( \chi : H \rightarrow \mathbb{C}^* \) which factors through \( H/(t^n - 1) \) must have order \( k \) for some divisor \( k \) of \( n \). The following is a combination of [BF08, Lemma 2.2] and [BF08, Theorem 3.3].

**Theorem 5.** Suppose \( \chi : H \rightarrow \mathbb{C}^* \) is a character that factors through \( H/(t^n - 1) \).

1. \( \alpha_{(n,\chi)} : \mathbb{Z} \ltimes H \rightarrow SL(n, \mathbb{C}) \) is irreducible if and only if the character \( \chi \) has order \( n \).

2. \( \alpha_{(n,\chi)} : \mathbb{Z} \ltimes H \rightarrow SL(n, \mathbb{C}) \) is irreducible if and only if the character \( \chi \) has order \( n \).

3. Given two characters \( \chi, \chi' : H \rightarrow \mathbb{C}^* \) of order \( n \), the representations \( \alpha_{(n,\chi)} \) and \( \alpha_{(n,\chi')} \) are conjugate if and only if \( \chi = t^k\chi' \) for some \( k \).

4. For any irreducible representation \( \alpha : \mathbb{Z} \ltimes H \rightarrow SL(n, \mathbb{C}) \) there exists a character \( \chi : H \rightarrow \mathbb{C}^* \) of order \( n \) such that \( \alpha \) is conjugate to \( \alpha_{(n,\chi)} \).
2.4. The adjoint representation corresponding to a metabelian representation. The following lemma allows us to decompose the adjoint representation corresponding to a metabelian representation into a direct sum of much simpler representations.

Lemma 6. Let $K$ be a knot, $n \in \mathbb{N}$ and $\chi: H_1(L_n) \to \mathbb{C}^*$ be a character. Suppose $\alpha = \alpha_{(n, \chi)}$ and let $\theta_1: \pi_1(N_K) \to \text{GL}(1, \mathbb{C})$ be the trivial representation. Setting $\alpha_n : \pi_1(N_K) \to \text{Aut}(\mathbb{C}[\mathbb{Z}/n])$ equal to the regular representation corresponding to the canonical projection map $\pi_1(N_K) \to \mathbb{Z} \to \mathbb{Z}/n$ and using $\text{ad} \alpha : \pi_1(N_K) \to \text{Aut}(sl(n, \mathbb{C}))$ to denote the adjoint representation, we have the following isomorphism of representations:

$$ad \alpha \oplus \theta_1 \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n, \chi_i)},$$

where $\chi_i$ is the character defined by $\chi_i(v) := \chi(v)^{-1} \chi(t^iv)$. Furthermore, if $\chi$ is a character of order $n$, then $\chi_1, \ldots, \chi_{n-1}$ are also characters of order $n$.

Proof. As before, we write $\pi = \pi_1(N_K)$. We denote by $\beta : \pi \to \text{Aut}(gl(n, \mathbb{C}))$ the adjoint representation of $\alpha$ on $gl(n, \mathbb{C})$, so $\beta(g)(A) = \alpha(g)A\alpha(g)^{-1}$ for $g \in \pi$ and $A \in gl(n, \mathbb{C})$. Note that $gl(n, \mathbb{C}) = sl(n, \mathbb{C}) \oplus \mathbb{C} \cdot I$. It follows immediately that $\beta = ad \alpha \oplus \theta_1$ splits off a trivial factor. It therefore suffices to show that

$$\beta \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n, \chi_i)}.$$

For $i = 0, \ldots, n-1$ we denote by $V_i$ the set of all matrices $(a_{jk})$ with $a_{jk} = 0$ unless $j-k \equiv i(n)$. It is not difficult to see that the action of $\pi$ on $gl(n, \mathbb{C})$ restricts to actions on $V_0, V_1, \ldots, V_{n-1}$. We equip $V_i$ with the ordered basis $\{e_{i+1,1}, e_{i+2,2}, \ldots, e_{i+n,n}\}$, where the indices are taken modulo $n$. The restriction of $\beta$ to $V_i$ can then be calculated with respect to this basis and it is straightforward to verify that it is given by $\beta_{(n, \chi_i)}$. Note that the character $\chi_0$ is the trivial character, and therefore $\beta_{(n, \chi_0)} = \alpha_n$. \qed

3. Twisted homology and cohomology groups

3.1. Definitions and basic properties. We recall the definition of twisted homology and cohomology and their basic properties. Let $(X, Y)$ be a pair of topological spaces, $V$ a finite dimensional complex vector space and $\alpha: \pi_1(X) \to \text{Aut}(V)$ a representation. Denote by $p: \tilde{X} \to X$ the universal covering and set $\tilde{Y} := p^{-1}(Y)$. Letting $\pi = \pi_1(X)$, we use the representation $\alpha$ to regard $V$ as a left $\mathbb{Z}[\pi]$–module. The chain complex $C_*(\tilde{X}, \tilde{Y})$ is also a left $\mathbb{Z}[\pi]$–module via deck transformations and we can form the twisted cohomology groups

$$H^*(X, Y; V_\alpha) = H_*(\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X}, \tilde{Y}), V)).$$
We write $b_i(X,Y;V_\alpha) = \dim_\mathbb{C} H^i(X,Y;V_\alpha)$. Using the natural involution $g \mapsto g^{-1}$ on the group ring $\mathbb{Z}[\pi]$, we can also view $C_*(\tilde{X},\tilde{Y})$ as a right $\mathbb{Z}[\pi]$–module and form the twisted homology groups
\[ H_*(X,Y;V_\alpha) = H_*(C_*(\tilde{X},\tilde{Y}) \otimes_{\mathbb{Z}[\pi]} V). \]

We write $b_i(X,Y;V_\alpha) = \dim_\mathbb{C} H_i(X,Y;V_\alpha)$.

Notice first off that the twisted cohomology and homology groups corresponding to isomorphic representations are isomorphic as $\mathbb{C}$–vector spaces. Note further that the 0–th twisted groups of a space can be computed immediately from the fundamental group (cf. [HS97, Section VI]):
\begin{align}
H^0(X;V_\alpha) &= \{ v \in V \mid \alpha(g)v = v \text{ for all } g \in \pi_1(X) \}, \\
H_0(X;V_\alpha) &= V/\{\alpha(g)v - v \mid v \in V, g \in \pi_1(X) \}.
\end{align}

Finally note that if $M$ is an $n$–manifold, then by Poincaré duality we have
\[ H_i(M;V_\alpha) \cong H^{n-i}(M,\partial M;V_\alpha) \text{ and } H_i(M,\partial M;V_\alpha) \cong H^{n-i}(M;V_\alpha). \]

We will several times make use of the following well–known lemma (cf. e.g. [FK06, Lemma 2.3]).

**Lemma 7.** Suppose that $V$ is equipped either with a bilinear non–singular form or with a hermitian non–singular form, and that $\alpha$ is orthogonal respectively unitary with respect to this form. Then
\[ H_i(X,Y;V_\alpha) \cong H^i(X,Y;V_\alpha) \]
for any $i$.

The next lemma shows that the hypotheses of the previous lemma is satisfied in the case of the adjoint representation.

**Lemma 8.** Let $\alpha : \pi \to \mathrm{SL}(n,\mathbb{C})$ be a representation and consider the corresponding adjoint representation $\text{ad} \alpha : \pi \to \text{Aut}(\mathfrak{sl}(n,\mathbb{C}))$. Then there exists a non–singular bilinear form on $\mathfrak{sl}(n,\mathbb{C})$ with respect to which $\text{ad} \alpha$ is orthogonal.

**Proof.** It is straightforward to verify that the form $(A,B) \mapsto \text{tr}(A \cdot B)$ has all the required properties. \hfill \square

### 3.2. Calculations

In this subsection we present some calculations of twisted homology and cohomology groups that will be used in the proofs of the main results.

**Lemma 9.** Let $K$ be a knot, $\chi : H_1(I_m) \to \mathbb{C}^*$ a character and $z \in U(1)$. Let $V = \mathbb{C}^n$ and $\alpha = \alpha_{(n,\chi,z)} : \pi_1(N_K) \to \text{Aut}(V)$, and set $\hat{\alpha}$ to be the restriction of $\alpha$ to $\pi_1(\partial N_K)$. If $z^n = 1$, then the following hold:
\begin{align}
b^0(\partial N_K;V_{\hat{\alpha}}) &= b_0(\partial N_K;V_{\hat{\alpha}}) = 1, \\
b^1(\partial N_K;V_{\hat{\alpha}}) &= b_1(\partial N_K;V_{\hat{\alpha}}) = 2, \\
b^2(\partial N_K;V_{\hat{\alpha}}) &= b_2(\partial N_K;V_{\hat{\alpha}}) = 1.
\end{align}
Proof. We let $\mu$ and $\lambda$ be the meridian and longitude of $\hat{K}$. Note that $\alpha(\lambda)$ is trivial and that $\alpha(\mu)$ is diagonal with eigenvalues $z, ze^{2\pi i/n}, \ldots, ze^{2\pi i(n-1)/n}$, which are distinct. Note that $\alpha(\mu)$ has precisely one eigenvalue which equals one. A direct calculation using Equation (1) now shows that $b_0(\partial N_K, V_\alpha) = 1$ and $b_0(\partial N_K, V_\alpha) = 1$. It follows from duality that $b_2(\partial N_K; V_\alpha) = 1$ and $b_2^*(\partial N_K; V_\alpha) = 1$. Since the Euler characteristic of the torus $\partial N_K$ is zero it now follows that $b_1(\partial N_K; V_\alpha) = b_1(\partial N_K; V_\alpha) = 2$. \hfill \Box

Lemma 10. Let $K$ be a knot. For $i = 1, \ldots, \ell$, let $\chi_i : H_1(L_n) \to \mathbb{C}^*$ be a non-trivial character and $z_i \in U(1)$ with $z_i^n = 1$. Let $V = \mathbb{C}^{n\ell}$ and consider the representation $\alpha = \bigoplus_{i=1}^{\ell} \alpha(m, \chi_i, z_i) : \pi_1(N_K) \to \text{Aut}(V)$. Then the following hold:

(i) $b^0(N_K; V_\alpha) = 0$,
(ii) if $\alpha$ is orthogonal or unitary with respect to a non-singular form on $V$, then $b^1(N_K; V_\alpha) \geq \ell$.

Proof. The first statement is an immediate consequence of Equation (1) and the assumption that $\chi_i$ are non-trivial. By Lemma 9 we have $b_1(\partial N_K; V_\alpha) = 2\ell$. Now consider the following short exact sequence

$$H^1(N_K; V_\alpha) \to H^1(\partial N_K; V_\alpha) \to H^2(N_K, \partial N_K; V_\alpha).$$

It follows that either $b^1(N_K; V_\alpha) \geq \ell$ or $b^2(N_K, \partial N_K; V_\alpha) \geq \ell$. But by Poincaré duality and by Lemma 7 the latter also equals $b^1(N_K; V_\alpha)$. \hfill \Box

4. Proofs of main results

4.1. Proof of Theorem 1. Set $\omega = e^{2\pi i/n}$ and recall the action of the cyclic group $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle$ on representations $\alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C})$ obtained by setting $(\sigma \cdot \alpha)(g) = \omega^{\varepsilon(g)} \alpha(g)$ for all $g \in \pi_1(N_K)$, where $\varepsilon : \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$.

We begin with the following useful lemma.

Lemma 11. Suppose $\alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C})$ is a representation whose associated character $\xi_\alpha \in X_n(N_K)$ is a fixed point of the $\mathbb{Z}/n$ action. Then up to conjugation, we have

$$\alpha(\mu) = \begin{pmatrix}
0 & \cdots & z \\
\vdots & \ddots & \vdots \\
z & \cdots & 0 \\
0 & \cdots & z
\end{pmatrix},$$

for some (in fact any) $z \in U(1)$ such that $z^n = (-1)^{n+1}$.

Proof. Let $c(t) = \det(\alpha(\mu) - tI)$ denote the characteristic polynomial of $\alpha(\mu)$, which we can write as

$$c(t) = (-1)^n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + 1.$$
Note that $c(t)$ is determined by the character $\xi_\alpha \in X_n(N_K)$, and so assuming $\xi_\alpha$ is a fixed point of the $\mathbb{Z}/n$ action, we conclude that $\alpha(\mu)$ and $\omega^k\alpha(\mu)$ have the same characteristic polynomials for all $k$. In particular,
\[
c(t) = \det(\omega^{-1}\alpha(\mu) - t I) = \det(\omega^{-1}\alpha(\mu) - (\omega^{-1}\omega)t I) = \det(\omega^{-1}I) \det(\alpha(\mu) - \omega t I) = \det(\alpha(\mu) - t\omega I) = c(\omega t).
\]
However, $\omega^k \neq 1$ unless $n|k$, and this implies $0 = c_{n-1} = c_{n-2} = \cdots = c_1$ and $c(t) = (-1)^nt^{n+1}$. In particular the matrix $\alpha(\mu)$ and the matrix appearing in Equation (2) have the same set of $n$ distinct eigenvalues. This implies that the two matrices are conjugate. □

In order to prove Theorem 1, we shall establish the following slightly more general result.

**Theorem 12.** The fixed point set of the $\mathbb{Z}/n$ action on $X_n(N_K)$ consists of characters $\xi_\alpha$ of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Theorem 5. In other words,
\[
X_n(N_K)^{\mathbb{Z}/n} = \{ \xi_\alpha \mid \alpha = \alpha_{(n,\chi)} \text{ for } \chi : H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^* \}.
\]

Notice that Theorem 1 can be viewed as the special case of Theorem 12 where $\alpha_{(n,\chi)}$ is irreducible. Notice further that not every reducible metabelian representation is of the form $\alpha_{(n,\chi)}$.

**Proof.** We first show that if $\alpha : \pi_1(N_K) \to SL(n, \mathbb{C})$ is given as $\alpha = \alpha_{(n,\chi)}$, then $\sigma \cdot \alpha$ is conjugate to $\alpha$. This of course implies that $\xi_\alpha = \xi_{\sigma \cdot \alpha}$.

Assume then that $\alpha = \alpha_{(n,\chi)}$. Then we have
\[
\alpha(\mu) = \begin{pmatrix} 0 & \cdots & z \\ z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & z & 0 \end{pmatrix},
\]
where $z$ satisfies $z^n = (-1)^{n+1}$. Further, $\alpha(g)$ is diagonal for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. By definition of $\sigma \cdot \alpha$, we see that
\[
(\sigma \cdot \alpha)(\mu) = \omega \alpha(\mu) = \begin{pmatrix} 0 & \cdots & \omega z \\ \omega z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \omega z & 0 \end{pmatrix}
\]
and that $(\sigma \cdot \alpha)(g) = \alpha(g)$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. It follows easily from Theorem 5 that $\sigma \cdot \alpha$ and $\alpha_{(n,\chi)}$ are conjugate; however it is easy to see this directly too. Simply
take

\[ P = \begin{pmatrix} 1 & \omega & 0 \\ \omega & \ddots & \omega \\ 0 & \omega^{n-1} \end{pmatrix}, \]

and compute that \( \sigma \cdot \alpha = P \alpha P^{-1} \) as claimed.

We now show the other implication, namely that each point \( \xi \in X_n(N_K)^{\mathbb{Z}/n} \) in the fixed point set can be represented as the character \( \xi = \xi_\alpha \) of a metabelian representation \( \alpha = \alpha_{(n, \chi)} \). Here \( \chi: H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^* \) is a character that factors through \( H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^n - 1) \), so it has order \( k \) for some \( k \) dividing \( n \). Recall that Theorem 5 tells us that \( \alpha_{(n, \chi)} \) is irreducible if and only if \( \chi \) has order \( n \).

By the general results on representation spaces and character varieties (see [LM85]), it follows that every point in the character variety \( X_n(N_K) \) can be represented as \( \xi_\alpha \) for some semisimple representation \( \alpha: \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \). Further, we see that two semisimple representations \( \alpha_1 \) and \( \alpha_2 \) determine the same character if and only if \( \alpha_1 \) is conjugate to \( \alpha_2 \). (This is evident from the fact that the orbits of the semisimple representations under conjugation are closed.)

Given \( \xi \in X_n(N_K)^{\mathbb{Z}/n} \), we can therefore suppose that \( \xi = \xi_\alpha \) for some semisimple representation \( \alpha \). Clearly \( \sigma \cdot \alpha \) is also semisimple, and since \( \xi_\alpha = \xi_{\sigma \cdot \alpha} \), we conclude that \( \alpha \) and \( \sigma \cdot \alpha \) are conjugate representations.

Lemma 11 implies \( \alpha(\mu) \) is conjugate to the matrix in Equation (2). It is convenient to conjugate \( \alpha \) so that \( \alpha(\mu) \) is diagonal, meaning that

\[ \alpha(\mu) = \begin{pmatrix} z & 0 \\ \omega z & \ddots \\ 0 & \ddots & \omega^{n-1} z \end{pmatrix}, \]

where \( z \) satisfies \( z^n = (-1)^{n+1} \).

Since the conjugacy class of \( \alpha \) is fixed under the \( \mathbb{Z}/n \) action, it follows that there exists a matrix \( A \in \text{SL}(n, \mathbb{C}) \) such that \( A\alpha A^{-1} = \sigma \cdot \alpha \). In other words, for all \( g \in \pi_1(N_K) \), we have

\[ A\alpha(g)A^{-1} = \omega^{\epsilon(g)}\alpha(g). \]

In the case of the meridian, this shows

\[ A\alpha(\mu) = \omega\alpha(\mu)A, \]
which implies $A = (a_{ij})$ satisfies $a_{ij} = 0$ unless $j = i + 1 \pmod{(n)}$. Thus, we see that

$$
A = \begin{pmatrix}
0 & \lambda_1 & 0 & \cdots & 0 \\
0 & 0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \lambda_{n-1} \\
\lambda_n & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

for some $\lambda_1, \ldots, \lambda_n$ satisfying $\lambda_1 \cdots \lambda_n = (-1)^{n+1}$.

It is completely straightforward to see that the characteristic polynomial of $A$ is given by

$$\det(A - tI) = (-1)^n(t^n - (-1)^{n+1}).$$

From this, we conclude that $A$ has as its eigenvalues the $n$ distinct $n$–th roots of $(-1)^{n+1}$. In particular, the subset of $\text{SL}(n, \mathbb{C})$ of matrices that commute with $A$ is just a copy of the unique maximal torus $T_A \cong (\mathbb{C}^*)^{n-1}$ containing $A$.

For any $g \in [\pi_1(N_K), \pi_1(N_K)]$, we have $\alpha(g) = (\sigma \cdot \alpha)(g)$. Thus it follows that $A\alpha(g)A^{-1} = \alpha(g)$, and this implies that $\alpha(g) \in T_A$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. This shows that the restriction of $\alpha$ to the commutator subgroup $[\pi_1(N_K), \pi_1(N_K)]$ is abelian, and we conclude from this that $\alpha$ is indeed metabelian. Notice that this, and an application of Theorem 5, completes the proof in the case $\alpha$ is irreducible.

In the general case, we see by Lemma 4 that $\alpha$ factors through $\mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Let $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Given a character $\chi : H \to \mathbb{C}^*$ we define the associated weight space $V_\chi$ by setting

$$V_\chi = \{ v \in \mathbb{C}^n \mid \chi(h) \cdot v = \alpha(h)v \text{ for all } h \in H \}.$$ 

Recall that $A \cdot \alpha(h) \cdot A^{-1} = \alpha(h)$ for any $h \in H$. It is straightforward so show that $A$ restricts to an automorphism of $V_\chi$. Since $H$ is abelian there exists at least one character $\chi : H \to \mathbb{C}^*$ such that $V_\chi$ is non–trivial. Given $i$ we denote by $t^i \chi$ the character given by $(t^i \chi)(h) = \chi(t^i h), h \in H$.

Note that $A$ has $n$ distinct eigenvalues and therefore is diagonalizable. Since $A$ restricts to an automorphism of $V_\chi$, there is an eigenvector $v$ of $A$ which lies in $V_\chi$. Let $\lambda$ be the corresponding eigenvalue. By the proof of [BF08, Theorem 2.3], the map $\alpha(\mu)$ induces an isomorphism $V_\chi \to V_{t^i \chi}$. We now calculate

$$A \cdot \alpha(\mu)v = (A\alpha(\mu)A^{-1}) \cdot Av = \omega \alpha(\mu) \cdot \lambda v = \lambda \omega \cdot \alpha(\mu)v,$$

i.e. $\alpha(\mu)v \in V_{t^i \chi}$ is an eigenvector of $A$ with eigenvalue $\omega \lambda$.

Iterating this argument, we see that $\alpha(\mu)^iv$ lies in $V_{t^i \chi}$ and is an eigenvector of $A$ with eigenvalue $\omega^i \lambda$. Since $\omega$ is a primitive $n$–th root of unity, the eigenvalues $\lambda, \omega \lambda, \ldots, \omega^{n-1} \lambda$ are all distinct, and this implies that the corresponding eigenvectors $v, \alpha(\mu)v, \ldots, \alpha(\mu)^{n-1}v$ form a basis for $\mathbb{C}^n$.

Let $m$ be the order of $\chi$, i.e. $m$ is the minimal number such that $\chi = t^m \chi$. By the above we have that $\mathbb{C}^n$ is generated by $V_\chi, V_{t_\chi}, \ldots, V_{t^{m-1} \chi}$. Since the characters
\(\chi, t\chi, \ldots, t^m\chi\) are pairwise distinct, it follows that \(\mathbb{C}^n\) is given as the direct sum \(V_\chi \oplus V_{t\chi} \oplus \cdots \oplus V_{t^{m-1}\chi}\).

We write \(k = \dim_{\mathbb{C}}(V_\chi)\) and note that \(n = km\). We note further that \(\alpha(\mu)^m\) has eigenvalues given by the set
\[
\{z^m, z^m e^{2\pi i/k}, \ldots, z^m e^{2\pi i(k-1)/k}\},
\]
and each eigenvalue has multiplicity \(m\). Clearly \(\alpha(\mu)^m\) restricts to an automorphism of \(V_{t^i\chi}\) for \(i = 0, \ldots, m-1\), and equally clearly we see that the restrictions all give conjugate representations. This implies that the restriction of \(\alpha(\mu)^m\) to \(V_\chi\) has eigenvalues in the set (3) above, each occurring with multiplicity 1. In particular we can find a basis \(\{v_1, \ldots, v_k\}\) for \(V_\chi\) in which the matrix of \(\alpha(\mu)^m\) has the form
\[
\begin{pmatrix}
0 & \cdots & z^m \\
& 0 & \cdots \\
& & \ddots \\
0 & \cdots & z^m
\end{pmatrix}.
\]

It is now straightforward to verify that with respect to the ordered basis
\[
\begin{align*}
v_1, & \quad z^{-1}\alpha(\mu)v_1, \quad \ldots, \quad z^{-(m-1)}\alpha(\mu)^{m-1}v_1, \\
v_2, & \quad z^{-1}\alpha(\mu)v_2, \quad \ldots, \quad z^{-(m-1)}\alpha(\mu)^{m-1}v_2, \\
\vdots & \quad \vdots \quad \ddots \quad \vdots \\
v_k, & \quad z^{-1}\alpha(\mu)w_k, \quad \ldots, \quad z^{-(m-1)}\alpha(\mu)^{m-1}v_k
\end{align*}
\]
\(\alpha\) is given by \(\alpha(n, \chi)\). \(\square\)

We pause briefly to explain how Theorem 12 can be applied to see that the twisted Alexander polynomials at metabelian representations must be a polynomial in \(t^n\).

In the following, we use \(\Delta_{K,i}^\alpha(t)\) to denote the \(i\)-th twisted Alexander polynomial for a given representation \(\alpha: \pi_1(N_K) \to \text{SL}(n, \mathbb{C})\) as presented in [FV09].

**Proposition 13.** Let \(\alpha\) be a metabelian representation of the form \(\alpha = \alpha_{n,\chi}: \pi_1(N_K) \to \text{SL}(n, \mathbb{C})\). Then
\[
\Delta_{K,0}^\alpha(t) = \begin{cases} 
1 - t^n, & \text{if } \chi \text{ is trivial,} \\
1, & \text{otherwise.}
\end{cases}
\]
Furthermore the twisted Alexander polynomial \(\Delta_{K,1}^\alpha(t)\) is actually a polynomial in \(t^n\).

**Remark 14.** This result addresses some questions raised by M. Hirasawa and K. Murasugi in their computations of the twisted Alexander polynomials in [HM09]. In particular, it gives a positive answer to Conjecture A from [HM09]. There are other approaches to prove Proposition 13; for example on p. 10 of [HKL08], C. Herald, P. Kirk, and C. Livingston prove the second statement using different techniques.

**Proof.** The first statement is obvious in the case that \(\chi\) is trivial. If \(\chi\) is non–trivial then the statement follows either from a direct calculation or from [FJV09].
We now turn to the proof of the second statement. For \( \theta \in U(1) \) and any representation \( \beta : \pi_1(N_K) \to \text{GL}(n, \mathbb{C}) \), define the \( \theta \)-twist of \( \beta \) to be the representation sending \( g \in \pi_1(N_K) \) to \( \theta^{\epsilon(g)} \beta(g) \), where \( \epsilon : \pi_1(N_K) \to \mathbb{Z} \) is determined by the orientation of \( K \). We denote the newly obtained representation by \( \beta^{\theta} : \pi_1(N_K) \to \text{GL}(n, \mathbb{C}) \).

Note that in case \( \alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) and \( \theta = e^{2\pi i/n} \) is an \( n \)-th root of unity, \( \alpha^{\theta} \) is again an \( \text{SL}(n, \mathbb{C}) \) representation. The proof of the proposition relies on the formula

\[
\Delta^{\beta^{\theta}}_{K,1}(t) = \Delta^\beta_{K,1}(\theta t).
\]

This formula is well-known and follows directly from the definition of the twisted Alexander polynomial. Equation (4) combines with Theorem 1 to complete the proof, as we now explain. Take \( \omega = e^{2\pi i/n} \). If \( \alpha^{\theta} = \alpha^{(n, \chi)} \) is metabelian, then Theorem 1 shows that its conjugacy class is fixed under the \( \mathbb{Z}/n \) action. In particular, since \( \alpha \) and \( \alpha^{\omega} \) are conjugate, Equation (4) shows that

\[
\Delta^\alpha_{K,1}(t) = \Delta^{\alpha^{\omega}}_{K,1}(t) = \Delta^\alpha_{K,1}(\omega t).
\]

Expanding \( \Delta^\alpha_{K,1}(t) = \sum a_i t^i \) and using the fact that \( t^k = (\omega t)^k \) if and only if \( k \) is a multiple of \( n \), this shows that \( a_k = 0 \) unless \( k \) is a multiple of \( n \) and this completes the proof.

### 4.2. Proof of Theorem 2

Assume now that \( \alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) is a representation and let \( \hat{\alpha} : \pi_1(\partial N_K) \to \text{SL}(n, \mathbb{C}) \) denote the restriction of \( \alpha \) to the boundary torus. Throughout this subsection, we will assume that the image of \( \hat{\alpha} \) contains a matrix with \( n \) distinct eigenvalues. Note that every irreducible metabelian representation of \( \pi_1(N_K) \) satisfies this condition.

Choose \( g \in \pi_1(\partial N_K) \) so that \( \alpha(g) \) has \( n \) distinct eigenvalues. Then this matrix is diagonalizable, and any other matrix that commutes with it must lie in the same maximal torus. Since \( \pi_1(\partial N_K) \cong \mathbb{Z} \oplus \mathbb{Z} \) is abelian, we see that the stabilizer subgroup of \( \hat{\alpha} \) under conjugation action is again this maximal torus. From this, Poincaré duality and Euler characteristic considerations, we conclude that

\[
\dim_{\mathbb{C}} H^0(\partial N_K; \text{sl}(n, \mathbb{C})_{ad, \hat{\alpha}}) = n - 1,
\]

\[
\dim_{\mathbb{C}} H^1(\partial N_K; \text{sl}(n, \mathbb{C})_{ad, \hat{\alpha}}) = 2(n - 1), \quad \text{and}
\]

\[
\dim_{\mathbb{C}} H^2(\partial N_K; \text{sl}(n, \mathbb{C})_{ad, \hat{\alpha}}) = n - 1.
\]

We now consider the long exact sequence in twisted cohomology associated with the pair \((N_K, \partial N_K)\). The inclusions

\[
(\partial N_K, \emptyset) \hookrightarrow (N_K, \emptyset) \hookrightarrow (N_K, \partial N_K)
\]

induce the following long exact sequence (coefficients in \( \text{sl}(n, \mathbb{C}) \) twisted by \( \alpha \) or \( \hat{\alpha} \) understood).
The next proposition shows that the image \( \mathcal{H}^1(N_K; \text{sl}(n, \mathbb{C})) \) has dimension \( n - 1 \), and we explain this by relating it to the general result that for a 3-manifold \( N \) with connected boundary \( \partial N \), there is a symplectic structure on the character variety \( X_n(\partial N) \) and the image \( X_n(N) \to X_n(\partial N) \) is Lagrangian, which is proved for representations into arbitrary reductive groups in the recent paper [Si09] of A. Sikora, under the assumption that \( \partial X \) is a connected surface of genus \( g \geq 2 \). We state and prove the analogous result only for the case of interest to us, namely \( \text{SL}(n, \mathbb{C}) \) representations of knot complements \( N_K \), but we remark that the argument can be generalized to other Lie groups \( G \) (e.g. \( G = SU(n) \)) and other 3-manifolds \( N \) with boundary \( \partial N \) a torus.

**Proposition 15.** If \( K \) is a knot and \( \alpha: \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) is a representation such that the image of the restriction \( \hat{\alpha}: \pi_1(\partial N_K) \to \text{SL}(n, \mathbb{C}) \) contains a matrix with \( n \) distinct eigenvalues, then the image

\[
\text{image} \left( \mathcal{H}^1(N_K; \text{sl}(n, \mathbb{C})) \xrightarrow{i_!} \mathcal{H}^1(\partial N_K; \text{sl}(n, \mathbb{C})) \right)
\]

has dimension \( n - 1 \) and is Lagrangian with respect to the symplectic structure \( \Omega \) defined below. It follows that

\[
\dim_{\mathbb{C}} \mathcal{H}^1(N_K; \text{sl}(n, \mathbb{C})) \geq n - 1.
\]

**Proof.** The fact that \( \dim_{\mathbb{C}} \text{image} \left( \mathcal{H}^1(N_K) \xrightarrow{i!} \mathcal{H}^1(\partial N_K) \right) = n - 1 \) follows easily from a diagram chase of the long exact sequence (5), using the fact that since the image of \( \hat{\alpha} \) contains an element with \( n \) distinct eigenvalues, we know \( H^0(\partial N_K; \text{sl}(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) = \mathbb{C}^{n-1} = H^2(\partial N_K; \text{sl}(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) \) and \( H^1(\partial N_K; \text{sl}(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) = \mathbb{C}^{2n-2} \).

The symplectic structure \( \Omega \) on \( \mathcal{H}^1(\partial N_K; \text{sl}(n, \mathbb{C})) \) is defined by composing the cup product with the symmetric bilinear pairing obtained by first multiplying the matrices and then taking the trace:

\[
\text{sl}(n, \mathbb{C}) \times \text{sl}(n, \mathbb{C}) \to \text{gl}(n, \mathbb{C}) \to \mathbb{C}
\]

\[
(A, B) \mapsto A \cdot B \mapsto \text{tr}(A \cdot B).
\]
We have already seen that the image \( H^1(N_K; \text{sl}(n, \mathbb{C})_{ad}) \) has dimension \( n - 1 \), so we just need to show that it is isotropic with respect to \( \Omega \).

Suppose \( x, y \in H^1(N_K; \text{sl}(n, \mathbb{C})_{ad}) \) and consider the long exact sequence (5) with untwisted coefficients in \( \mathbb{C} \). Let \( \cup \) denote the combined cup and matrix product, so \( x \cup y \in H^2(N_K; gl(n, \mathbb{C})_{ad}) \). Using the commutative diagram

\[
\begin{array}{ccc}
H^1(N_K; \text{sl}(n, \mathbb{C})_{ad}) \times H^1(N_K; \text{sl}(n, \mathbb{C})_{ad}) & \longrightarrow & H^2(N_K; gl(n, \mathbb{C})_{ad}) \\
\downarrow i^1 & & \downarrow i^2 \\
H^1(\partial N_K; \text{sl}(n, \mathbb{C})_{ad}) \times H^1(\partial N_K; \text{sl}(n, \mathbb{C})_{ad}) & \longrightarrow & H^2(\partial N_K; gl(n, \mathbb{C})_{ad}),
\end{array}
\]

we see that \( \Omega(i^1(x), i^1(y)) = tr(i^1(x) \cup i^1(y)) = tr i^2(x \cup y) \), so it is in the image of

\[
(6) \quad H^2(N_K; \mathbb{C}) \longrightarrow H^2(\partial N_K; \mathbb{C}),
\]

which by exactness of the third row of the long exact sequence (5), now taken with untwisted \( \mathbb{C} \) coefficients, equals the kernel of the surjection \( H^2(\partial N_K; \mathbb{C}) \rightarrow H^3(N_K, \partial N_K; \mathbb{C}) \). However, it is not difficult to compute \( H^3(N_K, \partial N_K; \mathbb{C}) = \mathbb{C} = H^2(\partial N_K; \mathbb{C}) \), and this implies the map in Equation (6) is the zero map. \( \blacksquare \)

In the special case that \( \dim_{\mathbb{C}} H^1(N_K; \text{sl}(n, \mathbb{C})_{ad}) = n - 1 \), we will prove integrability of all the tangent vectors and deduce that the corresponding point is a smooth point of the character variety.

In the next lemma, we denote by \( L_n \) the \( n \)-fold branched cover of \( \Sigma \) branched along the knot \( K \).

**Lemma 16.** Suppose \( \alpha: \pi_1(N_K) \rightarrow \text{SL}(n, \mathbb{C}) \) is an irreducible metabelian representation. If \( \dim_{\mathbb{C}} H^1(N_K; \text{sl}(n, \mathbb{C})_{ad}) = n - 1 \), then \( b_1(L_n) = 0 \). Consequently \( \alpha \) has finite image and is therefore conjugate to a unitary representation.

**Proof.** Recall that we can assume that \( \alpha = \alpha_{(m, \chi)} \) for some some character \( \chi: H_1(L_n) \rightarrow \mathbb{C}^* \). We denote by \( \theta_1: \pi_1(N_K) \rightarrow \text{GL}(1, \mathbb{C}) \) the trivial representation and we denote by \( \alpha_n: \pi_1(N_K) \rightarrow \text{Aut}(\mathbb{C}[\mathbb{Z}/n]) \) the regular representation corresponding to the canonical projection map \( \pi_1(N_K) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/n \). By Lemma 6 we have the following isomorphism of representations:

\[
ad \alpha + \theta_1 \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n, \chi_i)},
\]

where \( \chi_i, i = 1, \ldots, n - 1 \) are characters. Note that \( \theta_1, \alpha_n \) and \( ad \alpha \) are orthogonal (cf. Lemma 8). It follows that \( \beta := \bigoplus_{i=1}^{n-1} \beta_{(n, \chi_i)}: \pi_1(N_K) \rightarrow \text{GL}(n(n - 1), \mathbb{C}) \) is also
Suppose as a first step, we will use the hypothesis to show that $\dim \alpha T$ presented group $\pi$ as a special case. Before stating that result, we recall the following definition. A $\pi$-module by this observation with computations of the twisted cohomology of $R$ at $\alpha$ is a simple point in $X_n(K)$, and similarly for $\hat{\alpha}$ and $\xi$. For $\alpha$, this uses irreducibility, and for $\hat{\alpha}$, it follows by our hypothesis on the image of $\hat{\alpha}$.

Before beginning the argument, we introduce some notation. Given a finitely presented group $\pi$ and representation $\alpha: \pi \to SL(n, \mathbb{C})$, we use $H^*(\pi; sl(n, \mathbb{C})_{ad})$ to denote the cohomology of the group with coefficients in the $\pi$-module by $sl(n, \mathbb{C})_{ad}$.

In order to prove that $\hat{\alpha}$ is a simple point in $R_n(\partial N_K)$, we will compare the dimension of the cocycles $Z^1(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad})$ with the local dimension of $R_n(\partial N_K)$ at $\alpha$, which is defined to be the maximal dimension of the irreducible components of $R_n(\partial N_K)$ containing $\alpha$.

In [We64], Weil observed that there is a natural inclusion of the Zariski tangent space $T^*_\alpha(R_\pi) \hookrightarrow Z^1(\pi; sl(n, \mathbb{C})_{ad})$ into the space of cocycles, and we will combine this observation with computations of the twisted cohomology of $\pi_1(\partial N_K)$ and $\pi_1(\partial N_K)$ to complete the proof.

Because $\partial N_K$ is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$, we have isomorphisms

$$H^*(\partial N_K; sl(n, \mathbb{C})_{ad}) \to H^*(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad})$$

and as explained in [HP05, Lemma 3.1], the inclusion $N_K \hookrightarrow K(\pi_1(N_K), 1)$ induces maps

$$H^i(\pi_1(N_K); sl(n, \mathbb{C})_{ad}) \to H^i(N_K; sl(n, \mathbb{C})_{ad})$$

that are isomorphisms when $i = 0$ and 1 and injective when $i = 2$. 

Next we shall prove a more general result which implies the first part of Theorem 2 as a special case. Before stating that result, we recall the following definition. A point $\xi \in X$ in an affine algebraic variety is called a simple point if it is contained in a unique algebraic component of $X$ and is a smooth point of that component.

**Theorem 17.** Suppose $\alpha: \pi_1(N_K) \to SL(n, \mathbb{C})$ is an irreducible representation with $\dim H_1(N_K; sl(n, \mathbb{C})_{ad}) = n-1$ such that the image of the restriction $\hat{\alpha}: \pi_1(\partial N_K) \to SL(n, \mathbb{C})$ contains a matrix with $n$ distinct eigenvalues. Then $\xi_\alpha$ is a simple point in the character variety $X_n(N_K)$.

**Proof.** As a first step, we will use the hypothesis to show that $\xi_\hat{\alpha}$ is a simple point in $X_n(\partial N_K)$, then we will apply the deformation obstruction machinery described in [HPS01] to conclude that $\xi_\alpha$ is a simple point of $X_n(N_K)$. By Luna’s étale slice theorem [Lu73], we see that $\xi_\alpha$ is a simple point of $X_n(N_K)$ if and only if $\alpha$ is a simple point of $R_n(N_K)$, and similarly for $\hat{\alpha}$ and $\xi_{\hat{\alpha}}$. For $\alpha$, this uses irreducibility, and for $\hat{\alpha}$, it follows by our hypothesis on the image of $\hat{\alpha}$.

Before beginning the argument, we introduce some notation. Given a finitely presented group $\pi$ and representation $\alpha: \pi \to SL(n, \mathbb{C})$, we use $H^*(\pi; sl(n, \mathbb{C})_{ad})$ to denote the cohomology of the group with coefficients in the $\pi$-module by $sl(n, \mathbb{C})_{ad}$.

In order to prove that $\hat{\alpha}$ is a simple point in $R_n(\partial N_K)$, we will compare the dimension of the cocycles $Z^1(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad})$ with the local dimension of $R_n(\partial N_K)$ at $\alpha$, which is defined to be the maximal dimension of the irreducible components of $R_n(\partial N_K)$ containing $\alpha$.

In [We64], Weil observed that there is a natural inclusion of the Zariski tangent space $T^*_\alpha(R_\pi) \hookrightarrow Z^1(\pi; sl(n, \mathbb{C})_{ad})$ into the space of cocycles, and we will combine this observation with computations of the twisted cohomology of $\pi_1(\partial N_K)$ and $\pi_1(\partial N_K)$ to complete the proof.

Because $\partial N_K$ is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$, we have isomorphisms

$$H^*(\partial N_K; sl(n, \mathbb{C})_{ad}) \to H^*(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad})$$

and as explained in [HP05, Lemma 3.1], the inclusion $N_K \hookrightarrow K(\pi_1(N_K), 1)$ induces maps

$$H^i(\pi_1(N_K); sl(n, \mathbb{C})_{ad}) \to H^i(N_K; sl(n, \mathbb{C})_{ad})$$

that are isomorphisms when $i = 0$ and 1 and injective when $i = 2$. 

Orthogonal. By Lemma 10 we now have

$$\dim H^1(N_K; sl(n, \mathbb{C})_{ad}) = \dim H_1(N_K; \mathbb{C}[\mathbb{Z}/n]) - 1 + \dim \mathbb{C}[\mathbb{Z}/n]$$

$$= b_1(L_n) + 1 - 1 + \dim \mathbb{C}[\mathbb{Z}/n]$$

$$\leq b_1(L_n) + n - 1,$$

The condition $\dim H^1(N_K; sl(n, \mathbb{C})_{ad}) = n - 1$ now shows that $b_1(L_n) = 0$. Thus $H_1(L_n) = H_1(N_K; \mathbb{Z}[t^{\pm 1}]/(t^n - 1)$ is finite, and this implies $\alpha$ has finite image and is conjugate to a unitary representation. □
We equip the torus $\partial N_K$ with the standard CW–decomposition consisting of one 0–cell, two 1–cells and one 2–cell. It is straightforward to verify that the spaces of twisted 1-coboundaries and 1-cocycles satisfy

$$\dim_{\mathbb{C}} B^1(\partial N_K; sl(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) = n^2 - 1 - (n - 1) = n^2 - n$$

and

$$\dim_{\mathbb{C}} Z^1(\partial N_K; sl(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) = 2(n - 1) + n^2 - n = n^2 + n - 2.$$

Now observe that $\hat{\alpha}$ sits on an $n^2 + n - 2$ dimensional component, hence its local dimension is

$$\dim_{\alpha} R_n(\partial N_K) = n^2 + n - 2.$$

For arbitrary $\alpha \in R_n(\partial N_K)$, we have

$$\dim_{\alpha} R_n(\partial N_K) \leq \dim_{\mathbb{C}} T^{	ext{zar}}_{\alpha}(R_n(N_K)) \leq \dim_{\mathbb{C}} Z^1(\partial N_K; sl(n, \mathbb{C})_{\text{ad} \hat{\alpha}}).$$

In our case, we have equality throughout, and it follows that $\hat{\alpha}$ lies on a unique irreducible component of $R_n(\partial N_K)$ and is a smooth point of that component (see [Sh95, §2, Theorem 6]). This completes the proof that $\alpha$ is a simple point of $R_\alpha(\partial N_K)$.

In order to prove that $\xi_\alpha$ is a simple point of $X_n(N_K)$, we consider again the long exact sequence (5) in cohomology associated with the pair $(N_K, \partial N_K)$ and note that irreducibility of $\alpha$ implies that $H^0(N_K; sl(n, \mathbb{C})_{\text{ad} \alpha}) = 0$. It follows from Lemmas 7, 8 and Poincaré duality that $H^3(N_K, \partial N_K; sl(n, \mathbb{C})_{\text{ad} \alpha}) = 0$. Further, by hypothesis we have $H^1(N_K; sl(n, \mathbb{C})_{\text{ad} \alpha}) = \mathbb{C}^{n-1}$, and so we also have $H^2(N_K, \partial N_K; sl(n, \mathbb{C})_{\text{ad} \alpha}) = \mathbb{C}^{n-1}$ by Poincaré duality.

Since $H^1(\partial N_K; sl(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) = \mathbb{C}^{2(n-1)}$, it follows immediately that the middle row of (5)

$$0 \rightarrow H^1(N_K; sl(n, \mathbb{C})_{\text{ad} \alpha}) \rightarrow H^1(\partial N_K; sl(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) \rightarrow H^2(N_K, \partial N_K; sl(n, \mathbb{C})_{\text{ad} \alpha}) \rightarrow 0$$

is short exact. Thus $j^1 = 0$ and $j^2 = 0$, and further $i^1$ is injective and $i^2$ is an isomorphism.

We now use a powerful method for deforming representations that involves constructing formal deformations by computing the vanishing of all obstructions. This technique was first developed in [HPS01] for $\text{SL}(2, \mathbb{C})$ and later extended to other situations in [HP05, AHJ07]. Here we explain how to carry over their arguments to the $\text{SL}(n, \mathbb{C})$ case, and we use that approach to prove integrability of all elements in $Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{\text{ad} \alpha})$ under the present circumstances.

We begin by describing the general framework of formal deformation theory and the related obstructions. Suppose $\pi$ is a finitely presented group and $\alpha: \pi \rightarrow \text{SL}(n, \mathbb{C})$ is a representation. A formal deformation of $\alpha$ is a homomorphism $\alpha_\infty: \pi \rightarrow \text{SL}(n, \mathbb{C}[[t]])$ given by

$$\alpha_\infty(g) = \exp \left( \sum_{i=1}^{\infty} t^i a_i(g) \right) \alpha(g),$$
such that \( p_0(\alpha_\infty) = \alpha \), where \( p_0 : \text{SL}(n, \mathbb{C}[[t]]) \to \text{SL}(n, \mathbb{C}) \) is the homomorphism given by setting \( t = 0 \) and where \( a_i : \pi \to \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}, i = 1, \ldots, \), are 1-cochains with twisted coefficients. By [HPS01, Lemma 3.3], it follows that \( \alpha_\infty \) is a homomorphism if and only if \( a_1 \in \mathbb{Z}^1(\pi; \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) \) is a cocycle, and we call an element \( a \in \mathbb{Z}^1(\pi; \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) \) formally integrable if there is a formal deformation with leading term \( a_1 = a \).

Let \( a_1, \ldots, a_k \in C^1(\pi; \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) \) be cochains such that

\[
\alpha_k(g) = \exp \left( \sum_{i=1}^k t^i a_i(g) \right) \alpha(g)
\]

is a homomorphism into \( \text{SL}(n, \mathbb{C}[[t]]) \) modulo \( t^{k+1} \). Here, \( \alpha_k \) is called a formal deformation of order \( k \), and in this case by [HPS01, Proposition 3.1] there exists an obstruction class \( \omega_{k+1} := \omega_{k+1}^{(a_1, \ldots, a_k)} \in H^2(\pi; \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) \) with the following properties:

1. There is a cochain \( a_{k+1} : \pi \to \text{sl}(n, \mathbb{C}) \) such that

\[
\alpha_{k+1}(g) = \exp \left( \sum_{i=1}^{k+1} t^i a_i(g) \right) \alpha(g)
\]

is a homomorphism modulo \( t^{k+2} \) if and only if \( \omega_{k+1} = 0 \).

2. The obstruction \( \omega_{k+1} \) is natural, i.e. if \( \varphi : \pi' \to \pi \) is a homomorphism then

\[
\varphi^* \omega_k := \alpha_k \circ \varphi \text{ is also a homomorphism modulo } t^{k+1} \text{ and } \varphi^*(\omega_{k+1}^{(a_1, \ldots, a_k)}) = \omega_{k+1}^{(\varphi^* a_1, \ldots, \varphi^* a_k)}.
\]

**Lemma 18.** Let \( \alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) be an irreducible representation such that \( \dim_{\mathbb{C}} H^1(N_K; \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) = n - 1 \). If the image of the restriction \( \tilde{\alpha} : \pi_1(\partial N_K) \to \text{SL}(n, \mathbb{C}) \) contains an element with \( n \) distinct eigenvalues, then every cocycle \( a \in \mathbb{Z}^1(\pi_1(N_K); \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) \) is integrable.

**Proof.** Consider first the commutative diagram:

\[
\begin{array}{ccc}
H^2(\pi_1(N_K); \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) & \xrightarrow{i^*} & H^2(\pi_1(\partial N_K); \text{sl}(n, \mathbb{C})_{\text{ad} \hat{\alpha}}) \\
\downarrow & & \downarrow \cong \\
H^2(N_K; \text{sl}(n, \mathbb{C})_{\text{ad} \alpha}) & \xrightarrow{\cong} & H^2(\partial N_K; \text{sl}(n, \mathbb{C})_{\text{ad} \hat{\alpha}}).
\end{array}
\]

Here, the horizontal isomorphism on the bottom follows by consideration of the long exact sequence (5), and the vertical isomorphism on the right follows since \( \partial N_K \) is a \( K(\mathbb{Z} \oplus \mathbb{Z}, 1) \). Further, we know the vertical map on the left is an injection by [HP05, Lemma 3.3], and this shows \( i^* \) is an injection.
We now prove that every element \( a \in Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{ad}) \) is integrable. Suppose (by induction) that \( a_1, \ldots, a_k \in C^1(\pi; sl(n, \mathbb{C})_{ad}) \) are given so that
\[
\alpha_k(g) = \exp \left( \sum_{i=1}^k t^i a_i(g) \right) \alpha(g)
\]
is a homomorphism modulo \( t^{k+1} \). Then the restriction \( \hat{\alpha}_k : \pi_1(\partial N_K) \to SL(n, \mathbb{C}[\{t\}]) \) is also a formal deformation of order \( k \). On the other hand, \( \hat{\alpha}_k \) is a smooth point of \( R_n(\partial N_K) \), hence by [HPS01, Lemma 3.6], \( \hat{\alpha}_k \) extends to a formal deformation of order \( k + 1 \). Therefore
\[
0 = \omega_{k+1}^{(a_1, \ldots, a_k)} = i^* \omega_{k+1}^{(a_1, \ldots, a_k)}.
\]
As \( i^* \) is injective, the obstruction vanishes. \( \square \)

We are now ready to conclude the proof of Theorem 17. Using Lemma 18, we see that all cocycles in \( Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{ad}) \) are integrable. Applying Artin’s theorem [Ar68], we obtain from a formal deformation of \( \alpha \) a convergent deformation (see [HPS01, Lemma 3.6]). Thus \( \alpha \) is a smooth point of \( R_n(N_K) \) with local dimension \( \dim \alpha R_n(N_K) = n - 1 \). It follows that \( \alpha \) is a simple point of \( R_n(N_K) \) and this together with irreducibility of \( \alpha \) imply that \( \xi_\alpha \) is a simple point of \( X_n(N_K) \). \( \square \)

The proof of Theorem 17 is easily adapted to the \( SU(n) \) setting to prove the following statement, where we use \( X_{SU(n)}(N) \) to denote the character variety of \( SU(n) \) representations of \( \pi_1(N) \).

**Theorem 19.** Suppose \( \alpha : \pi_1(N_K) \to SU(n) \) is an irreducible representation with \( \dim \mathbb{R} H^1(N_K; su(n)_{ad}) = n - 1 \) such that the image of \( \hat{\alpha} : \pi_1(\partial N_K) \to SU(n) \) contains a matrix with \( n \) distinct eigenvalues. Then \( \xi_\alpha \) is a simple point in the character variety \( X_{SU(n)}(N_K) \).

We now complete the proof of Theorem 2. If \( \alpha \) is an irreducible metabelian representation with \( \dim \mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad}) = n - 1 \), then Lemma 16 applies and shows that \( \alpha \) has finite image and hence is conjugate to a unitary representation. Since \( \alpha(\mu) \) has \( n \) distinct eigenvalues, Theorem 17 applies and gives rise to a smooth complex \( (n - 1) \)-dimensional family of \( SL(n, \mathbb{C}) \) characters near \( \xi_\alpha \in X_n(N_K) \).

Note that, since \( \alpha \) is unitary and \( H^1(N_K; sl(n, \mathbb{C})_{ad}) \) is the complexification of \( H^1(N_K; su(n)_{ad}) \), we see that
\[
\dim \mathbb{R} H^1(N_K; su(n)_{ad}) = \dim \mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad})
\]
Thus Theorem 19 applies and gives rise to a smooth real \( (n - 1) \)-dimensional family of irreducible characters near \( \xi_\alpha \in X_{SU(n)}(N_K) \).

Note that Lemma 16 shows that \( b_1(L_n) = 0 \), and thus every irreducible metabelian representation \( \beta : \pi_1(N_K) \to SL(n, \mathbb{C}) \) factors through a finite group. In particular, this shows that up to conjugacy there are only finitely many irreducible metabelian \( SL(n, \mathbb{C}) \) representations, and their characters give rise to a finite collection of points in
the character variety $X^*_\alpha(N_K)$. It follows that we can take either of the two deformation families of conjugacy classes of irreducible representations so that $\xi_\alpha$ is the unique metabelian representation within the family.

4.3. Proof of Theorem 3.

Proof of Theorem 3. Let $K$ be a knot, $\alpha: \pi_1(N_K) \to \text{SL}(n, \mathbb{C})$ an irreducible metabelian representation, and $\varphi: \pi_1(N_K) \to \mathbb{Z}/n \ltimes H$ a homomorphism such that $\alpha$ factors through $\varphi$ and with $H$ is finite. Set $k = |H|.$

We first consider the cover $p: \tilde{N}_\varphi \to N_K$ corresponding to $\varphi$. Note that there exist precisely $k = |H|$ characters $H \to U(1)$. We denote this set by $\{\sigma_1, \ldots, \sigma_k\}$, where we assume that $\sigma_1$ is the trivial character. It is not difficult to see that the representation $\sigma_1 + \cdots + \sigma_k: H \to \text{Aut}(\mathbb{C}^k)$ is isomorphic to the regular representation $H \to \text{Aut}(\mathbb{C}[H])$. We denote the representation $\pi_1(N_K) \to \text{Aut}(\mathbb{C}[\mathbb{Z}/n \ltimes H])$ by $\varphi$ as well. Then it is straightforward to verify that

$$\varphi \cong \bigoplus_{i=1}^k \beta_{(n,\sigma_i)}.$$ 

In particular, setting $V = \mathbb{C}^{kn}$ and $U = \mathbb{C}^n$, we have

$$b_1(\tilde{N}_\varphi) = b_1(N; V_\varphi) = \sum_{i=1}^k b_1(N; U_{\beta_{(n,\sigma_i)}}).$$

Note that $\beta_{(n,\sigma_i)}$ is a unitary representation. It now follows immediately from Lemma 10 that $b_1(\tilde{N}_\varphi) \geq k$. Furthermore, if $b_1(\tilde{N}_\varphi) = k$ then it follows that $b_1(N; U_{\beta_{(n,\sigma_i)}}) = 1$ for any $i$. Statement (i) now follows immediately from Lemma 6.

We now turn to the proof of (ii). We write $T = \partial N_K$. Note that the image of the restriction $\bar{\varphi}: \pi_1(T) \to \mathbb{Z}/n \ltimes H$ has order $n$. In particular the preimage of $T$ under the covering $p: \tilde{N}_\varphi \to N_K$ has $k = |H|$ components. We denote the components by $T_1, \ldots, T_k$. Note that in each $T_i$ there exist simple closed curves $\mu_i$ and $\lambda_i$ such that $p|_{\mu_i}$ restricts to an $n$–fold cover of the meridian $\mu \subset T$ and such that $p|_{\lambda_i}$ restricts to a homeomorphism with the longitude $\lambda$ of $T$. Note that $\mu_i, \lambda_i$ form a basis for $H_1(T_i)$.

We now denote by $\tilde{\Sigma}_\varphi$ the result of gluing $k$ solid tori $S_1, \ldots, S_k$ to the boundary of $\tilde{N}_\varphi$ such that each $\mu_i$ bounds a disk in $S_i$. The projection map $p: \tilde{N}_\varphi \to N_K$ now extends in a canonical way to a covering map $\tilde{\Sigma}_\varphi \to \Sigma$, branched over $K$.

We finally turn to the proof of (iii). Consider the following Mayer–Vietoris sequence:

$$\bigoplus_{i=1}^k H_1(T_i) \to \bigoplus_{i=1}^k H_1(S_i) \oplus H_1(\tilde{N}_\varphi) \to H_1(\tilde{\Sigma}_\varphi) \to 0.$$ 

It follows immediately that

$$b_1(\tilde{\Sigma}_\varphi) \geq k + b_1(\tilde{N}_\varphi) - 2k = b_1(\tilde{N}_\varphi) - k.$$
In particular if $b_1(\hat{\Sigma}_\phi) = 0$, then $b_1(\tilde{N}_\phi) \leq k$. We can therefore now apply (i) to deduce that (iii) holds.

\[
\square
\]

References


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