METABELIAN SL$(n,\mathbb{C})$ REPRESENTATIONS OF KNOT GROUPS II: FIXED POINTS

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Abstract. Given a knot K in an integral homology sphere Σ with exterior N_K, there is a natural action of the cyclic group $\mathbb{Z}/n$ on the space of $\mathrm{SL}(n,\mathbb{C})$ representations of the knot group $\pi_1(N_K)$, and this induces an action on the $\mathrm{SL}(n,\mathbb{C})$ character variety. We identify the fixed points of this action in terms of characters of metabelian representations, and we apply this to show that the twisted Alexander polynomial $\Delta^\alpha_{K,1}(t)$ associated to an irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representation α is actually a polynomial in t^n.

1. Introduction

Suppose K is a knot. Throughout this paper we will always understand this to mean that K is an oriented simple closed curve in an integral homology 3-sphere Σ. We write $N_K = \Sigma^3 \setminus \tau(K)$, where $\tau(K)$ denotes an open tubular neighborhood of K.

The study of metabelian representations and metabelian quotients of knot groups goes back to the pioneering work of Neuwirth [Ne65], de Rham [dRh68], Burde [Bu67] and Fox [Fo70] (see also [BZ03, Section 14]). The theory was further developed by many authors, including Hartley [Ha79, Ha83], Livingston [Li95], Letsche [Le00], Lin [Lin01], Nagasato [Na07] and Jebali [Je08]. In [BF08] we proved a classification theorem for irreducible metabelian representations, and in this paper we continue our study of metabelian representations of knot groups.

We begin by introducing some terminology. Given a topological space M, let $R_n(M)$ be the space of $\mathrm{SL}(n,\mathbb{C})$ representations of $\pi_1(M)$ and $X_n(M)$ the associated character variety. We use ξ_α to denote the character of the representation $\alpha\colon \pi_1(M) \to \mathrm{SL}(n,\mathbb{C})$. We will often make use of the important fact that two irreducible representations determine the same character if and only if they are conjugate (see [LM85, Corollary 1.33]).

Now suppose K is a knot. There is an action of the group $\mathbb{Z}/n$ on the representation variety $R_n(N_K)$ given by twisting by the n–th roots of unity $\omega^k = e^{2\pi ik/n} \in U(1)$.

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(This is a special case of the more general twisting operation described in [LM85, Ch. 5].) More precisely, we write $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1\rangle$ and set $(\sigma \cdot \alpha)(g) = \omega^{\varepsilon(g)}\alpha(g)$ for each $g \in \pi_1(N_K)$, where $\varepsilon\colon \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$ is determined by the given orientation of the knot.

This constructs an action of $\mathbb{Z}/n$ on $R_n(N_K)$ which, in turn, descends to an action on the character variety $X_n(N_K)$. Our main result identifies the fixed points of $\mathbb{Z}/n$ in $X_n^*(N_K)$, the irreducible characters, as those associated to metabelian representations.

Theorem 1. *The character ξ_α of an irreducible representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is fixed under the $\mathbb{Z}/n$ action if and only if α is metabelian.*

In proving this result, we actually characterize the entire fixed point set $X_n(N_K)^{\mathbb{Z}/n}$ in terms of characters ξ_α of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Subsection 2.3 (see Theorem 4). When $n = 2$, it turns out that every metabelian $\mathrm{SL}(2, \mathbb{C})$ representation is dihedral and in this case Theorem 1 was first proved by F. Nagasato and Y. Yamaguchi (cf. [NY08, Proposition 4.8]).

As an application of Theorem 1, we prove a result about the twisted Alexander polynomials associated to metabelian representations. This result was first shown by C. Herald, P. Kirk and C. Livingston in [HKL08] using completely different methods. Our approach is elementary and quite natural, and it is explained in Section 3.2, where we apply it to give an answer to a question raised by Hirasawa and Murasugi in [HM09].

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2. The classification of metabelian representations of knot groups

In this section we recall some results from [BF08] regarding the classification of metabelian representations of knot groups.

2.1. **Preliminaries.** Given a group π, we shall write $\pi^{(n)}$ for the n–th term of the derived series of π. These subgroups are defined inductively by setting $\pi^{(0)} = \pi$ and $\pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}]$. The group π is called *metabelian* if $\pi^{(2)} = \{e\}$.

Suppose V is a finite dimensional vector space over $\mathbb{C}$. A representation $\varrho\colon \pi \to \mathrm{Aut}(V)$ is called *metabelian* if ϱ factors through $\pi/\pi^{(2)}$. The representation ϱ is called *reducible* if there exists a proper subspace $U \subset V$ invariant under $\varrho(\gamma)$ for all $\gamma \in \pi$. Otherwise ϱ is called *irreducible* or *simple*. If ϱ is the direct sum of simple representations, then ϱ is called *semisimple*.

Two representations $\varrho_1\colon \pi \to \operatorname{Aut}(V)$ and $\varrho_2\colon \pi \to \operatorname{Aut}(W)$ are called *isomorphic* if there exists an isomorphism $\phi\colon V \to W$ such that $\phi^{-1} \circ \varrho_1(g) \circ \phi = \varrho_2(g)$ for all $g \in \pi$.

2.2. **Metabelian quotients of knot groups.** Let $K \subset \Sigma^3$ be a knot in an integral homology 3-sphere. In the following we denote by $\widetilde{N}_K$ the infinite cyclic cover of N_K corresponding to the abelianization $\pi_1(N_K) \to H_1(N_K) \cong \mathbb{Z}$. Therefore $\pi_1(\widetilde{N}_K) = \pi_1(N_K)^{(1)}$ and

$$H_1(N_K; \mathbb{Z}[t^{\pm 1}]) = H_1(\widetilde{N}_K) \cong \pi_1(N_K)^{(1)}/\pi_1(N_K)^{(2)}.$$

The $\mathbb{Z}[t^{\pm 1}]$–module structure is given on the right hand side by $t^n \cdot g := \mu^{-n} g \mu^n$, where μ is a meridian of K.

For a knot K, we set $\pi := \pi_1(N_K)$ and consider the short exact sequence

$$1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1.$$

Since $\pi/\pi^{(1)} = H_1(N_K) \cong \mathbb{Z}$, this sequence splits and we get isomorphisms

$$\begin{array}{ccccc} \pi/\pi^{(2)} & \cong & \pi/\pi^{(1)} \ltimes \pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \\ g & \mapsto & (\mu^{\varepsilon(g)}, \mu^{-\varepsilon(g)} g) & \mapsto & (\varepsilon(g), \mu^{-\varepsilon(g)} g), \end{array}$$

where the semidirect products are taken with respect to the $\mathbb{Z}$ actions defined by letting $n \in \mathbb{Z}$ act by conjugation by μ^n on $\pi^{(1)}/\pi^{(2)}$ and by multiplication by t^n on $H_1(N_K; \mathbb{Z}[t^{\pm 1}])$.

2.3. **Irreducible metabelian** SL$(n,\mathbb{C})$ **representations of knot groups.** Let K be a knot. We write $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. The discussion of the previous section shows that irreducible metabelian SL$(n,\mathbb{C})$ representations of $\pi_1(N_K)$ correspond precisely to the irreducible SL$(n,\mathbb{C})$ representations of $\mathbb{Z} \ltimes H$.

Let $\chi\colon H \to \mathbb{C}^*$ be a character which factors through $H/(t^n - 1)$ and suppose $z \in S^1$ with $z^n = (-1)^{n+1}$. Then it follows from [BF08, Section 3] that, for $(j,h) \in \mathbb{Z} \ltimes H$, setting

$$\alpha_{(\chi,z)}(j,h) = \begin{pmatrix} 0 & & \dots & z \\ z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{n-1}h) \end{pmatrix}$$

defines an SL$(n,\mathbb{C})$ representation whose isomorphism type of this representation does not depend on the choice of z. In our notation we will not normally distinguish between metabelian representations of $\pi_1(N_K)$ and representations of $\mathbb{Z} \ltimes H$.

In the following we say that a character $\chi\colon H \to \mathbb{C}^*$ has *order* n if it factors through $H/(t^n - 1)$, but not through $H/(t^\ell - 1)$ for any $\ell < n$. Given a character $\chi\colon H \to \mathbb{C}^*$, let $t^i\chi$ be the character defined by $(t^i\chi)(h) = \chi(t^i h)$. Any character $\chi\colon H \to \mathbb{C}^*$ which factors through $H/(t^n - 1)$ must have order k for some divisor k of n. The following is a combination of [BF08, Lemma 2.2] and [BF08, Theorem 3.3].

Theorem 2. *Suppose $\chi\colon H \to \mathbb{C}^*$ is a character that factors through $H/(t^n-1)$.*

(i) *$\alpha_{(n,\chi)}\colon \mathbb{Z} \ltimes H \to \mathrm{SL}(n,\mathbb{C})$ is irreducible if and only if the character χ has order n.*
(ii) *Given two characters $\chi, \chi'\colon H \to \mathbb{C}^*$ of order n, the representations $\alpha_{(n,\chi)}$ and $\alpha_{(n,\chi')}$ are conjugate if and only if $\chi = t^k\chi'$ for some k.*
(iii) *For any irreducible representation $\alpha\colon \mathbb{Z} \ltimes H \to \mathrm{SL}(n,\mathbb{C})$ there exists a character $\chi\colon H \to \mathbb{C}^*$ of order n such that α is conjugate to $\alpha_{(n,\chi)}$.*

3. Main results

3.1. **Metabelian characters as fixed points.** Set $\omega = e^{2\pi i/n}$ and recall the action of the cyclic group $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1\rangle$ on representations $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ obtained by setting $(\sigma\cdot\alpha)(g) = \omega^{\varepsilon(g)}\alpha(g)$ for all $g \in \pi_1(N_K)$, where $\varepsilon\colon \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$.

We begin with the following lemma.

Lemma 3. *Suppose $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is a representation whose associated character $\xi_\alpha \in X_n(N_K)$ is a fixed point of the $\mathbb{Z}/n$ action. Then up to conjugation, we have*

$$\alpha(\mu) = \begin{pmatrix} 0 & & \dots & z \\ z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & 0 \end{pmatrix}, \tag{1}$$

for some (in fact any) $z \in U(1)$ such that $z^n = (-1)^{n+1}$.

Proof. Let $c(t) = \det(\alpha(\mu) - tI)$ denote the characteristic polynomial of $\alpha(\mu)$, which we can write as

$$c(t) = (-1)^n t^n + c_{n-1}t^{n-1} + \dots + c_1 t + 1.$$

Note that $c(t)$ is determined by the character $\xi_\alpha \in X_n(N_K)$, and so assuming ξ_α is a fixed point of the $\mathbb{Z}/n$ action, we conclude that $\alpha(\mu)$ and $\omega^k\alpha(\mu)$ have the same characteristic polynomials for all k. In particular,

$$\begin{aligned} c(t) &= \det(\alpha(\mu) - tI) \\ &= \det(\omega^{-1}\alpha(\mu) - tI) \\ &= \det(\omega^{-1}\alpha(\mu) - (\omega^{-1}\omega)tI) \\ &= \det(\omega^{-1}I)\det(\alpha(\mu) - \omega tI) \\ &= \det(\alpha(\mu) - t\omega I) = c(\omega t). \end{aligned}$$

However, $\omega^k \neq 1$ unless $n|k$, and this implies $0 = c_{n-1} = c_{n-2} = \dots = c_1$ and $c(t) = (-1)^n t^n + 1$. In particular the matrix $\alpha(\mu)$ and the matrix appearing in Equation (2) have the same set of n distinct eigenvalues. This implies that the two matrices are conjugate. $\square$

In order to prove Theorem 1, we establish the following more general result.

Theorem 4. *The fixed point set of the $\mathbb{Z}/n$ action on $X_n(N_K)$ consists of characters ξ_α of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Section 2.3. In other words,*

$$X_n(N_K)^{\mathbb{Z}/n} = \{\xi_\alpha \mid \alpha = \alpha_{(n,\chi)} \text{ for } \chi\colon\ H_1(N_K;\mathbb{Z}[t^{\pm1}]) \to \mathbb{C}^*\}.$$

Notice that Theorem 1 can be viewed as the special case of Theorem 4 where α is irreducible. (Recall that irreducible representations are conjugate if and only if they define the same character.) Notice further that not every reducible metabelian representation is of the form $\alpha_{(n,\chi)}$.

Proof. We first show that if $\alpha\colon\ \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is given as $\alpha = \alpha_{(n,\chi)}$, then $\sigma \cdot \alpha$ is conjugate to α. This of course implies that $\xi_\alpha = \xi_{\sigma\cdot\alpha}$.

Assume then that $\alpha = \alpha_{(n,\chi)}$. Then we have

$$\alpha(\mu) = \begin{pmatrix} 0 & \dots & & z \\ z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & 0 \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$. Further, $\alpha(g)$ is diagonal for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. By definition of $\sigma \cdot \alpha$, we see that

$$(\sigma \cdot \alpha)(\mu) = \omega\alpha(\mu) = \begin{pmatrix} 0 & \dots & & \omega z \\ \omega z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \omega z & 0 \end{pmatrix}$$

and that $(\sigma\cdot\alpha)(g) = \alpha(g)$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. It follows easily from Theorem 2 (2) that $\sigma \cdot \alpha$ and $\alpha_{(n,\chi)}$ are conjugate; however it is easy to see this directly too. Simply take

$$P = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \end{pmatrix},$$

and compute that $\sigma \cdot \alpha = P\alpha P^{-1}$ as claimed.

We now show the other implication, namely that each point $\xi \in X_n(N_K)^{\mathbb{Z}/n}$ in the fixed point set can be represented as the character $\xi = \xi_\alpha$ of a metabelian representation $\alpha = \alpha_{(n,\chi)}$, where $\chi\colon\ H_1(N_K;\mathbb{Z}[t^{\pm1}]) \to \mathbb{C}^*$ is a character that factors through $H_1(N_K;\mathbb{Z}[t^{\pm1}])/(t^n - 1)$, hence has order k for some k dividing n. (Note that Theorem 2 (1) tells us that $\alpha_{(n,\chi)}$ is irreducible if and only if χ has order n.)

By the general results on representation spaces and character varieties (see [LM85]), it follows that every point in the character variety $X_n(N_K)$ can be represented as ξ_α

for some semisimple representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$. Further, two semisimple representations α_1 and α_2 determine the same character if and only if α_1 is conjugate to α_2. (This is evident from the fact that the orbits of the semisimple representations under conjugation are closed.)

Given $\xi \in X_n(N_K)^{\mathbb{Z}/n}$, we can therefore suppose that $\xi = \xi_\alpha$ for some semisimple representation α. Clearly $\sigma \cdot \alpha$ is also semisimple, and since $\xi_\alpha = \xi_{\sigma\cdot\alpha}$, we conclude from the above that α and $\sigma \cdot \alpha$ are conjugate representations. This means that there exists a matrix $A \in \mathrm{SL}(n, \mathbb{C})$ such that $A\alpha A^{-1} = \sigma \cdot \alpha$, in other words, for all $g \in \pi_1(N_K)$, we have

$$A\alpha(g)A^{-1} = \omega^{\varepsilon(g)}\alpha(g). \tag{2}$$

Lemma 3 implies $\alpha(\mu)$ is conjugate to the matrix in Equation (2). It is convenient to conjugate α so that $\alpha(\mu)$ is diagonal, meaning that

$$\alpha(\mu) = \begin{pmatrix} z & & & 0 \\ & \omega z & & \\ & & \ddots & \\ 0 & & & \omega^{n-1}z \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$.

We now apply (2) to the meridian to conclude that

$$A\alpha(\mu) = \omega\alpha(\mu)A,$$

which implies $A = (a_{ij})$ satisfies $a_{ij} = 0$ unless $j = i + 1 \mod (n)$. Thus, we see that

$$A = \begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{n-1} \\ \lambda_n & 0 & \dots & 0 & 0 \end{pmatrix}$$

for some $\lambda_1, \dots, \lambda_n$ satisfying $\lambda_1 \cdots \lambda_n = (-1)^{n+1}$.

It is completely straightforward to see that the characteristic polynomial of A is given by

$$\det(A - tI) = (-1)^n(t^n - (-1)^{n+1}).$$

From this, we conclude that A has as its eigenvalues the n distinct n–th roots of $(-1)^{n+1}$. In particular, the subset of $\mathrm{SL}(n, \mathbb{C})$ of matrices that commute with A is just a copy of the unique maximal torus $T_A \cong (\mathbb{C}^*)^{n-1}$ containing A.

For any $g \in [\pi_1(N_K), \pi_1(N_K)]$, we have $\alpha(g) = (\sigma \cdot \alpha)(g)$. Thus it follows that $A\alpha(g)A^{-1} = \alpha(g)$, and this implies that $\alpha(g) \in T_A$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. This shows that the restriction of α to the commutator subgroup $[\pi_1(N_K), \pi_1(N_K)]$ is abelian, and we conclude from this that α is indeed metabelian. Notice that this, and an application of Theorem 2 (3), completes the proof in the case α is irreducible.

In the general case, it follows from the discussion in Section 2.2 that α factors through $\mathbb{Z} \ltimes H_1(N_K;\mathbb{Z}[t^{\pm 1}])$. Let $H = H_1(N_K;\mathbb{Z}[t^{\pm 1}])$. Given a character $\chi\colon H \to \mathbb{C}^*$ we define the associated weight space V_χ by setting

$$V_\chi = \{v \in \mathbb{C}^n \mid \chi(h)\cdot v = \alpha(h)v \text{ for all } h \in H\}.$$

Recall that $A \cdot \alpha(h) \cdot A^{-1} = \alpha(h)$ for any $h \in H$. It is straightforward so show that A restricts to an automorphism of V_χ. Since H is abelian there exists at least one character $\chi\colon H \to \mathbb{C}^*$ such that V_χ is non–trivial. For any i we denote by $t^i\chi$ the character given by $(t^i\chi)(h) = \chi(t^i h), h \in H$.

Note that A has n distinct eigenvalues and therefore is diagonalizable. Since A restricts to an automorphism of V_χ, there is an eigenvector v of A which lies in V_χ. Let λ be the corresponding eigenvalue. By the proof of [BF08, Theorem 2.3], the map $\alpha(\mu)$ induces an isomorphism $V_\chi \to V_{t\chi}$. We now calculate

$$A \cdot \alpha(\mu)v = (A\alpha(\mu)A^{-1}) \cdot Av = \omega\alpha(\mu) \cdot \lambda v = \lambda\omega \cdot \alpha(\mu)v,$$

i.e. $\alpha(\mu)v \in V_{t\chi}$ is an eigenvector of A with eigenvalue $\omega\lambda$.

Iterating this argument, we see that $\alpha(\mu)^i v$ lies in $V_{t^i\chi}$ and is an eigenvector of A with eigenvalue $\omega^i\lambda$. Since ω is a primitive n–th root of unity, the eigenvalues $\lambda, \omega\lambda, \dots, \omega^{n-1}\lambda$ are all distinct, and this implies that the corresponding eigenvectors $v, \alpha(\mu)v, \dots, \alpha(\mu)^{n-1}v$ form a basis for $\mathbb{C}^n$.

Let m be the order of χ, i.e. m is the minimal number such that $\chi = t^m\chi$. By the above we see that $\mathbb{C}^n$ is generated by $V_\chi, V_{t\chi}, \dots, V_{t^m\chi}$. Since the characters $\chi, t\chi, \dots, t^m\chi$ are pairwise distinct, it follows that $\mathbb{C}^n$ is given as the direct sum $V_\chi \oplus V_{t\chi} \oplus \cdots \oplus V_{t^{m-1}\chi}$.

We write $k = \dim_{\mathbb{C}}(V_\chi)$ and note that $n = km$. We note further that $\alpha(\mu)^m$ has eigenvalues given by the set

$$\{z^m, z^m e^{2\pi i/k}, \dots, z^m e^{2\pi i(k-1)/k}\}, \tag{3}$$

and each eigenvalue has multiplicity m. Clearly $\alpha(\mu)^m$ restricts to an automorphism of $V_{t^i\chi}$ for $i = 0, \dots, m-1$, and equally clearly we see that the restrictions all give conjugate representations. This implies that the restriction of $\alpha(\mu)^m$ to V_χ has eigenvalues in the set (3) above, each occurring with multiplicity 1. In particular we can find a basis $\{v_1, \dots, v_k\}$ for V_χ in which the matrix of $\alpha(\mu)^m$ has the form

$$\alpha(\mu^m) = \begin{pmatrix} 0 & \dots & & z^m \\ z^m & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z^m & 0 \end{pmatrix}.$$

It is now straightforward to verify that with respect to the ordered basis

$$\left\{\begin{array}{llll} v_1, & z^{-1}\alpha(\mu)v_1, & \dots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_1, \\ v_2, & z^{-1}\alpha(\mu)v_2, & \dots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_2, \\ \vdots & \vdots & \dots & \vdots \\ v_k, & z^{-1}\alpha(\mu)w_k, & \dots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_k \end{array}\right\},$$

α is given by $\alpha(n,\chi)$. □

3.2. **Application to twisted Alexander polynomials.** As an application, we now prove the following result regarding twisted Alexander polynomials of knots corresponding to metabelian representations. In the following, we use $\Delta^\alpha_{K,i}(t)$ to denote the i–th twisted Alexander polynomial for a given representation $\alpha\colon \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$ as presented in [FV09].

Proposition 5. *Let α be a metabelian representation of the form $\alpha=\alpha_{(n,\chi)}\colon \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$. Then*

$$\Delta^\alpha_{K,0}(t)=\begin{cases} 1-t^n, & \text{if } \chi \text{ is trivial,} \\ 1, & \text{otherwise.}\end{cases}$$

Furthermore the twisted Alexander polynomial $\Delta^\alpha_{K,1}(t)$ is actually a polynomial in t^n.

Remark 6. In their paper [HKL08], C. Herald, P. Kirk, and C. Livingston prove the same result using an entirely different approach (cf. p. 10 of [HKL08]). We also point out that Proposition 5 gives a positive answer to Conjecture A from a recent paper by M. Hirasawa and K. Murasugi (see [HM09]).

Proof. The proof of the first statement is not difficult. It is immediate when χ is trivial, and it follows by a direct calculation when χ is non–trivial.

We now turn to the proof of the second statement. For $\theta\in U(1)$ and any representation $\beta:\pi_1(N_K)\to \mathrm{GL}(n,\mathbb{C})$, define the θ-twist of β to be the representation sending $g\in\pi_1(N_K)$ to $\theta^{\varepsilon(g)}\beta(g)$, where $\varepsilon\colon \pi_1(N_K)\to\mathbb{Z}$ is determined by the orientation of K. We denote the newly obtained representation by $\beta_\theta\colon \pi_1(N_K)\to \mathrm{GL}(n,\mathbb{C})$. Note that in case $\alpha:\pi_1(N_K)\to\mathrm{SL}(n,\mathbb{C})$ and $\theta=e^{2\pi ik/n}$ is an n-th root of unity, α_θ is again an $\mathrm{SL}(n,\mathbb{C})$ representation. The proof of the proposition relies on the formula

$$\Delta^{\beta_\theta}_{K,1}(t)=\Delta^{\beta}_{K,1}(\theta t). \tag{4}$$

This formula is well-known and follows directly from the definition of the twisted Alexander polynomial. Equation (4) combines with Theorem 1 to complete the proof, as we now explain. Take $\omega=e^{2\pi i/n}$. If $\alpha=\alpha_{(n,\chi)}$ is metabelian, then Theorem 1 shows that its conjugacy class is fixed under the $\mathbb{Z}/n$ action. In particular, since α and α_ω are conjugate, Equation (4) shows that

$$\Delta^\alpha_{K,1}(t)=\Delta^{\alpha_\omega}_{K,1}(t)=\Delta^\alpha_{K,1}(\omega t).$$

Expanding $\Delta^{\alpha}_{K,1}(t) = \sum a_i t^i$ and using the fact that $t^k = (\omega t)^k$ if and only if k is a multiple of n, this shows that $a_k = 0$ unless k is a multiple of n and this completes the proof. □

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