Symplectic manifolds and fibered 3–manifolds

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Symplectic structures

Let W be a 4–manifold. A 2–form ω is called a symplectic structure on W if $d\omega = 0$ and if ω is non–degenerate, i.e. $\omega \wedge \omega \neq 0$ everywhere.

Question 1. When does $S^1 \times N^3$ support a symplectic structure?

Theorem. (Thurston 1976) If N fibers over S^1, then $S^1 \times N$ is symplectic.

Proof. Let $p : N \to S^1$ be a fibration. Define $\theta = p^*(dt)$. We can find a metric with respect to which θ is harmonic. We denote the Hodge dual of θ by $*\theta$.
Then

$$\omega = ds \wedge \theta + *\theta$$

is a symplectic form on $S^1 \times N$ since

$$\omega \wedge \omega = 2ds \wedge \theta \wedge *\theta.$$

Conjecture. $S^1 \times N$ is symplectic if and only if N fibers over S^1.

First fiberedness theorem.

Let ω be a symplectic structure on a 4–manifold, then let $K(\omega) \in H^2(W;\mathbb{Z})$ be the associated canonical class (defined as c_1 of the tangent bundle equipped with an almost complex structure compatible with ω).

Theorem A (F–Vidussi) If $S^1 \times N$ is symplectic with trivial canonical class, then N fibers over S^1.

By a result of Kronheimer a symplectic $S^1 \times N$ has trivial canonical class if and only if N has vanishing Thurston norm, e.g. if N is 0–framed surgery on a knot.

We now only consider the case $b_1(N) \geq 2$. Given a 3–manifold N we denote its multivariable Alexander polynomial by $\Delta_N \in \mathbb{Z}[t_1^{-1}, t_1, \ldots, t_b, t_b^{-1}]$.

Theorem. If $S^1 \times N$ is symplectic with trivial canonical class, then for any finite cover $\tilde{N}$ of N we have

$$\Delta_{\tilde{N}} = \pm 1.$$

Proof.
(1) Any finite cover is symplectic with trivial canonical class.
(2) By Taubes $SW_{S^1 \times \tilde{N}} = \pm 1$.
(3) By Meng–Taubes $\Delta_{\tilde{N}} = \pm 1$.

The following now implies Theorem A.

Theorem A'. If for any finite cover $\tilde{N}$ of N we have

$$\Delta_{\tilde{N}} = \pm 1,$$

then N fibers over S^1.

Proof. It follows from Turaev that for any prime p we have

$$\Delta_{\tilde{N}} = \pm 1 \Rightarrow b_1(\tilde{N}, \mathbb{F}_p) \leq 3.$$

We will show that if N does not fiber, then there exists $\tilde{N}$ with $b_1(\tilde{N}, \mathbb{F}_p) > 3$.

First note that if N is covered by a torus bundle, then N is already a torus bundle.

If N is not fibered we consider the following three cases.
(1) If N contains an incompressible torus, then by Kojima and Luecke there exists $\tilde{N}$ with $b_1(\tilde{N}, \mathbb{Z}) > 3$.
(2) If N is Seifert fibered without an incompressible torus, then $b_1(N) \leq 1$.
(3) If N is hyperbolic, then by the Lubotzky alternative either there exists $\tilde{N}$ with $b_1(\tilde{N}, \mathbb{F}_p) > 3$ or $\pi_1(N)$ is virtually soluble (but the latter would imply that a cover $\tilde{N}$ is a torus bundle)

This completes the proof of Theorems A' and A.

Second fiberedness theorem.

Given a group π we say that a subgroup $A \subset \pi$ is separable if for any $g \in \pi \setminus A$ there exists a homomorphism $\alpha : \pi \to G$, G a finite group, such that $\alpha(g) \notin \alpha(A)$.

Given a symplectic form ω on $S^1 \times N$ we can assume that $[\omega]$ lies in $H^2(S^1 \times N; \mathbb{Z})$. Under the Künneth decomposition

$$\begin{aligned} H^2(S^1 \times N; \mathbb{Z}) &= H^1(S^1; \mathbb{Z}) \otimes H^1(N; \mathbb{Z}) \oplus H^2(N; \mathbb{Z}) \\ &= H^1(N; \mathbb{Z}) \oplus H^2(N; \mathbb{Z}) \end{aligned}$$

we denote the component of $[\omega]$ in $H^1(N; \mathbb{Z})$ by $\phi = \phi(\omega)$. After rescaling if necessary we can assume that ϕ is primitive.

We say (N, ϕ) fibers if there exists a fibration $p : N \to S^1$ with $\phi = p^*(1)$.

Theorem B (F–Vidussi). Assume $S^1 \times N$ is symplectic. If $\phi(\omega) \in H^1(N;\mathbb{Z})$ is dual to an incompressible connected surface Σ such that $\pi_1(\Sigma) \subset \pi_1(N)$ is separable, then (N, ϕ) fibers over S^1.

Remark.
(1) Torus subgroups are separable, hence we get Theorem A back.
(2) Thurston conjectured that any f.g. subgroup of a hyperbolic 3–manifold is separable.
(3) Using work of E. Hamilton we can also show that for a graph manifold N we have $S^1 \times N$ is symplectic iff N fibers.

For a finite cover $p : \tilde{N} \to N$ we denote $p^*(\phi)$ by ϕ again. Given $(\tilde{N}, \phi)$ we consider the Alexander module $H_1(\tilde{N}, \mathbb{Z}[t^{\pm 1}])$ and the corresponding Alexander polynomial $\Delta_{\tilde{N},\phi} \in \mathbb{Z}[t^{\pm 1}]$.

Theorem. Assume $S^1 \times N$ is symplectic. Then $\Delta_{\tilde{N},\phi} \neq 0$ for any finite cover $\tilde{N}$.

Proof. (1) Taubes: The SW–invariants of $S^1 \times \tilde{N}$ are non–zero.
(2) Meng–Taubes: The multivariable Alexander polynomial $\Delta_{\tilde{N}} \neq 0$.
(3) With more care, $\Delta_{\tilde{N},\phi}$ (which is a specialization of $\Delta_{\tilde{N}}$) is non–zero as well.

Theorem B'. Assume $\Delta_{\tilde{N},\phi} \neq 0$ for any finite cover $\tilde{N}$. If $\phi \in H^1(N;\mathbb{Z})$ is dual to an incompressible connected surface Σ such that $\pi_1(\Sigma) \subset \pi_1(N)$ is separable, then (N,ϕ) fibers over S^1.

Remark. Since $\Delta_{N,\phi} \neq 0$ we can, by McMullen, always find an connected incompressible surface Σ dual to ϕ.

Proof. If $\Delta_{\tilde{N},\phi} \neq 0$ for any finite cover $\tilde{N}$, then one can use $\pi_1(N)$ residually finite to show that N is irreducible.

We have two embeddings

$$i_\pm : \Sigma \to N \setminus \Sigma \times (-1,1).$$

Since Σ is Thurston norm minimizing we know that $i_\pm : \pi_1(\Sigma) \to \pi_1(N \setminus \Sigma \times (-1,1))$ is injective. By Stallings' theorem Σ is a fiber if $\pi_1(\Sigma) \to \pi_1(N \setminus \Sigma)$ is an isomorphism.

What does our condition on Alexander polynomials tell us?

Given a homomorphism $\alpha : \pi_1(N) \to G$ to a finite group we get a finite cover $p : \tilde{N} \to N$. We let $\tilde{\Sigma} = p^{-1}(\Sigma)$ and $\widetilde{N \setminus \Sigma} = p^{-1}(N \setminus \Sigma)$.

Consider the long exact sequence

$$\begin{array}{lcl} & & H_1(\tilde{N}; \mathbb{Z}[t^{\pm 1}]) \\ \to H_0(\tilde{\Sigma}) \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{i_- t - i_+} & H_0(\widetilde{N \setminus \Sigma}) \otimes \mathbb{Z}[t^{\pm 1}] \\ \quad \to H_0(\tilde{N}; \mathbb{Z}[t^{\pm 1}]) & & \end{array}$$

The condition on $\Delta_{\hat{N},\phi}$ ensures that

$\mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_1(\tilde{N}) \otimes \mathbb{Z}[t^{\pm 1}]) = \mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\tilde{N}) \otimes \mathbb{Z}[t^{\pm 1}]) = 0.$

Hence

$\mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0 \tilde{\Sigma} \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]) = \mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\widetilde{N \setminus \Sigma}) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}])$

which implies that

$$\mathrm{rk}_{\mathbb{Z}}(H_0(\tilde{\Sigma}; \mathbb{Z})) = \mathrm{rk}_{\mathbb{Z}}(H_0(\widetilde{N \setminus \Sigma}))$$

But this implies that

$$\frac{|G|}{|\mathrm{Im}\{\pi_1(\Sigma) \to G\}|} = \frac{|G|}{|\mathrm{Im}\{\pi_1(N \setminus \Sigma) \to G\}|}.$$

We showed that if $\Delta_{\tilde{N},\phi} \neq 0$ for any finite cover $\tilde{N}$, then for any $\alpha : \pi_1(N) \to G$ we have

$$\mathrm{Im}\{\pi_1(\Sigma) \to G\} = \mathrm{Im}\{\pi_1(N \setminus \Sigma) \to G\}.$$

On the other hand, if Σ is not a fiber, then it follows from Stallings' theorem that $\pi_1(\Sigma) \neq \pi_1(N \setminus \Sigma)$. Therefore by separability of $\pi_1(\Sigma) \subset \pi_1(N)$ we can find $\alpha : \pi_1(N) \to G$ with

$$\mathrm{Im}\{\pi_1(\Sigma) \to G\} \neq \mathrm{Im}\{\pi_1(N \setminus \Sigma) \to G\}.$$

This concludes the proof of Theorems B' and B.

Manifolds with free S^1–action.

Assume that W is a 4–manifold with free circle action and orbit space N. We have a map $p_* : H^2(W;\mathbb{R}) \to H^1(N;\mathbb{R})$ (integration along the fiber).

Theorem C (F–Vidussi). Let W be a symplectic 4–manifold with free circle action such that the orbit space N is a graph manifold or that W has trivial canonical class. Then TFAE
(1) $c \in H^2(W;\mathbb{R})$ can be represented by a symplectic form.
(2) $c^2 \neq 0 \in H^4(W;\mathbb{R})$ and $p_*(c) \in H^1(N;\mathbb{R})$ can be represented by a non–degenerate closed 1–form.