ALEXANDER POLYNOMIALS OF PLANE ALGEBRAIC CURVES

Abstract. In this note we use Reidemeister torsion to study the relationship between the Alexander invariants of a curve complement and the Alexander invariants of the singularities. In particular we give a single approach to study the results of Cogolludo–Florens and Maxim–Leidy. We extend the results of Cogolludo–Florens to the multivariable case and we extend the results of Maxim–Leidy to the twisted case. The extensions in this note are very modest.

Caveat and Warning

(1) The results in this note should be taken with a grain of salt. I am not an algebraic geometer.
(2) This note is just the fruit of my attempts at understanding the work of Cogolludo–Florens and Leidy–Maxim. It's not at all clear whether my generalizations are of any use at all.
(3) Some of the material (especially Section 4) might also be considered algebraic overkill.

1. Introduction

Let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. Let $\mathbb{B}^4 \subset \mathbb{C}^2$ be a sufficiently large closed ball, in the sense that $\text{int}(\mathbb{B}^4) \setminus (\mathcal{C} \cap \text{int}(\mathbb{B}^4))$ is diffeomorphic to $\mathbb{C}^2 \setminus \mathcal{C}$. Such a ball exists by [Di92, Theorem 1.6.9]. Note that in particular all singularities of $\mathcal{C}$ lie in the interior of $\mathbb{B}^4$.

We denote $\text{int}(\mathbb{B})^4 \cap \mathcal{C}$ respectively $\mathbb{B}^4 \cap \mathcal{C}_i$ by $\mathcal{C}$ respectively by $\mathcal{C}_i$ again. Now let $X(\mathcal{C}) := \mathbb{B}^4 \setminus \nu\mathcal{C}$. We denote the meridians of $\mathcal{C}_1, \dots, \mathcal{C}_r$ by $\mu_1, \dots, \mu_r$.

The fundamental group $\pi_1(X(\mathcal{C}))$ can be in theory computed using a well–known approach of Zariski and van Kampen, we refer to [Di92, p. 127] for details. In practice though this algorithm is difficult to implement. It is therefore useful to get partial information on $\pi_1(X(\mathcal{C}))$ from more computable invariants.

This fundamental group is in general non–abelian, we refer [Di92, p. 129] for the simplest curves with non–abelian fundamental group which were already known to Zariski [Za29]. Non–abelian groups are in general very difficult to study. It is therefore useful to study abelian invariants extracted from the groups.

The one–variable Alexander polynomials of plane algebraic curves and their singularities were intensively studied by Libgober [Li82]. In particular he showed the following. Let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve and $H_1(X(\mathcal{C}) \to \mathbb{Z}$ the homomorphism

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given by sending each meridian to $1 \in \mathbb{Z}$. Denote the corresponding Alexander polynomial by $\Delta(X(\mathcal{C}))$. Then the homology of the r–fold cover of $\mathbb{C}^2$ branched along $\mathcal{C}$ is infinite if and only if an r–th root of unity is a zero of $\Delta(X(\mathcal{C}))$ (cf. [Li82, p. 840] and [Su74]).

Given a point $P = (x_P, y_P) \in \mathbb{C}^2$ and $\epsilon > 0$ we write $\mathbb{B}^4(P, \epsilon) = \{(x, y) \in \mathbb{C}^2 | |x - x_P|^2 + |y - y_P|^2 \leq \epsilon^2\}$ and $S^3(P, \epsilon) = \partial\mathbb{B}^4(P, \epsilon)$.

Now let $\text{Sing}(\mathcal{C}) := \{P_1, \dots, P_s\} \subset \mathbb{B}^4$ denote the set of singularities of $\mathcal{C}$. Then there exist $\epsilon_1, \dots, \epsilon_s > 0$ such that

(1) $\mathbb{B}^4(P_i, \epsilon_i)$ are pairwise disjoint,
(2) $\mathbb{B}^4(P_i, \epsilon_i) \subset \text{int}(\mathbb{B}^4)$,
(3) $\mathbb{B}^4(P_i, \epsilon_i) \setminus \mathcal{C} \cap \mathbb{B}^4(P_i, \epsilon_i)$ is the cone on $S^3(P_i, \epsilon_i) \setminus \mathcal{C} \cap S^3(P_i, \epsilon_i)$.

Such ϵ_i exist by Thom's first isotopy lemma (cf. [Di92, Section 5] for details). Furthermore let $S^3_\infty = \partial\mathbb{B}^4$. Let $L_i := S^3_i \cap \mathcal{C}$ and write $X(L_i) := S^3_i \setminus \nu L_i$ for $i = 1, \dots, s, \infty$.

Now let L be the link at a singularity or the L 'at infinity'. It is a well–known result that the links $L_i, i = 1, \dots, s, \infty$ are iterated torus links (cf. e.g. [Di92, Proposition 2.2.6]). If L is the link at a singularity P, then the number of components of L can be read off from the Puiseux expansion. It also equals the number of irreducible components of the analytic germ of $(X(\mathcal{C}), P)$ (cf. [Di92, p. 103]).

Burau [Bu32] showed that the link at the singularity is determined by its one–variable Alexander polynomial if the link has only one component. If L has more than one singularity, then it is in general not determined by its one–variable Alexander polynomial (cf. [Di92, p. 44]).

We also point out that Loeser and Vaquié [LV90] gave an explicit formula for the on–variable Alexander polynomial $\Delta(X(\mathcal{C}))$ in terms of the number of irreducible components, the degree of $\mathcal{C}$ and the type and location of the singularities. In particular they give examples which showed that the Alexander polynomial of $X(\mathcal{C})$ is can not determined just from the number of irreducible components, the degree of $\mathcal{C}$ and the type (but *not* the location) of the singularities.

Libgober [Li82] first showed that $\Delta(X(\mathcal{C})) \in \mathbb{Z}[t^{\pm 1}]$ divides the product of the Alexander polynomials of the singularities. This result was refined by Degtyarev [De04, p. 205] who showed one restrict oneself to a certain subset of the singularities. Another refinement is given by [CF05, Corollary 1.2] which shows that in some sense, up to a minor extra term, $\Delta(X(\mathcal{C}))$ divides the product of the Alexander polynomial of the link at infinity and the product of the Alexander polynomials of the singularities twice. Furthermore in [CF05, Theorem 1.1] this is extended to twisted one–variable Alexander polynomials.

The multivariable Alexander polynomial of a link and a curve complement is a finer invariant. For example the link type of algebraic links are completely determined by their multivariable Alexander polynomials (cf. [Ya84, Theorem A] for links and [Bu34] for links with 2 components) (cf. also [Di92, p. 44]). The multivariable

Alexander polynomial of an algebraic link can be computed explicitly given a description as an iterated torus link (cf. [EN85, Theorem 12.1]). Note that any such multivariable Alexander polynomial is a product of terms $(t_1^{n_1} \cdot \dots \cdot t_r^{n_r} - 1)^k, k \in \mathbb{Z}$.

We now state our two main results. Let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. We denote the meridians of $\mathcal{C}_1, \dots, \mathcal{C}_r$ in $X(\mathcal{C})$ by $\mu_1, \dots, \mu_r$. Let $\psi : H_1(X(\mathcal{C}); \mathbb{Z}) \to \mathbb{Z}^m$ be an epimorphism. Furthermore let $\mathrm{Sing}(\mathcal{C}) := \{P_1, \dots, P_s\} \subset \mathbb{B}^4$ denote the set of singularities of $\mathcal{C}$. Let L_i and $X(L_i) := S_i^3 \setminus \nu L_i$ for $i = 1, \dots, s, \infty$ as above.

The following extends results of Cogolludo–Florens to the multivariable case (Corollary 7.2).

Theorem 1.1. *Let R be a commutative UFD. Let $\Lambda = R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be the multivariable polynomial ring of rank m. Furthermore let $\alpha : \pi_1(X(\mathcal{C})) \to GL(R[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ be a unitary ψ–compatible representation. Assume that $m \geq 2$ and that ψ is an epimorphism. Then*

$$\Delta_1^\alpha(X(\mathcal{C})) \cdot \overline{\Delta_1^\alpha(X(\mathcal{C}))} \in R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$$

divides

$$\prod_{i=1}^{r} \det(id - \alpha(\mu_i))^{\max\{0, s_i - \chi(\mathcal{C}_i)\}} \prod_{i \in \{1, \dots, s, \infty\}} \Delta_1^\alpha(X(L_i)) \in R[t_1^{\pm 1}, \dots, t_m^{\pm 1}].$$

The following theorem is Theorem 6.3 in the text. It can be viewed as a generalization of the results of Leidy–Maxim to the twisted case (cf. also Theorem 6.4 for a variation on their degree formula).

Theorem 1.2. *Let $\mathbb{K}[t^{\pm 1}]$ be a skew Laurent polynomial ring with skew quotient field $\mathbb{K}(t)$. Let $\alpha : \pi_1(X(\mathcal{C})) \to GL(\mathbb{K}[t^{\pm 1}], d)$ be a unitary ψ–compatible representation. Let $\mathcal{B}$ be any $\mathbb{K}[t^{\pm 1}]$–basis for $H_2(X(\mathcal{C}); \mathbb{K}(t))$. Assume that $H_1(X(L_i); \mathbb{K}(t)) = 0$ for $i = 1, \dots, s, \infty$. Then we have the following equality in $K_1(\mathbb{K}(t))/K_1(\mathbb{K}[t^{\pm 1}])$:*

$$\lambda \cdot \prod_{i \in \{1, \dots, s, \infty\}} \Delta^\alpha(X(L_i)) = \Delta^\alpha(X(\mathcal{C})) \cdot \overline{\Delta^\alpha(X(\mathcal{C}))} \cdot I(\mathcal{B}),$$

where

$$\lambda := \prod_{i=1}^{r} (id - \alpha(\mu_i))^{s_i - \chi(\mathcal{C}_i)}$$

and $s_i := \#Sing(\mathcal{C}) \cap \mathcal{C}_i$.

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Notations and conventions: All homology groups and all cohomology groups are with respect to $\mathbb{Z}$–coefficients, unless it specifically says otherwise. For a link L

in S^3, $X(L)$ denotes the exterior of L in S^3. (That is, $X(L) = S^3 \setminus \nu L$ where νL is an open tubular neighborhood of L in S^3). An arbitrary (commutative) field is denoted by $\mathbb{F}$. For a ring R we denote by $R^\times$ the units of R.

2. Plane algebraic curves and their topology

The following summarizes some well–known results.

Theorem 2.1. (1) *$\pi_1(X)$ is normally generated by the meridians of the irreducible components and $H_1(X)$ is a free abelian group of rank r with basis given by the meridians of the irreducible components.*
(2) *X is homotopy equivalent to a 2–complex.*
(3) *If $\mathcal{C}$ intersects the line at infinity transversely, then $\pi_1(\partial X) \to \pi_1(X)$ is surjective.*

Proof. The first statement follows from the fact that by gluing in disks at the meridians we kill the fundamental group. The statement about the first homology group follows from Lefschetz duality (cf. [Li82, p. 835]). For the second statement we refer to [Li86] for a proof (cf. also [Di92, Theorem 1.6.8]. The last statement follows from applying the Lefschetz hyperplane theorem (cf. e.g. [Di92, p. 25]) to a hyperplane 'close to infinity' (cf. also [LM05, p. 9]). □

Let M be a 3–manifold and $\phi : H_1(M;\mathbb{Z}) \to \mathbb{Z}$ a homomorphism. We say that (M, ϕ) fibers over S^1 if the homotopy class of maps $M \to S^1$ determined by $\phi \in H^1(M;\mathbb{Z}) = [M, S^1]$ contains a representative that is a fiber bundle over S^1. Milnor [Mi68, Theorem 4.8] showed that for $i = 1, \dots, s, \infty$ $(X(L_i), \phi_i)$ fibers over S^1 for $\phi_i : H_1(X(L_i);\mathbb{Z}) \to H_1(X;\mathbb{Z}) \to \mathbb{Z}$ where the last map is induced by sending all meridians to 1. The Milnor number associated to the singularity P_i is defined as $\mu(\mathcal{C}, P_i) = \dim(H_1(F_i;\mathbb{Q}))$, where F_i is the fiber of $X(L_i)$. We write $\mu(\mathcal{C}, P_\infty)$ for the Euler characteristic of the fiber of $X(L_\infty)$.

Given an algebraic curve $\mathcal{D}$ we define $\chi(\mathcal{D})$ to be the Euler characteristic of the normalized curve, i.e. the curve without singularities obtained from $\mathcal{D}$ by blow–ups. Note that $\chi(\mathcal{D})$ can be computed as follows: Let $\mathcal{D}'$ be the result of first removing balls around the singularities, and let $\mathcal{D}''$ be the result of gluing in disks to all the boundary components of $\mathcal{D}'$. Then $\mathcal{D}''$ is topologically equivalent to $\mathcal{D}$ blown up at the singularities, in particular

$$\chi(\mathcal{D}) = \chi(\mathcal{D}'').$$

Since gluing in a disk increases the Euler characteristic by one we also get the following useful formula

$$\chi(\mathcal{D}) = \chi(\mathcal{D}') + b_0(\partial \mathcal{D}'). \tag{1}$$

Let $\mathcal{C} \subset \mathbb{C}^2$ again be an algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. Then the discussion above can be used to show that $\chi(\mathcal{C}) = \sum_{i=1}^r \chi(\mathcal{C}_i)$.

3. Reidemeister torsion

For the remainder of the paper we will only consider associative rings R with $1 \neq 0$ with the property that if $r \neq s$, then R^r is not isomorphic to R^s. This is for example the case if R is a skew field.

For such a ring R define $\mathrm{GL}(R) := \lim\limits_{\to} \mathrm{GL}(R, n)$, where we have the following maps in the direct system: $\mathrm{GL}(R, n) \to \mathrm{GL}(R, n+1)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. We define $K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$. In particular $K_1(R)$ is an abelian group. For details we refer to [Mi66] or [Tu01].

Let $\mathbb{K}$ be a (skew) field and A a square matrix over $\mathbb{K}$. After elementary row operations we can arrange that A is represented by a 1×1–matrix (d). Then the Dieudonné determinant $\det(A) \in \mathbb{K}^\times_{ab} = \mathbb{K}^\times/[\mathbb{K}^\times, \mathbb{K}^\times]$ (where $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$) is defined to be d. Note that the Dieudonné determinant is invariant under elementary row operations and that $A = \det(A) \in K_1(\mathbb{K})$. The Dieudonné determinant induces an isomorphism $\det : K_1(\mathbb{K}) \to \mathbb{K}^\times_{ab}$. We refer to [Ro94, Theorem 2.2.5 and Corollary 2.2.6] for more details.

Let X be any CW–complex, by this we will always mean a finite connected CW–complex. Denote the universal cover of X by $\tilde{X}$. We view $C_*(\tilde{X})$ as a right $\mathbb{Z}[\pi_1(X)]$–module via deck transformations. Let R be a ring. Let $\alpha : \pi_1(X) \to \mathrm{GL}(R, d)$ be a representation, this equips R^d with a left $\mathbb{Z}[\pi_1(X)]$–module structure. We can therefore consider the right R–module chain complex $C^\alpha_*(X, R^d) := C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^d$. We denote its homology by $H^\alpha_i(X; R^d)$. We drop the notation α when the representation is clear from the context.

We now assume that $H^\alpha_*(X; R^d)$ is a free R–module. Let $\mathcal{B}$ be a basis for $H^\alpha_*(X; R^d)$. We can define the Reidemeister torsion $\tau^\alpha(X, \mathcal{B}) \in K_1(R)/\pm\alpha(\pi_1(X))$. If $H^\alpha_*(X; R^d) = 0$, then we drop the notation $\mathcal{B}$.

Since $\tau^\alpha(X, \mathcal{B})$ only depends on the homeomorphism type of X and the choice of a basis $\mathcal{B}$ we can define $\tau^\alpha(M, \mathcal{B})$ for a manifold M by picking any CW–structure for M. We refer to the excellent book of Turaev [Tu01] for filling in the details.

4. Multivariable Laurent polynomial rings

By a multivariable skew Laurent polynomial ring of rank m over $\mathbb{K}$ we mean a ring R which is an algebra over a skew field $\mathbb{K}$ with unit (i.e. we can view $\mathbb{K}$ as a subring of R) together with a decomposition $R = \oplus_{\alpha \in \mathbb{Z}^m} V_\alpha$ such that the following hold:

(1) V_α is a one–dimensional $\mathbb{K}$–vector space,
(2) $V_\alpha \cdot V_\beta = V_{\alpha+\beta}$.

In particular R is $\mathbb{Z}^m$–graded. Note that $V_{(0,\dots,0)} = \mathbb{K}$ and that properties (1) and (2) imply that R is a domain.

The example to keep in mind is a commutative Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Let $t^\alpha := t_1^{\alpha_1} \cdot \dots \cdot t_m^{\alpha_m}$ for $\alpha = (\alpha_1, \dots, \alpha_m)$, then $V_\alpha = \mathbb{F}t^\alpha, \alpha \in \mathbb{Z}^m$ has the required properties.

Let R be a multivariable skew Laurent polynomial ring of rank m over $\mathbb{K}$. To make our subsequent definitions and arguments easier to digest we will always pick $t^\alpha \in V_\alpha \setminus \{0\}$ for $\alpha \in \mathbb{Z}^m$ such that $t^{-\alpha} = (t^\alpha)^{-1}$ for all $\alpha \in \mathbb{Z}^m$. We get the following properties

(1) $t^\alpha t^{\tilde{\alpha}} t^{-(\alpha+\tilde{\alpha})} \in \mathbb{K}^\times$ for all $\alpha, \tilde{\alpha} \in \mathbb{Z}^m$, and
(2) $t^\alpha \mathbb{K} = \mathbb{K} t^\alpha$ for all α.

If $m = 1$ then we can and will always pick $t^{(n)} \in V_{(n)}$ such that $t^{(n)} = (t^{(1)})^n$ for any $n \in \mathbb{Z}$.

We normally denote a multivariable skew Laurent polynomial ring of rank m over $\mathbb{K}$ suggestively by $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We will always assume that we chose $t^\alpha, \alpha \in \mathbb{Z}^m$. The argument of [DLMSY03, Corollary 6.3] can be used to show that any such Laurent polynomial ring is a (left and right) Ore domain. We denote the quotient field of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ by $\mathbb{K}(t_1, \dots, t_m)$.

In the following assume that $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ is equipped with an involution $r \mapsto \overline{r}$ such that $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$ for all $r, s \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This extends to an involution on $\mathbb{K}(t_1, \dots, t_m)$ and to an involution on $K_1(\mathbb{K}(t_1, \dots, t_m))$ via $\overline{(a_{ij})} := (\overline{a_{ji}})$ for $(a_{ij}) \in K_1(\mathbb{K}(t_1, \dots, t_m))$. Note that $\det(\overline{A}) = \overline{\det(A)}$ for any $A \in K_1(\mathbb{K}(t_1, \dots, t_m))$.

5. The main theorem

5.1. **Compatible homomorphisms.** Let X be a manifold and let $\psi : H_1(X) \to \mathbb{Z}^m$ be an epimorphism. Let $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be a multivariable skew Laurent polynomial ring of rank m as in Section 4. A representation $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ is called *ψ–compatible* if for any $g \in \pi_1(X)$ we have $\alpha(g) = At^{\psi(g)}$ for some $A \in \mathrm{GL}(\mathbb{K}, d)$. This generalizes definitions in [Tu02] and [Fr05]. We denote the induced representation $\pi_1(X) \to \mathrm{GL}(\mathbb{K}(t_1, \dots, t_m), d)$ by α as well.

The following will be a useful lemma:

Lemma 5.1. *Let X be a manifold and $\psi : H_1(X; \mathbb{Z}) \to \mathbb{Z}^m$ be a non–trivial map and $\alpha : \pi_1(X) \to \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ a ψ–compatible homomorphism to a multivariable skew Laurent polynomial ring. Then $H_0(X; \mathbb{K}(t_1, \dots, t_m)) = 0$.*

Proof. We can give X a CW–structure with one zero cell and r one cells. Denote the universal cover by $\tilde{X}$. Then picking appropriate lifts of the cells of X to $\tilde{X}$ we see that the map $C_1(\tilde{X}) \to C_0(\tilde{X})$ is represented by a matrix of the form

$$\begin{pmatrix} 1 - g_1 & 1 - g_2 & \dots & 1 - g_r \end{pmatrix}$$

where $g_1,\dots,g_r \in \pi_1(X)$ generate $\pi_1(X)$. Since ψ is non–trivial and since α is ψ–compatible it follows that $\alpha(1-g_j) \neq 0 \in \mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ for some j, but then $\alpha(1-g_j)$ is a unit in $\mathbb{K}(t_1,\dots,t_m)$. It therefore follows that $H_0(X;\mathbb{K}(t_1,\dots,t_m)) = 0$. □

5.2. **Unitary representations and intersection pairings.** We explain carefully the definition of the intersection pairings with twisted coefficients. This is slightly delicate even in the case that R is a commutative ring.

In this section let R be a (possibly non–commutative) ring with involution $r \mapsto \overline{r}$ such that $\overline{ab} = \overline{b}\cdot\overline{a}$. Let V be a right R–module together with a non–singular R–sesquilinear inner product $\langle\, ,\, \rangle : V \times V \to R$. This means that for all $v,w \in V$ and $r \in R$ we have

$$\langle vr, w\rangle = \langle v,w\rangle r, \quad \langle v, wr\rangle = \overline{r}\langle v,w\rangle$$

and $\langle\, ,\, \rangle$ induces via $v \mapsto (w \mapsto \langle v,w\rangle)$ an R–module isomorphism $V \cong \mathrm{Hom}_R(V,R)$. Here we view $\mathrm{Hom}_R(V,R)$ as right R–module homomorphisms where R gets the right R–module structure given by involuted left multiplication. Furthermore consider $\mathrm{Hom}_R(V,R)$ as a right R–module via right multiplication in the target R.

We say that $A \in \mathrm{GL}(V,R)$ is *unitary* if

$$\langle Av, Aw\rangle = \langle v,w\rangle$$

for all $v,w \in V, g \in \pi_1(M)$.

Let X be an n–manifold and $\pi := \pi_1(X)$. Let $\alpha : \pi \to \mathrm{GL}(V)$ a unitary representation. This representation α can be used to define a left $\mathbb{Z}[\pi]$–module structure on V. Denote the universal cover of X by $\tilde{X}$. Let $V' = V$ as R–modules equipped with the right $\mathbb{Z}[\pi]$–module structure given by $v \cdot g := \alpha(g^{-1})v$ for $v \in V$ and $g \in \pi$. Then the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X}), V') & \to & \mathrm{Hom}_R\big(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} V, R\big) \\ f & \mapsto & ((c\otimes w) \mapsto \langle f(c), w\rangle)\,. \end{array}$$

is a well–defined right R–module homomorphism (note that we need that α is unitary) and gives rise to isomorphisms $H^*(X;V') \to H_*(\mathrm{Hom}_R\big(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} V, R\big))$. The evaluation homomorphism now induces a homomorphism of right R–modules

$$H^i(X;V') \to \mathrm{Hom}_R(H_i(X;V),R),$$

here we equip $H_*(-,V), H^*(-,V)$ with the right R–module structures given on V. If R is a skew field then the universal coefficient theorem for chain complexes over the field R applied to the R–complex $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} V$ shows that the evaluation homomorphism is in fact an isomorphism.

Combining with the Poincaré duality isomorphism (cf. [Wa99]) we get the following maps

$$H_i(X;V) \to H_i(X,\partial X;V) \cong H^{n-i}(X;V') \to \mathrm{Hom}_R(H_{n-i}(X;V),R).$$

We therefore get an R–sesquilinear pairing

$$H_i(X;V) \times H_{n-i}(X;V) \to R.$$

If $n = 2i$ then the pairing is hermitian. It is called the intersection pairing of X with V–coefficient system.

For the remainder of this paper we equip R^d with the non–singular R–sesquilinear inner product $\langle v, w\rangle = \overline{w}^t v$. Note that a matrix in $A = (a_{ij}) \in \mathrm{GL}(R, d)$ is unitary if $A\overline{A}^t = \mathrm{id}$ where $\overline{A} = (\overline{a}_{ij})$.

5.3. **The main theorem.** For the remainder of this section let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. We denote the meridians of $\mathcal{C}_1, \dots, \mathcal{C}_r$ in $X(\mathcal{C})$ by $\mu_1, \dots, \mu_r$. Let $\psi : H_1(X(\mathcal{C}); \mathbb{Z}) \to \mathbb{Z}^m$ be an epimorphism. Let $\Lambda = \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be a multivariable skew Laurent polynomial ring of rank m with quotient field $Q(\Lambda) = \mathbb{K}(t_1, \dots, t_m)$. Furthermore let $\alpha : \pi_1(X(\mathcal{C})) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ be a unitary ψ–compatible representation.

Now let $\mathrm{Sing}(\mathcal{C}) := \{P_1, \dots, P_s\} \subset \mathbb{B}^4$ denote the set of singularities of $\mathcal{C}$. Let L_i and $X(L_i) := S_i^3 \setminus \nu L_i$ for $i = 1, \dots, s, \infty$ as in Section 2. We denote the induced representations $\pi_1(X(L_i)) \to \pi_1(X(\mathcal{C})) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ by α as well.

Lemma 5.2. (1) *If $H_1(X(L_j); Q(\Lambda)^d) = 0$ for $j = \{1, \dots, s, \infty\}$, then for all i we have $H_i(X(L_j); Q(\Lambda)^d) = 0$ for $j = \{1, \dots, s, \infty\}$ and $H_i(\partial X(\mathcal{C}); Q(\Lambda)^d) = 0$.*
(2) *If $H_i(\partial X(\mathcal{C}); Q(\Lambda)^d) = 0$ for all i, then $H_i(X(\mathcal{C}); Q(\Lambda)^d) = 0$ for all $i \neq 2$.*

Proof. (1) We refer to Section 8.
(2) Since $X(\mathcal{C})$ is homotopy equivalent to a 2–complex by Theorem 2.1 it follows that $H_i(X(\mathcal{C}); Q(\Lambda)) = 0$ for $i > 2$. Since ψ is non–trivial and since α is ψ–compatible it follows from Lemma 5.1 that $H_0(X(\mathcal{C}); Q(\Lambda)) = 0$. Consider the exact sequence
$$H_1(\partial X(\mathcal{C}); Q(\Lambda)) \to H_1(X(\mathcal{C}); Q(\Lambda)) \to H_1(X(\mathcal{C}), \partial X(\mathcal{C}); Q(\Lambda)).$$
It remains to show that $H_1(X(\mathcal{C}), \partial X(\mathcal{C}); Q(\Lambda)^d) = 0$. But $H_1(X(\mathcal{C}), \partial X(\mathcal{C}); Q(\Lambda)) \cong H^3(X(\mathcal{C}); (Q(\Lambda)^d)')$ which is zero since $X(\mathcal{C})$ is homotopy equivalent to a 2–complex.

□

Now pick any basis $\mathcal{B}$ for $H_2(X(\mathcal{C}); Q(\Lambda)^d)$. Then denote by $I(\mathcal{B})$ the matrix corresponding to the intersection form
$$H_2(X(\mathcal{C}); Q(\Lambda)^d) \times H_2(X(\mathcal{C}); Q(\Lambda)^d) \to Q(\Lambda)$$
and to the basis $\mathcal{B}$. Note that $I(\mathcal{B})$ is a matrix defined over $Q(\Lambda)$. Since $Q(\Lambda)$ is flat over Λ we have a natural isomorphism
$$H_2(X(\mathcal{C}); Q(\Lambda)^d) \cong H_2(X(\mathcal{C}); \Lambda^d) \otimes_\Lambda Q(\Lambda).$$
We say that a basis $\mathcal{B}$ is defined over Λ if all elements in $\mathcal{B}$ are of the form $v \otimes 1$ with $v \in H_2(X(\mathcal{C}); \Lambda^d)$.

Lemma 5.3. *If $\mathcal{B}$ is defined over Λ, then $I(\mathcal{B})$ is a matrix defined over Λ.*

Proof. There exists an intersection pairing

$$H_2(X(\mathcal{C});\Lambda^d)\times H_2(X(\mathcal{C});\Lambda^d)\to\Lambda.$$

It follows from the definitions that intersection pairings are functorial, i.e. we have a commutative diagram

$$\begin{array}{ccccc} H_2(X(\mathcal{C});\Lambda^d) & \times & H_2(X(\mathcal{C});\Lambda^d) & \to & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ H_2(X(\mathcal{C});Q(\Lambda)^d) & \times & H_2(X(\mathcal{C});Q(\Lambda)^d) & \to & Q(\Lambda). \end{array}$$

The lemma is now immediate. □

The following theorem shows the relationship between the Reidemeister torsions of the singularities and the curve complement of an algebraic curve.

Theorem 5.4. *Let $\mathcal{B}$ be any basis for $H_2(X(\mathcal{C});Q(\Lambda))$. Assume that $H_1(X(L_i);Q(\Lambda))=0$ for all i. Then we have the following equality in $K_1(Q(\Lambda))/\alpha(\pi_1(X(\mathcal{C})))$:*

$$\lambda\cdot\prod_{i\in\{1,\dots,s,\infty\}}\tau^\alpha(X(L_i))=\tau^\alpha(X(\mathcal{C}),\mathcal{B})\cdot\overline{\tau^\alpha(X(\mathcal{C}),\mathcal{B})}\cdot I(\mathcal{B}),$$

where

$$\lambda:=\prod_{i=1}^r(id-\alpha(\mu_i))^{s_i-\chi(\mathcal{C}_i)}$$

and $s_i:=\#Sing(\mathcal{C})\cap\mathcal{C}_i$.

Note that λ is defined over Λ if $s_i-\chi(\mathcal{C}_i)\geq 0$ for all i. This is the case if none of the components $\mathcal{C}_i$ is a line, or if any component $\mathcal{C}_i$ has at least one singularity. We postpone the proof to Section 8. The commutative one–variable case is the main result in [CF05, Theorem 5.6].

6. One–variable Laurent polynomial rings

6.1. **Alexander polynomials.** Let X be a CW–complex and let $\psi\in H^1(X;\mathbb{Z})$ non–trivial. Let $\mathbb{K}$ be a skew field and let $\mathbb{K}[t^{\pm1}]$ be a skew Laurent polynomial ring. Let $\alpha:\pi_1(X)\to \mathrm{GL}(\mathbb{K}[t^{\pm1}],d)$ be a ψ–compatible representation.

The $\mathbb{K}[t^{\pm1}]$–modules $H_1(X;\mathbb{K}[t^{\pm1}]^d)$ are called twisted (non–commutative) Alexander modules. Similar modules were studied in [Co04], [Ha05], [Fr05]. The rings $\mathbb{K}[t^{\pm1}]$ are principal ideal domains (PID) since $\mathbb{K}$ is a skew field. We can therefore decompose

$$H_i^\alpha(X;\mathbb{K}[t^{\pm1}]^d)\cong\mathbb{K}[t^{\pm1}]^f\oplus\bigoplus_{i=1}^l\mathbb{K}[t^{\pm1}]/(p_i(t))$$

for some $f\geq 0$ and $p_i(t)\in\mathbb{K}[t^{\pm1}]\setminus\{0\}$ for $i=1,\dots,l$. We define $\Delta_i^\alpha(X):=\prod_{i=1}^l p_i(t)\in\mathbb{K}[t^{\pm1}]\setminus\{0\}$. Note that this differs from the definition in [Fr05] where we set $\Delta_i^\alpha(X)=0$ if $f>0$.

$\Delta_i^\alpha(X)$ is called the (twisted) *Alexander polynomial* of (X,α). Note that $\Delta_i^\alpha(X) \in \mathbb{K}[t^{\pm 1}]$ has a high degree of indeterminacy. For example writing the $p_i(t)$ in a different order will give a different Alexander polynomial. We refer to [Co04, p. 367] and [Fr05, Theorem 3.1] for a discussion of the indeterminacy of $\Delta_i^\alpha(X)$.

In the case of one–dimensional representations we can determine $\Delta_0^\alpha(X)$. We call $\psi \in H^1(X;\mathbb{Z})$ *primitive* if the corresponding map $\psi : H_1(X;\mathbb{Z}) \to \mathbb{Z}$ is surjective.

Lemma 6.1. *Let X be a CW–complex, $\psi \in H^1(X;\mathbb{Z})$ primitive. Let $\alpha : \pi_1(X) \to GL(\mathbb{K}[t^{\pm 1}],1)$ be a ψ–compatible one–dimensional representation. If $Im(\alpha(\pi_1(X))) \subset \mathbb{K}[t^{\pm 1}]$ is cyclic, then $\Delta_0^\alpha(X) = at - 1$ for some $a \in \mathbb{K}$. Otherwise $\Delta_0^\alpha(X) = 1$.*

Proof. This statement follows easily from considering the chain complex for X and from well–known properties of PID's. □

Let X be a CW–complex of dimension k and let $\psi \in H^1(M;\mathbb{Z})$ non–trivial and let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{K}[t^{\pm 1}],d)$ be a ψ–compatible representation. Note that $H_*(X;\mathbb{K}(t)^d) = H_*(X;\mathbb{K}[t^{\pm 1}]^d) \otimes_{\mathbb{K}[t^{\pm 1}]} \mathbb{K}(t)$. We write $FH_*(K;\mathbb{K}[t^{\pm 1}]^d)$ for the quotient of $H_*(X;\mathbb{K}[t^{\pm 1}]^d)$ by its maximal $\mathbb{K}[t^{\pm 1}]$–torsion submodule. Note that $FH_*(K;\mathbb{K}[t^{\pm 1}]^d)$ is a free $\mathbb{K}[t^{\pm 1}]$–module since $\mathbb{K}[t^{\pm 1}]$ is a PID. Note that the map $H_*(X;\mathbb{K}[t^{\pm 1}]^d) \to H_*(X;\mathbb{K}(t)^d)$ also induces a map $FH_*(X;\mathbb{K}[t^{\pm 1}]^d) \to H_*(X;\mathbb{K}(t)^d)$.

We say that a basis $\mathcal{B} = \{\mathcal{B}_0,\dots,\mathcal{B}_k\}$ for $H_*(X;\mathbb{K}(t)^d)$ is a $\mathbb{K}[t^{\pm 1}]$*–basis* if there exists a basis $\tilde{\mathcal{B}}_i$ for $FH_i(X;\mathbb{K}[t^{\pm 1}]^d)$ such that $\mathcal{B}_i$ is the image of $\tilde{\mathcal{B}}_i$ under $H_*(X;\mathbb{K}[t^{\pm 1}]^d) \to H_*(X;\mathbb{K}(t)^d)$.

Theorem 6.2. *Let X be a finite CW–complex which is homotopy equivalent to a 2–complex. Let $\psi \in H^1(X;\mathbb{Z})$ non–trivial and let $\alpha : \pi_1(X) \to GL(\mathbb{K}[t^{\pm 1}],d)$ be a ψ–compatible representation. Let $\mathcal{B}$ be a $\mathbb{K}[t^{\pm 1}]$–basis for $H_*(X;\mathbb{K}(t)^d)$ defined over $\mathbb{K}[t^{\pm 1}]$. Then $\tau^\alpha(M,\mathcal{B}) \in K_1(\mathbb{K}(t))/K_1(\mathbb{K}[t^{\pm 1}])$ is independent of the choice of basis $\mathcal{B}$. Furthermore*

$$\tau^\alpha(X,\mathcal{B}) = \Delta_1^\alpha(X)\Delta_0^\alpha(X)^{-1} \in K_1(\mathbb{K}(t))/K_1(\mathbb{K}[t^{\pm 1}]).$$

This theorem was proved in [Tu86, p. 174] and [KL99] in the commutative case.

Proof. Let $\mathcal{B}' = \{\mathcal{B}'_0,\dots,\mathcal{B}'_k\}$ be an alternative $\mathbb{K}[t^{\pm 1}]$–basis for $H_i(X;\mathbb{K}(t))$. Let $\tilde{\mathcal{B}}_i$ and $\tilde{\mathcal{B}'}_i$ be the corresponding bases for $FH_i(X;\mathbb{K}[t^{\pm 1}]^d)$. Then $\tilde{\mathcal{B}}_i = A_i\tilde{\mathcal{B}'}_i$ where A_i is a matrix defined over $\mathbb{K}[t^{\pm 1}]$ and invertible over $\mathbb{K}[t^{\pm 1}]$. Clearly

$$\tau^\alpha(X,\mathcal{B}) = \prod_i A_i^{(-1)^{i+1}}\tau^\alpha(X,\mathcal{B}') \in K_1(\mathbb{K}(t))/\pm\alpha(\pi_1(X)).$$

It follows that $\tau^\alpha(X,\mathcal{B}) = \tau^\alpha(X,\mathcal{B}') \in K_1(\mathbb{K}(t))/K_1(\mathbb{K}[t^{\pm 1}])$.

The argument in [KL99] can easily be adapted to the non–commutative case to prove that

$$\tau^\alpha(X,\mathcal{B}) = \prod_i \Delta_i^\alpha(X)^{(-1)^{i+1}}.$$

Let Y be a 2–complex homotopy equivalent to X. Since the Alexander polynomial is a homotopy invariant it is enough to study the Alexander polynomials of Y. Since Y is a 2–complex it follows that $\Delta_i^\alpha(X) = 1$ for $i > 2$. Furthermore

$$H_2(Y; \mathbb{K}[t^{\pm 1}]^d) \subset C_2(Y; \mathbb{K}[t^{\pm 1}]^d),$$

in particular $H_2(Y; \mathbb{K}[t^{\pm 1}]^d)$ is torsion–free, hence $\Delta_2^\alpha(X) = 1$. □

6.2. **One–variable version of Theorem 5.4.** Let X be a CW–complex and let $\psi \in H^1(X; \mathbb{Z})$ non–trivial and let $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{K}[t^{\pm 1}], d)$ be a ψ–compatible representation. We define

$$\Delta^\alpha(X) := \Delta_1^\alpha(X) \cdot \Delta_0^\alpha(X)^{-1}.$$

Then the following theorem follows immediately from Theorems 5.4 and 6.2.

Theorem 6.3. *Let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. Let $\psi : H_1(X; \mathbb{Z}) \to \mathbb{Z}$ be an epimorphism. Let $\alpha : \pi_1(X(\mathcal{C})) \to GL(\mathbb{K}[t^{\pm 1}], d)$ be a unitary ψ–compatible representation. Let $\mathcal{B}$ be any $\mathbb{K}[t^{\pm 1}]$–basis for $H_2(X(\mathcal{C}); \mathbb{K}(t))$. Assume that $H_1(X(L_i); \mathbb{K}(t)) = 0$ for $i = 1, \dots, s, \infty$. Then we have the following equality in $K_1(\mathbb{K}(t))/K_1(\mathbb{K}[t^{\pm 1}])$:*

$$\lambda \cdot \prod_{i \in \{1, \dots, s, \infty\}} \Delta^\alpha(X(L_i)) = \Delta^\alpha(X(\mathcal{C})) \cdot \overline{\Delta^\alpha(X(\mathcal{C}))} \cdot I(\mathcal{B}),$$

where

$$\lambda := \prod_{i=1}^{r} (id - \alpha(\mu_i))^{s_i - \chi(\mathcal{C}_i)}$$

and $s_i := \#Sing(\mathcal{C}) \cap \mathcal{C}_i$.

A slightly different version in the untwisted commutative case (i.e. $\mathbb{K}[t^{\pm 1}] = \mathbb{Q}[t^{\pm 1}]$) was first shown by Libgober [Li82]. The commutative case is the main result in [CF05, Theorem 5.6].

6.3. **The degree of the one–variable Alexander polynomial.** Let $\mathbb{K}[t^{\pm 1}]$ be a Laurent polynomial ring. Let $f(t) \in \mathbb{K}[t^{\pm 1}] \setminus \{0\}$. For $f(t) = \sum_{i=m}^{n} a_i t^i \in \mathbb{K}[t^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f(t)) := n - m$. Note that $\deg(\Delta_i^\alpha(X))$ is well–defined (cf. [Co04]). This extends to $\deg : \mathbb{K}(t) \setminus \{0\} \to \mathbb{Z}$ via $\deg(f(t)g(t)^{-1}) = \deg(f(t)) - \deg(g(t))$. Since deg is a homomorphism to an abelian group this induces a homomorphism $\deg : \mathbb{K}(t)^\times_{ab} \to \mathbb{Z}$. Using the Dieudonné determinant we get a homomorphism $\deg : K_1(\mathbb{K}(t)) \to \mathbb{Z}$. For a homomorphism $\mathbb{Z}[\pi_1(X)] \xrightarrow{\alpha} \mathbb{K}[t^{\pm 1}] \hookrightarrow \mathbb{K}(t)$ it also passes to $\deg : K_1(\mathbb{K}(t))/\pm \pi_1(X) \to \mathbb{Z}$. For more details about this degree homomorphism, we refer to [Fr05, Section 3.3].

If we only consider the degrees we get the following result

Theorem 6.4. *Assume we are in the setting of Theorem 5.4. Let* $\psi : H_1(X(\mathcal{C});\mathbb{Z}) \to \mathbb{Z}$ *be the homomorphism given by* $\psi(\mu_i) = 1$ *for all* i*. Let* $\alpha : \pi_1(X) \to GL(\mathbb{K}[t^{\pm 1}], d)$ *be a unitary* ψ*–compatible representation. Then* $H_*(X(L_i), \mathbb{K}(t)) = 0$ *for* $i = 1, \dots, s, \infty$ *and for any* $\mathbb{K}[t^{\pm 1}]$*–basis* $\mathcal{B}$ *for* $H_2(X(\mathcal{C}), Q(\Lambda)^d)$ *we have*

$$deg(\tau^\alpha(X(\mathcal{C}), \mathcal{B})) \le \frac{d}{2}\Big(-\chi(\mathcal{C}) + \sum_{i\in\{1,\dots,s,\infty\}} (\mu(\mathcal{C}, P_i) - 1) + \sum_{i=1}^{r} \#Sing(\mathcal{C}) \cap \mathcal{C}_i\Big).$$

In particular

$$deg(\Delta_1(X(\mathcal{C}))) \le \frac{d}{2}\Big(2 -\chi(\mathcal{C}) + \sum_{i\in\{1,\dots,s,\infty\}} (\mu(\mathcal{C}, P_i) - 1) + \sum_{i=1}^{r} \#Sing(\mathcal{C}) \cap \mathcal{C}_i\Big).$$

Note that $\sum_{i=1}^r \#\mathrm{Sing}(\mathcal{C}) \cap \mathcal{C}_i$ can be viewed as the number of singularities, each counted with a weight (namely the number of sheets that meet at the singularity).

Proof. For $i = 1, \dots, s, \infty$ it follows from [Mi68, Theorem 4.8] that $(X(L_i), H_1(X(L_i);\mathbb{Z}) \to H_1(X(\mathcal{C});\mathbb{Z}) \xrightarrow{\psi} \mathbb{Z})$ fibers over S^1. It then follows from [Ha05, Fr05] that $H_*(X(L_i), \mathbb{K}(t)) = 0$ and

$$\deg(\tau^\alpha(X(L_i))) = d\,(\mu(\mathcal{C}, P_i) - 1).$$

Now let $\mathcal{B}$ be a $\mathbb{K}[t^{\pm 1}]$–basis for $H_2(X(\mathcal{C}); \mathbb{K}(t))$. In particular we can therefore apply Theorem 5.4 and we get

$$\lambda \cdot \prod_{i\in\{1,\dots,s,\infty\}} \tau^\alpha(X(L_i)) = \tau^\alpha(X(\mathcal{C}), \mathcal{B}) \cdot \overline{\tau^\alpha(X(\mathcal{C}), \mathcal{B}))} \cdot I(\mathcal{B}),$$

in $K_1(\mathbb{K}(t))/\pm\alpha(\pi_1(X))$ where $\lambda = \prod_{i=1}^r (\mathrm{id} - \alpha(\mu_i))^{s_i - \chi(\mathcal{C}_i)}$ and $s_i = \#\mathrm{Sing}(\mathcal{C}) \cap \mathcal{C}_i$.

Clearly $I(\mathcal{B})$ is a matrix defined over $\mathbb{K}[t^{\pm 1}]$. Since $\mathbb{K}[t^{\pm 1}]$ is a PID we can diagonalize $I(\mathcal{B})$ over $\mathbb{K}[t^{\pm 1}]$ using elementary row and column operations. This shows that $\det(I(\mathcal{B})) \in \mathbb{K}(t)^\times_{ab}$ can be represented by an element in $\mathbb{K}[t^{\pm 1}]$. In particular it follows that $\deg(I(\mathcal{B})) \ge 0$.

Furthermore note that

$$\begin{aligned}
\deg(\tau^\alpha(X(\mathcal{C}), \mathcal{B})) &= \deg(\det(\tau^\alpha(X(\mathcal{C}), \mathcal{B}))) \\
&= \deg(\overline{\det(\tau^\alpha(X(\mathcal{C}), \mathcal{B}))}) \\
&= \deg(\det(\overline{\tau^\alpha(X(\mathcal{C}), \mathcal{B})})) \\
&= \deg(\overline{\tau^\alpha(X(\mathcal{C}), \mathcal{B})}).
\end{aligned}$$

Furthermore

$$\deg\Big(\prod_{i=1}^r (\mathrm{id} - \alpha(\mu_i))^{s_i - \chi(\mathcal{C}_i)}\Big) = d\sum_{i=1}^r (s_i - \chi(\mathcal{C}_i))$$

since α is ψ–compatible and $\psi(\mu_i) = 1$ for all i. Note that by equation (1) we have

$$\sum_{i=1}^r (s_i - \chi(\mathcal{C}_i)) = \sum_{i=1}^r s_i \quad - \chi(\mathcal{C}).$$

From the fact that the degree map is a homomorphism it now follows that

$$2\deg(\tau^\alpha(X(\mathcal{C}),\mathcal{B})) \le d(s-\chi(\mathcal{C})) + d \sum_{i\in\{1,\dots,s,\infty\}} (\mu(\mathcal{C},P_i)-1).$$

The first inequality is now immediate. The second equality follows from Theorem 6.2 and the observation that the argument in Lemma 5.1 can be used to show that $\deg(\Delta_0^\alpha(X(\mathcal{C}))) \le d$. □

Remark. If $\mathcal{C}$ intersects the line at infinity transversely, then Leidy and Maxim [LM05] prove two upper bounds on $\deg(\Delta_1(X(\mathcal{C})))$ in the untwisted case, one just involving the singularities, and one just involving the link at infinity. Combined they give (in the case $d=1$) a slightly better bound than ours. Note that if $\mathcal{C}$ intersects the line at infinity transversely, then L_∞ is just the Hopf link on d components, its one–variable Alexander polynomial is $(t^d-1)^{d-2}(t-1)$ (cf. [Ok02, p. 9]) and has degree $d(d-2)+1$. So the higher–order Alexander polynomials have degree $d(d-2)$.

Remark. Theorem 6.3 has the advantage over the bounds in [LM05] that it is an equality, i.e. the study of the degrees of $\Delta^\alpha(X(\mathcal{C}))$ is in some way equivalent to the study of the degree of $I(\mathcal{B})$.

7. Commutative representations

Note that there is no multivariable Alexander polynomial defined over non–commutative multivariable Laurent polynomial rings. The reason is that if A is a matrix defined over a multivariable skew Laurent polynomial ring Λ of rank $m \ge 2$, then $\det(A) \in Q(\Lambda)^\times/[Q(\Lambda)^\times, Q(\Lambda)^\times]$ is in general not represented by an element in Λ.

If we want to study multivariable Alexander polynomials we therefore have to go to the commutative setting.

7.1. **Torsion invariants.** Let R be a commutative Noetherian unique factorization domain (henceforth UFD). An example of R to keep in mind is $\mathbb{F}[t_1^\pm, t_2^\pm, \dots, t_n^\pm]$, a (multivariable) Laurent polynomial ring over a field $\mathbb{F}$. For a finitely generated R-module A, we can find a presentation

$$R^r \xrightarrow{P} R^s \to A \to 0$$

since R is Noetherian. Let $i \ge 0$ and suppose $s-i \le r$. We define $E_i(A)$, *the i-th elementary ideal* of A, to be the ideal in R generated by all $(s-i)\times(s-i)$ minors of P if $s-i>0$ and to be R if $s-i \le 0$. If $s-i>r$, we define $E_i(A)=0$. It is known that $E_i(A)$ does not depend on the choice of a presentation of A (cf. [CF77, p. 101] together with [Li97, Theorem 6.1]).

Since R is a UFD there exists a unique smallest principal ideal of R that contains $E_0(A)$. A generator of this principal ideal is defined to be the *order of* A and denoted by $\mathrm{ord}(A) \in R$. The order is well-defined up to multiplication by a unit in R. Note that A is not R-torsion if and only if $\mathrm{ord}(A)=0$. For more details, we refer to [Hi02].

Given a UFD R we denote its quotient field by $Q(R)$. We use the determinant to identify $K_1(Q(R))$ with $Q(R)^\times$.

7.2. **Twisted Alexander invariants.** Let X be a CW–complex. Let $\psi : \pi_1(X) \to \mathbb{Z}^m$ be a homomorphism. We do not demand that ψ is surjective. Let R be a commutative unique factorization domain (UFD), e.g. $R = \mathbb{Z}, R = \mathbb{C}, R = \mathbb{Z}/p$. Henceforth $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and $R(t_1, \dots, t_m)$ will denote the usual commutative Laurent polynomial ring and its quotient field.

Let $\alpha : \pi_1(M) \to \mathrm{GL}(R, k)$ be a representation. Using α and ψ, we define a left $\mathbb{Z}[\pi_1(M)]$-module structure on $R^d \otimes_{\mathbb{Z}} \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] =: R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]^d$ as follows:

$$g \cdot (v \otimes p) := (\alpha(g) \cdot v) \otimes (t^{\psi(g)} p)$$

where $g \in \pi_1(M)$ and $v \otimes p \in R^d \otimes_{\mathbb{Z}} \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] = R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]^d$ and $t^{\psi(g)}$ is in multiindex notation. Note that $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ is a UFD again.

We can therefore consider the $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$-modules $H_i^\alpha(X; R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]^d)$. If X has finitely many cells in dimension i then these modules are finitely generated over $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ since $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ is Noetherian.

The *i-th (twisted) Alexander polynomial of* (X, ψ, α) is defined to be $\mathrm{ord}(H_i^\alpha(X; R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]^d)) \in R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and denoted by $\Delta_i^\alpha(X)$. Note that twisted Alexander polynomials are well-defined up to multiplication by a unit in $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We drop α when α is the trivial representation to $\mathrm{GL}(\mathbb{Q}, 1)$. We also define

$$\Delta^\alpha(X) := \Delta_1^\alpha(X) \cdot (\Delta_0^\alpha(X))^{-1}.$$

Remark. Let X be a compact manifold. Then $\pi_1(X)$ is finitely presented and we can obtain the Eilenberg–MacLane space $K(\pi_1(X), 1)$ by adding cells of dimension greater than or equal to 3 to X. This does not change the two lowest homology groups, in particular

$$H_i^\alpha(X; R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]^d) \cong H_i^\alpha(K(\pi_1(X), 1); R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]^d)$$

In particular $\Delta_i^\alpha(X)$ only depends on $\pi_1(X)$ for $i = 0, 1$. Furthermore given a presentation of the fundamental group Fox calculus [Fo53, Fo54, CF77]) can be used to compute $\Delta_i^\alpha(K(\pi_1(X), 1))$ for $i = 0, 1$.

7.3. **Multivariable version of Theorem 5.4.** In the following let R be a commutative UFD.

Theorem 7.1. *Assume we are in the setting of Theorem 5.4. Let* $\psi : H_1(X(\mathcal{C}); \mathbb{Z}) \to \mathbb{Z}^m$ *be a non–trivial homomorphism and let* $\alpha : \pi_1(X) \to GL(R[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ *be a unitary* ψ*–compatible representation such that* $H_1(X(L_i), R(t_1, \dots, t_m)) = 0$ *for* $i = 1, \dots, s, \infty$. *Let*

$$\lambda := \prod_{i=1}^{r} \det(id - \alpha(\mu_i))^{s_i - \chi(\mathcal{C}_i)}$$

and $s_i := \#Sing(\mathcal{C}) \cap \mathcal{C}_i$. *Then there exists* $p \in \Lambda$ *such that*

$$p \cdot \Delta^\alpha(X(\mathcal{C})) \cdot \overline{\Delta^\alpha(X(\mathcal{C}))} = \lambda \cdot \prod_{i\in\{1,\dots,s,\infty\}} \Delta^\alpha(X(L_i)).$$

Corollary 7.2. *Assume we are in the setting of Theorem 5.4 and that* $m \geq 2$ *and that* ψ *is an epimorphism. Then*

$$\Delta_1^\alpha(X(\mathcal{C})) \cdot \overline{\Delta_1^\alpha(X(\mathcal{C}))} \in R[t_1^{\pm1}, \dots, t_m^{\pm1}]$$

divides

$$\prod_{i=1}^r \det(id - \alpha(\mu_i))^{\max\{0, s_i - \chi(\mathcal{C}_i)\}} \prod_{i\in\{1,\dots,s,\infty\}} \Delta_1^\alpha(X(L_i)) \in R[t_1^{\pm1}, \dots, t_m^{\pm1}].$$

Proof. An easy argument shows that $\Delta_0^\alpha(X(\mathcal{C})) = 1$ since $m \geq 2$ and since ψ is an epimorphism (cf. [FK05b, Lemma 6.2] for details). The corollary now follows immediately from Theorem 7.1. □

Remark. If $\mathcal{C}$ intersects the line at infinity transversely, then using the ideas of [LM05] and the main result of [KSW05] and using one can also show that $\Delta_1^\alpha(X(\mathcal{C}))$ divides $\prod_{i\in\{1,\dots,s\}} \Delta_1^\alpha(X(L_i))$ (up to some terms of the form $\det(1 - At_i)$) and that $\Delta_1^\alpha(X(\mathcal{C}))$ divides $\Delta_1^\alpha(X(L_\infty))$.

Often it is difficult to determine $\pi_1(X(\mathcal{C}))$ and therefore it is difficult to find a presentation. But we can always easily determine $H_1(X(\mathcal{C}))$ and we can therefore always study the untwisted multivariable Alexander polynomial. Therefore perhaps the most interesting application of Corollary 7.2 is the untwisted case:

Corollary 7.3. *Assume we are in the setting of Theorem 5.4,* $m \geq 2$ *and that* $\psi : H_1(X(\mathcal{C}); \mathbb{Z}) \to \mathbb{Z}^r$ *is the canonical isomorphism. Then*

$$\Delta_1(X(\mathcal{C})) \cdot \overline{\Delta_1(X(\mathcal{C}))} \in R[t_1^{\pm1}, \dots, t_m^{\pm1}]$$

divides

$$\prod_{i=1}^r (1 - t_i)^{\max\{0, s_i - \chi(\mathcal{C}_i)\}} \prod_{i\in\{1,\dots,s,\infty\}} \Delta_1(X(L_i)) \in R[t_1^{\pm1}, \dots, t_m^{\pm1}].$$

We point out that the maps $H_1(X(L_i)) \to H_1(X(\mathcal{C}))$ are injective (this needs a reference!), in particular the Alexander polynomial of $X(L_i)$ corresponding to $H_1(X(L_i)) \to H_1(X(\mathcal{C})) \cong \langle t_1, \dots, t_r \rangle$ is indeed the multivariable Alexander polynomial of L_i.

The proof of Theorem 7.1 will require the remainder of this section.

We first consider $\tau^\alpha(X(L_i))$, $i = 1, \dots, s, \infty$. Since $H_1(X(L_i), R(t_1, \dots, t_m)^d) = 0$ it follows from Lemma 5.2 that $H_*(X(L_i), R(t_1, \dots, t_m)^d) = 0$. We can therefore apply [Tu01, Theorem 4.7] to get

$$\tau^\alpha(X(L_i)) = \prod_{i=0}^{3} \left(\Delta_i^\alpha(X(L_i))\right)^{(-1)^{i+1}}.$$

But it follows from [FK05, Corollary 4.3 and Proposition 4.13] (in the case that L_i has one component) and [FK05b, Lemmas 6.2 and 6.5] (in the case that L_i has more than one component) that $\Delta_i^\alpha(X(L_i)) = 1$ for $i = 2, 3$. In particular $\tau^\alpha(X(L_i)) = \Delta^\alpha(X(L_i))$.

In the following we denote the ring $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ by Λ and the ring $R[t_1^{\pm 1}, \dots, t_{i-1}^{\pm 1}, t_{i+1}^{\pm 1}, \dots, t_m^{\pm 1}]$ by Λ_i.

Lemma 7.4. *For any i we can find a basis $\mathcal{B}$ for $H_2(X(\mathcal{C}), Q(\Lambda))$ defined over Λ such that*

$$\tau^\alpha(X(\mathcal{C}), \mathcal{B}) \cdot q_i = \Delta^\alpha_{X(\mathcal{C})} \in Q(\Lambda)^\times/\Lambda^\times$$

with $q_i \in \Lambda_i$.

Proof. Let $h : X(\mathcal{C}) \to Y$ be a homotopy equivalence with Y a 2–complex which exists by Theorem 2.1. Then for any basis $\mathcal{B}$ for $H_2(X(\mathcal{C}); Q(\Lambda)^d)$ it follows from [Tu01, Theorem 9.1] that

$$\tau^\alpha(X(\mathcal{C}), \mathcal{B}) = \tau^\alpha(Y, h_*(\mathcal{B})) \cdot u$$

for $u = \det(\alpha(A))$ where A is an invertible matrix over $\mathbb{Z}[\pi_1(X(\mathcal{C}))]$. Clearly u is a unit in Λ. Since Alexander polynomials are homotopy invariants this shows that it is enough to show the lemma for Y.

Consider $C_*(Y; \Lambda^d)$, this is a complex of free based Λ–modules of length two. Denote the rank of $C_j(Y; \Lambda^d)$ by n_j. We use the bases to identify $C_j(Y; \Lambda^d)$ with Λ^{n_j}. Denote the matrices corresponding to the boundary maps by A_i.

Let $Q_i := Q(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_m)[t_i^{\pm 1}]$. Note that any element in Q_i is of the form fg^{-1} with $f \in \Lambda, g \in \Lambda_i \setminus \{0\}$. Note that $Q(\Lambda)$ and Q_i are flat over Λ. Consider

$$K := \operatorname{Ker}\{\Lambda^{n_2} \otimes_\Lambda Q_i \xrightarrow{A_2} \Lambda^{n_1} \otimes_\Lambda Q_i\} \subset \Lambda^{n_2} \otimes_\Lambda Q_i.$$

This is a torsion–free Q_i–module, and since Q_i is a PID it follows that K is in fact a free Q_i–module. We can pick $b_1, \dots, b_r \in K$ which form a Q_i–basis for K. Note that after multiplication by units in Q_i we can assume that $b_1, \dots, b_r \in K \cap \Lambda^{n_2}$. Let B be the $n_2 \times r$–matrix with columns $b_1, \dots, b_r$.

Claim. The (principal) ideal in Q_i generated by the $r \times r$–minors of B is generated by an element $\Delta \in \Lambda_i$.

Denote by $\{m_1, \dots, m_N\}$ ($N = \binom{n_2}{r}$) the set of all the $r \times r$–minors of B. Furthermore given $g_1, \dots, g_l \in \Lambda$ we denote by $(g_1, \dots, g_l)_\Lambda$ respectively $(g_1, \dots, g_l)_{Q_i}$ the ideal generated by $g_1, \dots, g_l$ in Λ respectively Q_i. Let $\Delta' \in \Lambda$ be a greatest common divisor of $m_1,\dots,m_N$. Then $(\Delta')_\Lambda$ is the smallest principal ideal containing $(m_1,\dots,m_N)_\Lambda$. Since Q_i is a PID we can find $g\in Q_i$ such that

$$(m_1,\dots,m_N)_{Q_i}=(g)_{Q_i}.$$

Note that after multiplication by a unit we can assume that $g\in\Lambda$ and we can assume that $g\in\Lambda$ is not divisible by any non–unit in $\Lambda_i\subset\Lambda$. Since $(m_1,\dots,m_N)_{Q_i}=(g)_{Q_i}$ we can write $m_i=g\frac{f_i}{g_i}$ with $f_i\in\Lambda, g_i\in\Lambda_i$. Hence $m_ig_i=gf_i\in\Lambda$. But since $g\in\Lambda$ is not divisible by any non–unit in Λ_i it follows that $f_i=g_ih_i$ for some $h_i\in\Lambda$. In particular g divides m_i in Λ, i.e. g is a common divisor of $m_1,\dots,m_N$. By definition of Δ' it follows that

$$(\Delta')_\Lambda\subset(g)_\Lambda.$$

Putting everything together we get

$$(\Delta')_{Q_i}\subset(g)_{Q_i}=(m_1,\dots,m_N)_{Q_i}\subset(\Delta')_{Q_i}.$$

It follows that

$$\begin{aligned}(\Delta')_{Q_i}&=(m_1,\dots,m_N)_{Q_i}\\&=(E_{n_2-r}(Q_i^{n_2}/BQ_i^r))\\&=(E_{n_2-r}(Q_i^{n_2}/K))\\&=(E_{n_2-r}(A_2Q_i^{n_2})).\end{aligned}$$

But $A_2(Q_i^{n_2})$ is a free Q_i–module of rank n_2-r, hence $(E_{n_2-r}(A_2Q_i^{n_2}))=Q_i$. It follows that $\Delta\in Q_i^\times$. In particular $\Delta'\in\Lambda\cap Q_i^\times$. But clearly this implies that $\Delta'=t_i^l\Delta$ for some $\Delta\in\Delta_i$. The claim now follows from the observation that $(\Delta')_{Q_i}=(\Delta)_{Q_i}$.

Now denote by D_* the following Λ–complex of free based modules:

$$0\to\Lambda^r\xrightarrow{B}\Lambda^{n_2}\xrightarrow{A_2}\Lambda^{n_1}\xrightarrow{A_1}\Lambda^{n_0}\to 0.$$

Note that for $i=0,1$ we have $H_i(D_*\otimes Q(\Lambda))=H_i(Y;Q(\Lambda)^d)=0$. Furthermore $H_i(D_*\otimes Q(\Lambda))=0$ for $i=2,3$ by construction. It follows that $D_*\otimes Q(\Lambda)$ is acyclic and we can therefore apply [Tu01, Theorem 4.7]. We get

$$\tau(D\otimes_\Lambda Q(\Lambda))=\prod_{i=0}^{2}\operatorname{ord}(H_i(D))^{(-1)^{i+1}}.$$

Let $\mathcal{B}=\{b_1,\dots,b_r\}\subset H_2(Y;\Lambda^d)\subset H_2(Y;Q(\Lambda)^d)$. This is a Λ–basis for $H_2(Y;Q(\Lambda)^d)$. It follows easily from the definitions that $\tau(D\otimes_\Lambda Q(\Lambda))=\tau(Y,\mathcal{B})$.

It follows from [Tu01, Lemma 4.11] that $\operatorname{ord}(H_2(D))=\Delta$, hence

$$\prod_{i=0}^{2}\operatorname{ord}(H_i(D))^{(-1)^{i+1}}=\Delta^{-1}\Delta_1^\alpha(Y)(\Delta_0^\alpha(Y))^{-1}.$$

This concludes the proof of the lemma. □

For $i = 1, \dots, m$ pick a basis $\mathcal{B}_i$ for $H_2(X(\mathcal{C}), Q(\Lambda))$ defined over Λ as in the above lemma. Let q_i as in the above lemma and let $p_i = \det(I(\mathcal{B}_i)) \in \Lambda$. Then for $i = 1, \dots, m$ we have

$$\frac{p_i}{q_i}\Delta^\alpha(X(\mathcal{C})) \cdot \overline{\Delta^\alpha(X(\mathcal{C}))} = \lambda \cdot \prod_{i\in\{1,\dots,s,\infty\}} \Delta^\alpha(X(L_i)).$$

But since $\gcd(q_1, \dots, q_m) = 1$ we get the required result.

8. Proof of Theorem 5.4

Let $\mathcal{C} \subset \mathbb{C}^2$ be an algebraic curve. We use the notation from Section 2. Pick an open tubular neighborhood $\nu\mathcal{C}$ of $\mathcal{C}$ such that $\nu\mathcal{C} \cap S^3_i$ is an open tubular neighborhood for $L_i \subset S^3_i$ for $i = 1, \dots, s, \infty$. We write $\nu L_i := \nu\mathcal{C} \cap S^3_i$ and $X(L_i) = S^3 \setminus \nu L_i$. Let $T_i, i = 1, \dots, s, \infty$ be the boundaries of $S^3_i \setminus \nu L_i$. Now consider $F := \mathcal{C} \cap (\mathbb{B}^4 \setminus \cup_{i=1}^s \mathrm{int}(\mathbb{B}^4_i))$. Clearly $\partial X(\mathcal{C}) \cap (\mathbb{B}^4 \setminus \cup_{i=1}^s \mathrm{int}(\mathbb{B}^4_i)) \cong F \times S^1$. We can therefore write

$$\partial X(\mathcal{C}) := (F \times S^1) \cup_{T_1\cup\dots\cup T_s\cup T_\infty} \bigcup_{i=1,\dots,s,\infty} X(L_i).$$

With this setup we can now proof the first part of Lemma 5.2. So let $\psi : H_1(X(\mathcal{C})) \to \mathbb{Z}^m$ be an epimorphism. Let Λ be a multivariable skew Laurent polynomial ring of rank m with quotient field $Q(\Lambda)$ and let $\alpha : \pi_1(X(\mathcal{C})) \to \mathrm{GL}(\Lambda, d)$ be a ψ–compatible representation such that $H_1(X(L_j); Q(\Lambda)^d) = 0$ for $j = \{1, \dots, s, \infty\}$. It follows from standard arguments (cf. e.g. [FK05b, Lemmas 6.2 and 6.3]) that $H_i(X(L_j); Q(\Lambda)^d) = 0$ for $j = \{1, \dots, s, \infty\}$ and for all i.

We have to show that $H_i(\partial X(\mathcal{C}); Q(\Lambda)^d) = 0$ for all i. Let $F_i := \mathcal{C}_i \setminus (\cup \mathrm{int}(\mathbb{B}^4_i)) \cap \mathcal{C}_i$ for $i = 1, \dots, r$. Then $F_1, \dots, F_r$ are the connected components of F. For $i \in \{1, \dots, r\}$ we have $F_i \subset \mathcal{C}_i$, in particular $\psi : H_1(F_i \times S^1) \to H_1(X(\mathcal{C})) \to \mathbb{Z}^m$ factors through $H_1(\mathcal{C}_i \times S^1)$. Since $\psi(\mu_i)$ is non–trivial we can apply the argument of the proof of [CF05, Theorem 5.6] to show that

$$\tau^\alpha(F_i \times S^1) = (\mathrm{id} - \alpha(\mu_i))^{-\chi(F_i)}.$$

In particular $H_*(F_i \times S^1, Q(\Lambda)^d) = 0$.

Denote the components of T_i by $T_i^1, \dots, T_i^{o_i}$. Clearly ψ is non–trivial on any T_i^j. Using the standard cell decomposition of the torus we can easily see that $H_*(T_i^j, Q(\Lambda)^d) = 0$ and in fact $\tau^\alpha(T_i) = 1 \in K_1(Q(\Lambda))/ \pm \alpha(\pi_1(X(\mathcal{C})))$ (c.f. e.g. [Ki96, Proposition 4.4]). The first statement of Lemma 5.2 now follows from the Meyer–Vietoris sequence for $\partial X(\mathcal{C})$ for the above decomposition of $\partial X(\mathcal{C})$.

The proof of Theorem 5.4 builds on the above discussion. By equation (1) we have $\chi(F_i) + s_i = \chi(\mathcal{C}_i)$. It therefore follows from the Meyer–Vietoris sequence and from

[Tu01, Theorem 3.4] that

$$\tau^\alpha(\partial X(\mathcal{C})) = \prod_{i=1}^{r} (\mathrm{id} - \alpha(\mu_i))^{s_i - \chi(\mathcal{C}_i)} \cdot \prod_{i\in\{1,\dots,s,\infty\}} \tau^\alpha(X(L_i)).$$

Now let $\mathcal{B}$ be any basis for $H_2(X(\mathcal{C}); Q(\Lambda)^d)$. We equip $\mathrm{Hom}_{Q(\Lambda)}(H_2(X(\mathcal{C}), Q(\Lambda)^d), Q(\Lambda))$ with the corresponding dual basis. Let $\mathcal{B}'$ be the corresponding basis of $H_2(X(\mathcal{C}), \partial X(\mathcal{C}); Q(\Lambda)^d)$ under the isomorphisms

$$H_2(X(\mathcal{C}), \partial X(\mathcal{C}); Q(\Lambda)^d) \cong H^2(X(\mathcal{C}), (Q(\Lambda)^d)') \cong \mathrm{Hom}_{Q(\Lambda)}(H_2(X(\mathcal{C}), Q(\Lambda)^d), Q(\Lambda)).$$

Note that with the bases $\mathcal{B}$ and $\mathcal{B}'$ the long exact sequence H

$$\dots \to H_i(\partial X(\mathcal{C}); Q(\Lambda)^d) \to H_i(X(\mathcal{C}); Q(\Lambda)^d) \to H_i(X(\mathcal{C}), \partial X(\mathcal{C}); Q(\Lambda)^d) \to \dots$$

becomes a based acyclic complex. By [Mi66, Theorem 3.2] we now have the following equality in $K_1(Q(\Lambda))/\pm\alpha(\pi_1(X))$

$$\tau^\alpha(X(\mathcal{C}), \mathcal{B}) = \tau^\alpha(X(\mathcal{C}), \partial X(\mathcal{C}), \mathcal{B}') \cdot \tau^\alpha(\partial X(\mathcal{C})) \cdot \tau(H).$$

But it follows immediately from the definitions that $\tau(H) = \det(I(\mathcal{B}))$. It remains to show that

$$\tau^\alpha(X(\mathcal{C}), \partial X(\mathcal{C}), \mathcal{B}') = \overline{\tau^\alpha(X(\mathcal{C}), \mathcal{B})}^{-1}.$$

But this follows from an argument as in the proof of [Tu01, Theorem 14.1 and Corollary 14.2]. Note that this only works because $\mathcal{B}'$ corresponds to $\mathcal{B}$ under Poincaré duality.

9. Examples of unitary ψ–compatible representations

9.1. **Skew fields of group rings.** A group G is called locally indicable if for every finitely generated subgroup $U \subset G$ there exists a non–trivial homomorphism $U \to \mathbb{Z}$.

Theorem 9.1. *Let G be a locally indicable and amenable group and let R be a subring of $\mathbb{C}$. Then $R[G]$ is an Ore domain, in particular it embeds in its classical right ring of quotients $\mathbb{K}(G)$.*

It follows from [Hi40] that $R[G]$ has no zero divisors. The theorem now follows from [Ta57] or [DLMSY03, Corollary 6.3].

A group G is called poly–torsion–free–abelian (PTFA) if there exists a filtration

$$1 = G_0 \subset G_1 \subset \dots \subset G_{n-1} \subset G_n = G$$

such that G_i/G_{i-1} is torsion free abelian. It is well–known that PTFA groups are amenable and locally indicable (cf. [St74]). The group rings of PTFA groups played an important role in [COT03], [Co04] and [Ha05].

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