L^2–invariants and commensurability

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Hilbert Γ–modules

For a (countable) group Γ we have the Hilbert space

$$l^2(\Gamma) = \{\varphi : \Gamma \to \mathbb{C} | \sum_{g\in\Gamma} |\varphi(g)|^2 < \infty\}.$$

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for some Hilbert space H with trivial Γ–action.
Example. Let Γ a finite group, $V = \mathbb{C}$ with the trivial Γ–action, then consider

$$\begin{array}{rcl} V = \mathbb{C} & \to & \mathbb{C} \otimes l^2(\Gamma) = \mathbb{C} \otimes \mathbb{C}[\Gamma] = \mathbb{C}[\Gamma] \\ 1 & \mapsto & \frac{1}{\sqrt{|\Gamma|}} \sum_{g\in\Gamma} g. \end{array}$$

L^2–dimension

Definition. Given a Hilbert Γ–module V pick an embedding $V \subset H \otimes l^2(\Gamma)$ and an orthonormal basis $\{v_i\}$ for H. Let

$$\Pi : H \otimes l^2(\Gamma) \to V$$

be the orthogonal projection, then

$$\dim_\Gamma(V) := \sum_i \langle \Pi(v_i \otimes e), v_i \otimes e \rangle.$$

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Example 1. Let $V = (l^2(\Gamma))^n$. We have $V = \mathbb{C}^n \otimes l^2(\Gamma)$. Let v_i be an ONB for $\mathbb{C}^n$. Then

$$\dim_\Gamma((l^2(\Gamma))^n) = \sum_{i=1}^n \langle v_i \otimes e, v_i \otimes e \rangle = n.$$

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Example 2. Let $\Gamma = \{g_1, \dots, g_n\}$ finite and $V = \mathbb{C}$ with trivial Γ–action. We have

$$\begin{array}{rcl} V = \mathbb{C} & \to & \mathbb{C} \otimes \mathbb{C}[\Gamma] = \mathbb{C}[\Gamma] \\ 1 & \mapsto & \frac{1}{\sqrt{|\Gamma|}} \sum g_i. \end{array}$$

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Properties: Restriction and Induction

Restriction. Let $V \subset H \otimes l^2(\Gamma)$ be a Hilbert Γ–module and $\tilde{\Gamma} \subset \Gamma$ a finite index subgroup. Then V is also a $\tilde{\Gamma}$–module via

$$V \subset H \otimes l^2(\Gamma) \cong H \otimes \mathbb{C}[\Gamma/\tilde{\Gamma}] \otimes l^2(\tilde{\Gamma})$$

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Induction. If $\Gamma \subset \tilde{\Gamma}$ and V is a Γ–module, then

$$\overline{V \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[\tilde{\Gamma}]}$$

is a $\tilde{\Gamma}$–module via

$$\overline{V \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[\tilde{\Gamma}]} \subset H \otimes l^2(\tilde{\Gamma})$$

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$$\dim_{\tilde{\Gamma}}(\overline{V \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[\tilde{\Gamma}]}) = \dim_{\Gamma}(V).$$

L^2–Betti numbers

Let X be a finite CW–complex together with a homomorphism $\varphi : \pi_1(X) \to \Gamma$. Let $\tilde{X}$ be the φ–cover of X. Then

$$C_*(X; l^2(\Gamma)) = C_*(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} l^2(\Gamma)$$

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which is a Hilbert Γ–module. Now define

$$b_p^{(2)}(X; l^2(\Gamma)) = \dim_{\Gamma} H_p(X; l^2(\Gamma)),$$

the p–th L^2–Betti number of (X, φ).

Properties of L^2–Betti numbers

Assume we have $(X, \varphi : \pi_1(X) \to \Gamma)$.

1. If $\Gamma = \{e\}$, then $b_p^{(2)}(X, \varphi) = b_p(X)$.

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3. If Γ is a subgroup of a group $\tilde{\Gamma}$, then
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4. If $\tilde{X}$ is a finite cover of X of order n such that $\mathrm{Ker}(\varphi) \subset \pi_1(\tilde{X})$, then
$$b_p^{(2)}(\tilde{X}, \pi_1(\tilde{X}) \to \Gamma) = n \cdot b_p^{(2)}(X, \pi_1(X) \to \Gamma).$$
(Restriction)

Commensurability (the good news)

Two 3–manifolds are called commensurable if they have diffeomorphic finite covers.

Observation. If an n_1–fold cover of M_1 is diffeomorphic to an n_2–fold cover of M_2, then

$$n_1 \cdot b_p^{(2)}(M_1, id) = n_2 \cdot b_p^{(2)}(M_2, id).$$

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Proof. Let $\tilde{M}$ be the common cover, then

$$\begin{aligned} n_i b_p^{(2)}(M_i, id) &= b_p^{(2)}(\tilde{M}, \pi_1(\tilde{M}) \to \pi_1(M)) \\ &= b_p^{(2)}(\tilde{M}, \pi_1(\tilde{M})), \end{aligned}$$

by Restriction and Induction.

Commensurability (the bad news)

Unfortunately the L^2–Betti numbers give meaningless commensurability invariants.

Atiyah conjecture. Let M be a closed manifold, then $b_p^{(2)}(M, \varphi)$ is rational for any $\varphi : \pi_1(M) \to \Gamma$.

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Atiyah conjecture. Let M be a closed manifold, then $b_p^{(2)}(M, \varphi)$ is rational for any $\varphi : \pi_1(M) \to \Gamma$.

More bad news: If M is a prime 3–manifold with infinite $\pi_1(M)$ and with empty or toroidal boundary, then $b_p^{(2)}(M, \mathrm{id}) = 0$.

The L^2–torsion

The vanishing of the L^2–Betti numbers is perhaps even a blessing in disguise since
"vanishing homology implies existence of Reidemeister torsion"
Indeed, "if $b_*^{(2)}(M,\varphi)=0$, [and if the Novikov–Shubin invariant is positive] then there exists the L^2–torsion $\tau(M,\varphi)\in\mathbb{R}$".

Properties of the L^2–torsion

Assume we have $(X, \varphi : \pi_1(X) \to \Gamma)$.

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Immediate consequence: If an n_1–fold cover of M_1 is diffeomorphic to an n_2–fold cover of M_2, then
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So the L^2–torsions of commensurable 3–manifolds are rationally dependent.
But the invariant $\tau(M, \mathrm{id})$ equals the sum of the volumes of the hyperbolic pieces of M (up to a factor of $-\frac{1}{6\pi}$).

The von Neumann ρ–invariant

Let M be a closed oriented 3–manifold and $\varphi : \pi_1(M) \to \Gamma$ a homomorphism, then the von Neumann ρ–invariant $\rho(M, \varphi) \in \mathbb{R}$ is defined. The von Neumann ρ–invariant is the L^2–signature defect, i.e. the difference between the L^2–signature and the ordinary signature of a bounding 4–manifold.

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But: its computation is completely beyond the reach of current methods.

Cyclic commensurability of knots

We now restrict to cyclic commensurability of knots. For a knot $K \subset S^3$ we denote the n–fold cyclic cover of $S^3 \setminus \nu K$ by $X(K)_n$. We denote the 0–surgery on K by $M(K)$.

Theorem. If $X(K_1)_{n_1} = X(K_2)_{n_2}$ and furthermore $b_1(X(K_i)_{n_i}) = 1$, then

$$\begin{aligned} n_1 \cdot \tau(X(K_1), \mathbb{Z}) &= n_2 \cdot \tau(X(K_2), \mathbb{Z}) \\ n_1 \cdot \rho(M(K_1), \mathbb{Z}) &= \pm n_2 \cdot \rho(M(K_2), \mathbb{Z}) \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

This follows from Induction and Restriction as before since a cover with $b_1 = 1$ has a unique (up to sign) homomorphism to $\mathbb{Z}$.

Let K be a knot and A a Seifert matrix. Then

$$\rho(M(K),\mathbb{Z}) = \int_{z\in S^1} \operatorname{sign}(A(1-z)+A^t(1-z^{-1})).$$

This number depends on the zeroes of $\Delta_K(t)=\det(At-A^t)$ on the unit circle and the twisted signatures of K.
Furthermore $\tau(X(K),\mathbb{Z})$ equals the Mahler measure of $\Delta_K(t)$, i.e.

$$\tau(X(K),\mathbb{Z}) = \int_{z\in S^1} \ln|\Delta_K(z)|.$$

Given a Seifert matrix A for K we have

$$\begin{aligned}\Delta_K(t) &= \det(At-A^t)\\ &= (1-t)^{-1}\det(A(t-1)+A^t(t^1-1)).\end{aligned}$$

Hence

$$\tau(X(K),\mathbb{Z}) = \int_{z\in S^1} \ln|\det(A(1-z)+A^t(1-\bar{z}))|.$$

(here we use that the Mahler measure of $t-1$ equals 1). This

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Using $z = t + t^{-1}$ we can compute the zeros of $\Delta_{K_i}(t)$. We get

$$\tau(X(K_1), \mathbb{Z}) = \ln\left(\frac{1}{4}\left(7 + \sqrt{13} + \sqrt{46 + 14\sqrt{13}}\right)\right)$$

$$\tau(X(K_1), \mathbb{Z}) = \ln\left(\frac{1}{4}\left|-7 - 3\sqrt{13} - \sqrt{150 + 42\sqrt{13}}\right|\right)$$

It is now straightforward to check that indeed

$$6\tau(X(K_2), \mathbb{Z}) = 8\tau(X(K_1), \mathbb{Z}),$$

Examples

The knots $K_1 = 9_{48}$ and $K_2 = 12^n_{642}$ are cyclically commensurable, in fact $X(K_1)_8 = X(K_2)_6$. (examples by W. Neumann).We have

$$\begin{aligned} \Delta_{K_1}(t) &= t^4 - 7t^3 + 11t^2 - 7t + 1, \\ \Delta_{K_2}(t) &= t^4 + 7t^3 - 15t^2 + 7t + 1. \end{aligned}$$

Using $z = t + t^{-1}$ we can compute the zeros of $\Delta_{K_i}(t)$. We get

$$\tau(X(K_1), \mathbb{Z}) = \ln\left(\frac{1}{4}\left(7 + \sqrt{13} + \sqrt{46 + 14\sqrt{13}}\right)\right)$$

$$\tau(X(K_1), \mathbb{Z}) = \ln\left(\frac{1}{4}\left|-7 - 3\sqrt{13} - \sqrt{150 + 42\sqrt{13}}\right|\right)$$

It is now straightforward to check that indeed

$$6\tau(X(K_2), \mathbb{Z}) = 8\tau(X(K_1), \mathbb{Z}),$$

We also compute

$$\begin{aligned} \rho(M(K_1), \mathbb{Z}) &\approx 1.645123.... \\ \rho(M(K_2), \mathbb{Z}) &\approx 1.806503.... \end{aligned}$$