THE DECATEGORIFICATION OF SUTURED FLOER HOMOLOGY

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Abstract. We define a torsion invariant for balanced sutured manifolds and show that it agrees with the Euler characteristic of sutured Floer homology. The torsion is easily computed and shares many properties of the usual Alexander polynomial.

1. Introduction

Sutured Floer homology is an invariant of balanced sutured manifolds introduced by the second author [Ju06]. It is an offshoot of the Heegaard Floer homology of Ozsváth and Szabó [OS04a], and contains knot Floer homology [OS04c, Ra03] as a special case. The Euler characteristics of these homologies are torsion invariants of three–manifolds; for example, the Euler characteristic of the Heegaard Floer homology HF^+ is given by Turaev's refined torsion [Tu97, Tu02] and the Euler characteristic of knot Floer homology is given by the Alexander polynomial. In this paper, we investigate the torsion invariant which is the Euler characteristic of sutured Floer homology.

To make a more precise statement, we recall some basic facts about sutured Floer homology. Given a balanced sutured manifold (M, γ) and a relative Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$, the sutured Floer homology is an finitely generated abelian group $SFH(M, \gamma, \mathfrak{s})$. *A priori*, $SFH(M, \gamma, \mathfrak{s})$ is relatively $\mathbb{Z}/2$ graded; to fix an absolute $\mathbb{Z}/2$ grading, we must specify a homology orientation ω of the pair $(M, R_-(\gamma))$.

Following Turaev, we define a torsion invariant $T_{(M,\gamma)}$ for weakly balanced sutured manifolds, which is essentially the maximal abelian torsion of the pair $(M, R_-(\gamma))$. Our construction is generally very close to Turaev's, but we use handle decompositions of sutured manifolds in place of triangulations. This makes it easier to define a correspondence between lifts and Spin^c structures. $T_{(M,\gamma)}$ is a function which assigns a number to each $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$. The torsion function is well-defined up to a global factor of ± 1; again, to fix the sign, we must specify a homology orientation ω of $(M, R_-(\gamma))$.

Theorem 1. *Let (M, γ) be a balanced sutured manifold. Then for any $\mathfrak{s} \in Spin^c(M, \gamma)$ and homology orientation ω of $H_1(M, R_-(\gamma))$,*

$$T_{(M,\gamma,\omega)}(\mathfrak{s}) = \chi(SFH(M, \gamma, \mathfrak{s}, \omega)).$$

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Related invariants have also been studied by Goda and Sakasai [GS08] in the case of homology products, and by Wehrli (in preparation).

It is often convenient to combine the torsion invariants of (M,γ) into a single generating function. To do so, we fix an affine isomorphism $\iota : \mathrm{Spin}^c(M,\gamma) \to H_1(M)$, and write

$$\tau(M,\gamma) = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)} T_{(M,\gamma)}(\mathfrak{s})[\iota(\mathfrak{s})].$$

$\tau(M,\gamma)$ is an element of the group ring $\mathbb{Z}[H_1(M)]$. It is well defined up to multiplication by elements of the form $\pm[h]$ for $h \in H_1(M)$. This invariant is best thought of as a generalization of the Alexander polynomial to sutured manifolds. Many classical properties of the Alexander polynomial have analogues for $\tau(M,\gamma)$.

For example, if Y is a three-manifold with toroidal boundary, then its Alexander polynomial $\Delta(Y)$ is by definition an invariant of $\pi_1(Y)$. Similarly, we have

Proposition 2. *If $H_1(M)$ is free then $\tau(M,\gamma)$ is determined by the map*

$$\pi_1(R_-(\gamma)) \to \pi_1(M).$$

A well-known theorem of McMullen [Mc02] says that $\Delta(Y)$ gives a lower bound on the Thurston norm of Y. There is a natural extension of the Thurston norm to sutured manifolds due to Scharlemann [Sc89]. For $\alpha \in H_2(M,\partial M)$, let $x^s(\alpha)$ denote its sutured Thurston norm.

Theorem 3. *Let $\mathcal{S} \subset Spin^c(M,\gamma)$ be the support of $SFH(M,\gamma)$. Then*

$$\max_{\mathfrak{s},\mathfrak{t}\in\mathcal{S}} \langle \mathfrak{s}-\mathfrak{t}, \alpha\rangle \leq x^s(\alpha).$$

As an immediate consequence, we have

Corollary 4. *Let $S \subset H_1(M)$ be the support of $\tau(M,\gamma)$. Then*

$$\max_{s,t\in S}(s-t)\cdot\alpha \leq x^s(\alpha).$$

In analogy with the situation for closed manifolds, it is tempting to guess that one always has equality in Theorem 3, but an example of Cantwell and Conlon [CC06] shows that this is not the case.

The unit Thurston norm ball of a link complement is always centrally symmetric. We demonstrate that S and $\mathcal{S}$ can be centrally asymmetric in general.

Another well-known property of the Alexander polynomial is that $\Delta_K(1) = 1$ for any $K \subset S^3$. This may be generalized as follows. Let $p_* : H_1(M) \to H_1(M,R_-)$ be the natural map. For any group G there is a canonical element $I_G \in \mathbb{Z}[G]$ defined by

$$I_G = \begin{cases} \sum_{g\in G} g & |G| < \infty, \\ 0 & |G| = \infty. \end{cases}$$

Proposition 5. $p_*(\tau(M,\gamma)) = \pm I_{H_1(M,R_-)}$.

For $K \subset S^3$ we have $\Delta_K(t) = \tau(M,\gamma)$ for some (M,γ) with $H_1(M,R_-) = 0$. In this case $I_{H_1(M,R_-)} = 1$, and we recover the fact that $\Delta_K(1) = \pm 1$.

Definition. A balanced sutured manifold (M,γ) is a *sutured L-space* if $SFH(M,\gamma)$ is torsion-free and is supported in a single $\mathbb{Z}/2$ homological grading.

Examples of such manifolds are easy to find; *e.g.* if $\Sigma \subset S^3$ is a Seifert surface of an alternating knot, then the complement of a neighborhood of Σ is a sutured L-space. As a corollary of Proposition 5, we have

Corollary 6. *If (M,γ) is a sutured L-space, then for each $\mathfrak{s} \in Spin^c(M,\gamma)$, the group $SFH(M,\gamma,\mathfrak{s})$ is either trivial or isomorphic to $\mathbb{Z}$.*

In the last section we compute the torsion for a variety of examples, including pretzel surface complements and for all sutured manifolds complementary to Seifert surfaces of knots with ≤ 9 crossings. In all these examples, the sutured Floer homology is easily determined from the torsion. As an application, we give a simple example of a phenomenon first demonstrated by Goda [Go94]: there exist sutured manifolds whose total space is a handlebody, but which are not disk decomposable.

The paper is organized as follows. In section 2, we recall the relevant facts about sutured Floer homology. Section 3 contains the definition of the torsion, and section 4 explains how to compute it using Fox calculus. Section 5 contains the proof of Theorem 1. In section 6, we discuss some algebraic properties of the torsion, including Proposition 5. Section 7 discusses the relation between SFH and the sutured Thurston norm. Finally, section 8 is devoted to examples.

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Convention. All 3–manifolds are understood to be oriented and compact. All homology groups are with integral coefficients unless otherwise specified.

2. Sutured Floer homology

In this section, we recall some relevant facts about sutured manifolds and sutured Floer homology. For full details, we refer the reader to [Ju06].

2.1. **Balanced sutured manifolds.** For our purposes, a *sutured manifold* (M,γ) is a compact oriented 3-manifold M with boundary together with a set $s(\gamma)$ of oriented simple closed curves on ∂M, called *sutures*. We fix a closed tubular neighborhood $\gamma \subset \partial M$ of the sutures: γ is a union of pairwise disjoint annuli. Finally, we require that each component of $R(\gamma) = \partial M \setminus \text{Int}(\gamma)$ be oriented, and that this orientation is coherent with respect to $s(\gamma)$, i.e., if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma;\mathbb{Z})$ as some suture. Define $R_+(\gamma)$ (respectively $R_-(\gamma)$) to be those components of $R(\gamma)$ whose orientation is consistent (opposite to) the orientation on ∂M induced by M.

The notion of a sutured manifold is due to Gabai [Ga83]; the description given above is slightly less general than Gabai's, in that we have omitted the possibility of toroidal sutures.

Example 2.1. Let R be a compact oriented surface with no closed components. Then there is an induced orientation on ∂R. Let $M = R \times [-1, 1]$, define $\gamma = \partial R \times [-1, 1]$, finally put $s(\gamma) = \partial R \times 0$. The pair (M, γ) is called a *product sutured manifold.*

Example 2.2. Suppose Y is a closed three-manifold. Let $M = Y \setminus B^3$, and let $s(\gamma)$ be a neighborhood of the equator in B^3. We denote the resulting sutured manifold by $Y(1)$.

Example 2.3. Suppose that L is a link in the three-manifold Y. Then the sutured manifold $Y(L) = (M, \gamma)$ is given by $M = Y \setminus N(L)$, and for each component L_0 of L we take $s(\gamma) \cap \partial N(L_0)$ to be two oppositely oriented meridians of L_0.

Example 2.4. Let K be a null-homologous knot in a closed three-manifold M, and let R be a Seifert surface for K. If $U \simeq \mathrm{Int}(R) \times (-1, 1)$ is a regular neighborhood of $\mathrm{Int}(R)$, then the complement $M \setminus U$ is a sutured manifold with a single suture $\gamma = \partial R \times [-1, 1]$. The curve $s(\gamma)$ is a parallel copy of K. Then (M, γ) is called the sutured manifold complementary to R, and is denoted by $M(R)$.

Definition 2.1. [Ju06] A *weakly balanced sutured manifold* is a sutured manifold (M, γ) such that for each component M_0 of M we have

$$\chi(R_+(\gamma) \cap M_0) = \chi(R_-(\gamma) \cap M_0).$$

A *balanced sutured manifold* is a weakly balanced sutured manifold (M, γ) such that M has no closed components and the map $\pi_0(\gamma) \to \pi_0(\partial M)$ is surjective. Finally, we say that (M, γ) is *strongly balanced* if it is balanced and for each component V of ∂M we have $\chi(R_+(\gamma) \cap V) = \chi(R_-(\gamma) \cap V)$.

Balanced sutured manifolds were defined in [Ju06]. The examples given above are both balanced. A straightforward argument using Poincaré duality shows that for any sutured manifold we have $2\chi(M) = \chi(R_-(\gamma)) + \chi(R_+(\gamma))$, in particular for a weakly balanced sutured manifold we have

$$\chi(M, R_-(\gamma)) = \chi(M, R_+(\gamma)) = 0.$$

Sutured Floer homology is only defined for balanced sutured manifolds. However, we can define the torsion for any weakly balanced sutured manifold.

2.2. **Spin^c–structures on sutured manifolds.** Suppose that (M, γ) is a sutured manifold. Let v_0 be a nowhere zero vector field along ∂M that points into M along $R_-(\gamma)$, points out of M along $R_+(\gamma)$, and on γ is given by the gradient of the height function $s(\gamma) \times [-1, 1] \to [-1, 1]$.

Definition 2.2. Let v and w be vector fields on M that agree with v_0 on ∂M. We say that v and w are homologous if there is an open ball $B \subset \text{Int}(M)$ such that $v|(M \setminus B)$ is homotopic to $w|(M \setminus B)$ rel ∂M through nowhere zero vector fields. We define $\text{Spin}^c(M, \gamma)$ to be the set of homology classes of nowhere zero vector fields v on M such that $v|\partial M = v_0$.

A priori, this definition appears to depend on the choice of v_0. However, the space of such vector fields is contractible, so there is a canonical identification between equivalance classes coming from different choices of v_0. In the case of a closed, oriented 3–manifold the definition is equivalent to the standard definition given in terms of bundles (cf. [Tu97]).

Lemma 2.3. *$\text{Spin}^c(M, \gamma) \neq \emptyset$ if and only if (M, γ) is weakly balanced. Furthermore, there exists a free and transitive action of $H^2(M, \partial M; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$ on the set $\text{Spin}^c(M, \gamma)$.*

Proof. An analogous argument as in the proof of [Ju08b, Proposition 3.5] implies that $\text{Spin}^c(M, \gamma) \neq \emptyset$ if and only if (M, γ) is weakly balanced. Let $\mathfrak{s}_1, \mathfrak{s}_2 \in \text{Spin}^c(M, \gamma)$. It follows from obstruction theory that $\text{Spin}^c(M, \gamma)$ is an affine space over $H^2(M, \partial M; \mathbb{Z})$, since nowhere zero vector fields can be thought of as sections of the unit sphere bundle STM. Then $\mathfrak{s}_1 - \mathfrak{s}_2$ is the first obstruction to homotoping vector fields representing $\mathfrak{s}_1$ and $\mathfrak{s}_2$.

If v is a representative of $\mathfrak{s}$ and the simple closed curve c represents $h \in H_1(M; \mathbb{Z})$ then an explicit representative of $\mathfrak{s} + h$ can be obtained by Reeb turbularization, which is described in [Tu90, p.639]. □

2.3. **Sutured Floer homology.** We now sketch the construction of $SFH(M, \gamma)$. Our starting point is a Heegaard diagram adapted to the pair (M, γ).

Definition 2.4. A *balanced sutured Heegaard diagram*, in short a balanced diagram, is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where Σ is a compact oriented surface with boundary and $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_d\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_d\}$ are two sets of pairwise disjoint simple closed curves in $\text{Int}(\Sigma)$ such that $\pi_0(\partial\Sigma) \to \pi_0(\Sigma \setminus \bigcup \boldsymbol{\alpha})$ and $\pi_0(\partial\Sigma) \to \pi_0(\Sigma \setminus \bigcup \boldsymbol{\beta})$ are surjective.

Note that the restrictions on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ imply that Σ has no closed components and that the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both linearly independent in $H_1(\Sigma)$.

Every balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ uniquely defines a sutured manifold (M, γ) using the following construction. Let M be the 3-manifold obtained from $\Sigma \times [-1, 1]$ by attaching 2–handles along the curves $\alpha_i \times \{-1\}$, $i = 1, \dots, d$ and $\beta_j \times \{1\}$, $j = 1, \dots, d$. The sutures are defined by taking $\gamma = \partial\Sigma \times [-1, 1]$ and $s(\gamma) = \partial\Sigma \times \{0\}$. Equivalently, (M, γ) can be constructed from the product sutured manifold $R_-(\gamma) \times [-1, 1]$ by first adding d one–handles to $R_-(\gamma) \times \{1\}$, and then d two–handles. The Heegaard surface Σ is the upper boundary of the manifold obtained by adding the one–handles. The

α curves are the belt circles of the one–handles, and the β curves are the attaching circles of the two–handles.

The following proposition combines [Ju06, Proposition 2.9] and [Ju06, Proposition 2.13].

Proposition 2.5. *The sutured manifold corresponding to a balanced diagram is balanced, and for every balanced sutured manifold there exists a balanced diagram defining it.*

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a balanced diagram for (M, γ) then the α and β curves define totally real tori $\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_d$ and $\mathbb{T}_\beta = \beta_1 \times \ldots \times \beta_d$ in the symmetric product $\mathrm{Sym}^d(\Sigma)$. We can suppose that $\bigcup \boldsymbol{\alpha}$ and $\bigcup \boldsymbol{\beta}$ intersect transversally. Then $SFH(M, \gamma)$ is the homology of a chain complex whose generators are the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. More concretely, an element of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a set $\mathbf{x} = \{x_1, \ldots, x_d\}$, where each x_i is in $\alpha_j \cap \beta_k$, and each α and β curve is represented exactly once among the x_i's. Still more concretely, for each permutation $\sigma \in S_d$ we define

$$(\mathbb{T}_\alpha \cap \mathbb{T}_\beta)_\sigma = \{(x_1, \ldots, x_d) \,|\, x_i \in \alpha_i \cap \beta_{\sigma(i)}, i = 1, \ldots, d\}.$$

Then

$$\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \bigcup_{\sigma \in S_d} (\mathbb{T}_\alpha \cap \mathbb{T}_\beta)_\sigma.$$

The differential in the chain complex is defined by counting rigid holomorphic disks in $\mathrm{Sym}^d \Sigma$. Since we are mostly interested in the Euler characteristic of SFH, we will have little need to understand these disks; in fact, the only place they appear is in the proof of Proposition 2.8. For the full definition of the differential, the interested reader is referred to [Ju06].

2.4. **Orientations and Grading.** Next, we consider the homological grading on the sutured Floer chain complex. In its simplest form, this grading is a relative $\mathbb{Z}/2$ grading given by the sign of intersection in $\mathrm{Sym}^d(\Sigma)$ — two generators have the same grading if the corresponding intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ have the same sign. To fix the sign of intersection, or equivalently, to turn this relative $\mathbb{Z}/2$ grading into an absolute one, we must orient the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. (The orientation of Σ, and thus of $\mathrm{Sym}^d(\Sigma)$ is determined by the orientation of M. To be precise, Σ is always oriented as the boundary of the compression body determined by the α curves.)

Choosing an orientation of $\mathbb{T}_\alpha$ is the same as choosing a generator of $\Lambda^d(A)$, where $A \subset H_1(\Sigma)$ is the d-dimensional subspace spanned by the α's. Similarly, an orientation of $\mathbb{T}_\beta$ is specified by a choice of generator for $\Lambda^d(B)$, where B is the subspace of $H_1(\Sigma)$ spanned by the β's. To fix the sign of intersection, we must orient the tensor product $\Lambda^d(A) \otimes \Lambda^d(B)$. The choice of orientation can be expressed in terms of a homology orientation for the pair (M, R_-):

Lemma 2.6. *Specifying an orientation of $\Lambda^d(A) \otimes \Lambda^d(B)$ is equivalent to specifying an orientation of $H_*(M, R_-)$.*

Proof. By Proposition 2.5 the pair (M, γ) is homotopy equivalent to a CW pair (X, Y) such that $X \setminus Y$ is a union of d one-cells and d two-cells. Orienting $H_*(M, R_-)$ is equivalent to orienting $C_*(X, Y)$. This complex is supported in dimensions 1 and 2, with $C_1(X, Y) \cong A$ and $C_2(X, Y) \cong B$. □

2.5. **Generators and Spin^c structures.** An important property of the sutured Floer chain complex is that it decomposes as a direct sum over Spin^c structures. Definition 4.5 of [Ju06] explains how to assign a Spin^c structure $\mathfrak{s}(\mathbf{x})$ to each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ so that if the boundary $d(\mathbf{x}) = \sum a_i \mathbf{y}_i$ ($a_i \neq 0$), then $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\mathbf{y}_i)$ for all i. The exact mechanics of this assignment do not concern us at the moment, but we will need to know how to compute the difference between the Spin^c structures assigned to two generators.

Given $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, pick a path θ along the α_i's from $\mathbf{x}$ to $\mathbf{y}$. More precisely, θ is a singular 1-chain supported on the α's with $\partial\theta = \sum_{i=1}^n y_i - \sum_{i=1}^n x_i$. Similarly choose a path η from $\mathbf{x}$ to $\mathbf{y}$ along the β's. The difference $\theta - \eta$ represents an element of $H_1(\Sigma)$. If θ' is a different path from $\mathbf{x}$ to $\mathbf{y}$ along the α's, the difference $\theta - \theta'$ is a linear combination of the α_i in $H_1(\Sigma)$. Similarly if η' is another path from $\mathbf{x}$ to $\mathbf{y}$ then $\eta' - \eta$ is a linear combination of β's. Thus $\theta - \eta$ represents a well defined element of $H_1(\Sigma)/L$, where L is the subspace spanned by the α's and β's. We write

$$\mathbf{x} - \mathbf{y} = [\theta - \eta] \in H_1(\Sigma)/L \cong H_1(M).$$

Lemma 2.7. [Ju06, Lemma 4.7] *We have* $\mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y}) = \mathbf{x} - \mathbf{y}$ *in* $H_1(M; \mathbb{Z})$.

We can now make a precise statement about the right-hand side of the equation appearing in Theorem 1. Let (M, γ) be a balanced sutured manifold, $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ a relative Spin^c structure, and choose a homology orientation ω for $H_*(M, R_-)$. Then the sutured Floer homology $SFH(M, \gamma, \mathfrak{s}, \omega)$ has a well-defined Euler characteristic. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a balanced diagram for (M, γ), this Euler characteristic can be computed as

$$\chi(SFH(M, \gamma, \mathfrak{s}, \omega)) = \sum_{\{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid \mathfrak{s}(\mathbf{x}) = \mathfrak{s}\}} m(\mathbf{x}),$$

where $m(\mathbf{x})$ denotes the intersection sign $\mathbb{T}_\alpha \cdot \mathbb{T}_\beta$ at $\mathbf{x}$.

In practice, it is convenient to combine the Euler characteristics corresponding to different Spin^c structures into a single generating function, which we view as an element of the group ring $\mathbb{Z}[H_1(M)]$. For this fix an affine isomorphism $\iota : \mathrm{Spin}^c(M, \gamma) \to H_1(M)$ and let

$$\chi(SFH(M, \gamma, \omega)) = \sum_{\{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\}} m(\mathbf{x})[\iota(\mathbf{x})].$$

$\chi(SFH(M, \gamma, \omega))$ is well-defined up to multiplication by an element in $H_1(M)$, viewed as a unit in $\mathbb{Z}[H_1(M)]$.

2.6. **Duality.** Let (M, γ) be a balanced sutured manifold, and denote by $(M, -\gamma)$ the same manifold but with the orientation of the suture $s(\gamma)$ reversed. The effect of this is to reverse the roles of R_+ and R_-: $R_\pm(-\gamma) = R_\mp(\gamma)$. In this subsection we show that the groups $SFH(M, \gamma)$ and $SFH(M, -\gamma)$ are isomorphic, and that they are dual to $SFH(-M, \gamma)$, and $SFH(-M, -\gamma)$. This essentially follows the same way as for ordinary Heegaard Floer homology, though it has not appeared in print before in the case of sutured Floer homology.

If $\mathfrak{s} \in \text{Spin}^c(M, \gamma)$ is represented by a nowhere vanishing vector field v, then $-v$ defines a Spin^c–structure on $(M, -\gamma)$, we denote its homology class by $-\mathfrak{s}$.

Proposition 2.8. *Let (M, γ) be a balanced sutured manifold and let $\mathfrak{s} \in Spin^c(M, \gamma)$. Then*

$$SFH(M, \gamma, \mathfrak{s}) \cong SFH(M, -\gamma, -\mathfrak{s})$$

as relatively graded groups, and hence

$$SFH(-M, \gamma, \mathfrak{s}) \cong SFH(-M, -\gamma, -\mathfrak{s}).$$

Moreover, $SFH(M, \gamma, \mathfrak{s})$ and $SFH(-M, \gamma, \mathfrak{s})$ are the homologies of dual chain complexes.

Proof. First let us fix our orientation conventions. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a sutured diagram defining (M, γ) then the orientation of M is given on $\Sigma \times [-1, 1]$ by taking the product orientation, where $[-1, 1]$ is oriented from -1 to 1. This orientation naturally extends to the attached 2-handles. Then $s(\gamma) = \partial\Sigma \times \{0\}$ is oriented as the boundary of $\Sigma \times \{0\}$, outward normal first. With these choices, $R_\pm(\gamma)$ is obtained from $\Sigma \times \{\pm 1\}$ by doing surgery along the feet of the 2-handles.

Given (M, γ), choose an admissible sutured diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defining it. Then $(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defines $(-M, -\gamma)$, because the product orientation on $\Sigma \times [-1, 1]$ is reversed, furthermore now the suture is $-\partial\Sigma \times \{0\}$.

If we flip the roles of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we get the sutured diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$. Call the sutured manifold defined by it (N, ν). Then we can define a homeomorphism $h\colon N \to M$ which maps $\Sigma \times [-1, 1] \subset N$ to $\Sigma \times [-1, 1] \subset M$ using the formula $h(s, t) = (s, -t)$ and extends to the 2-handles naturally. This map h is orientation reversing from N to M, but preserves the orientation of the sutures. Thus $(N, \nu) = (-M, \gamma)$.

Combining the observations of the previous two paragraphs we see that $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ defines $(M, -\gamma)$. The chain complexes $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$, and $CF(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ all have the same generators, namely $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Let $\mathbf{x}, \mathbf{y}$ be generators of $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ that are connected by a rigid pseudo-holomorphic Whitney disc $u\colon \mathbb{D} \to \text{Sym}^d(\Sigma)$. Then $-u$ is a pseudo-holomorphic disc connecting $\mathbf{x}$ to $\mathbf{y}$ in $CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and also in $CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$, whereas in $CF(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ the intersection points $\mathbf{x}, \mathbf{y}$ are connected by u. Thus

$$CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) = CF(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$$

and

$$CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}) = CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}),$$

and the chain complex $CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is dual to $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. To get the refined statement involving the Spin^c-structures, observe that if $\mathbf{x}$ is a generator of $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathbf{x}'$ is the corresponding generator of $CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$, then $h_*(\mathfrak{s}(\mathbf{x}')) = -\mathfrak{s}(\mathbf{x})$. On the other hand, the Spin^c-structure assigned to $\mathbf{x}$ in $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and in $CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ can be represented by the same vector field on M. □

3. The definition of the torsion function

In this section we first define the maximal abelian torsion of a pair of finite CW–complexes. We then define the torsion invariant for weakly balanced sutured 3–manifolds. Our approach follows closely the ideas of Turaev exposed in [Tu97, Tu98, Tu01, Tu02].

3.1. **Lifts and Euler structures.** Let (X, Y) be a pair of finite k-dimensional CW–complexes with $Y \subset X$ and X connected. We write $H = H_1(X)$ and view H as a multiplicative group. For $0 \leq i \leq k$ denote by $\{c_{i1}, \dots, c_{in_i}\}$ the set of i–cells in $X \setminus Y$. We write $K = \{(i,j) \,|\, 0 \leq i \leq k,\ 1 \leq j \leq n_i\}$. Let $\pi : \hat{X} \to X$ be the universal abelian cover of X and write $\hat{Y} = \pi^{-1}(Y)$.

Definition 3.1. A *lift* l from (X, Y) to $(\hat{X}, \hat{Y})$ is a choice for every $(i,j) \in K$ of a cell $l(i,j) = \hat{c}_{ij}$ in $\hat{X}$ lying over c_{ij}. Note that if l' is any other lift then for every $(i,j) \in K$ there is a $g_{ij} \in H$ such that $l'(i,j) = g_{ij} \cdot \hat{c}_{ij}$. We say that l and l' are equivalent if

$$\prod_{(i,j)\in K} g_{ij}^{(-1)^i} \in H$$

is trivial. We denote the set of equivalence classes of lifts by $\mathrm{Lift}(X, Y)$.

We now define an action of H on $\mathrm{Lift}(X, Y)$. Let $h \in H$ and suppose that $\mathfrak{l} \in \mathrm{Lift}(X, Y)$ is represented by a lift l. Then $h \cdot \mathfrak{l}$ is represented by the lift l' such that $l'(0,1) = h \cdot l(0,1)$ and $l'(i,j) = l(i,j)$ for $(i,j) \in K \setminus \{(0,1)\}$. Note that if $X \neq Y$ then H acts freely and transitively on $\mathrm{Lift}(X, Y)$. In particular, given $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathrm{Lift}(X, Y)$ we get a well-defined element $\mathfrak{l}_1 - \mathfrak{l}_2 \in H$. On the other hand, if $X = Y$ then $|\mathrm{Lift}(X, Y)| = 1$.

Definition 3.2. For each cell c_{ij} pick a point p_{ij} in c_{ij}. An *Euler chain* for (X, Y) is a one–dimensional singular chain θ in X with

$$\partial\theta = \sum_{(i,j)\in K} (-1)^i p_{ij}.$$

Given two Euler chains θ, η we define $\theta - \eta \in H$ to be the homology class of the 1–cycle $\theta - \eta$. Two Euler chains θ, η are called equivalent if $\theta - \eta$ is trivial in H. We call an equivalence class of Euler chains an *Euler structure* and denote the set of Euler structures by $\mathrm{Eul}(X, Y)$.

Note that $\mathrm{Eul}(X,Y) \neq \emptyset$ if and only if $\chi(X,Y) = 0$. Furthermore, if $\mathrm{Eul}(X,Y) \neq \emptyset$ then H acts freely and transitively on $\mathrm{Eul}(X,Y)$.

Definition 3.3. Suppose that $\chi(X,Y) = 0$. Then we define a map

$$E\colon \mathrm{Lift}(X,Y) \to \mathrm{Eul}(X,Y)$$

as follows. Pick a point $\hat{p} \in \hat{X}$. Suppose that $\mathfrak{l} \in \mathrm{Lift}(X,Y)$ and choose a lift l representing $\mathfrak{l}$. For every $(i,j) \in K$ connect $\hat{p}$ and a point $\hat{p}_{ij} \in l(i,j)$ with a path $\hat{\theta}_{ij}$ such that $\partial\hat{\theta}_{ij} = (-1)^i(\hat{p}_{ij} - \hat{p})$. Let $\hat{\theta} = \sum_{(i,j)\in K} \hat{\theta}_{ij}$. Then $\theta = \pi(\hat{\theta})$ is an Euler chain since $\chi(X,Y) = 0$. The Euler structure $\mathfrak{e}$ represented by θ only depends on $\mathfrak{l}$, so we define $E(\mathfrak{l}) = \mathfrak{e}$.

If $X \neq Y$ then the map E is an H-equivariant bijection. If $X = Y$ then $\mathrm{Eul}(X,Y)$ is canonically isomorphic to H, and the image of the unique element of $\mathrm{Lift}(X,Y)$ under E is $0 \in H$.

3.2. **Torsion of CW–complexes.** We continue with the notation from the previous section. Let $\mathfrak{l} \in \mathrm{Lift}(X,Y)$ be a lift represented by l. Furthermore, let φ be a ring homomorphism $\mathbb{Z}[H] \to \mathbb{F}$ to a commutative field $\mathbb{F}$. In this section we recall the definition of the Reidemeister–Turaev torsion $\tau^\varphi(X,Y,\mathfrak{l}) \in \mathbb{F}$.

Consider the chain complex $C_*(X,Y;\mathbb{F}) = C_*(\hat{X},\hat{Y};\mathbb{Z}) \otimes_{\mathbb{Z}[H]} \mathbb{F}$. Here H acts via deck transformations on $\hat{X}$ and hence on $C_*(\hat{X},\hat{Y};\mathbb{Z})$, and H acts on $\mathbb{F}$ via φ. If this complex is not acyclic; i.e., if the twisted homology groups $H_*(X,Y;\mathbb{F})$ do not vanish, then we set $\tau^\varphi(X,Y,\mathfrak{l}) = 0 \in \mathbb{F}$. If the complex is acyclic, then we can define the torsion $\tau^\varphi(X,Y,\mathfrak{l}) \in \mathbb{F} \setminus \{0\}$ as follows.

The cells $\{\, l(i,j)\colon (i,j) \in K \,\}$ define a basis of $C_*(\hat{X},\hat{Y};\mathbb{Z})$ as a complex of free $\mathbb{Z}[H]$–modules and hence give a basis for $C_*(X,Y;\mathbb{F})$ as a complex of $\mathbb{F}$–modules. We now define $\tau^\varphi(X,Y,\mathfrak{l}) \in \mathbb{F}$ to be the torsion of the based acyclic complex $C_*(X,Y;\mathbb{F})$. It is straightforward to check that this is independent of the choice of representative lift corresponding to $\mathfrak{l} \in \mathrm{Lift}(X,Y)$. We refer to [Tu01] for an excellent introduction to the torsion of based complexes.

The torsion $\tau^\varphi(X,Y,\mathfrak{l}) \in \mathbb{F}$ is well–defined up to sign; this indeterminacy comes from the fact that we do not have a canonical ordering of the basis. We can eliminate this indeterminacy by equipping (X,Y) with a homology orientation ω, i.e., an orientation of the vector space $H_*(X,Y;\mathbb{R}) = \oplus_{i\geq 0} H_i(X,Y;\mathbb{R})$. Then we can get a well–defined element $\tau^\varphi(X,Y,\mathfrak{l},\omega) \in \mathbb{F}$. We refer to [Tu02, Section K] for details.

Now suppose that $\chi(X,Y) = 0$. For $\mathfrak{e} \in \mathrm{Eul}(X,Y)$ we define $\tau^\varphi(X,Y,\mathfrak{e},\omega)$ to be $\tau^\varphi(X,Y,E^{-1}(\mathfrak{e}),\omega)$ if $X \neq Y$. If $X = Y$ recall that $\mathrm{Eul}(X,Y)$ is canonically identified with H. For $h \in H = \mathrm{Eul}(X,Y)$ let $\tau^\varphi(X,Y,h,\omega) = h$ if ω is the positive orientation of $H_*(X,Y) = 0$, and $\tau^\varphi(X,Y,h,\omega) = -h$ otherwise.

3.3. **The maximal abelian torsion of a CW–complex.** We continue with the notation from the previous sections. In particular, let (X,Y) be a pair of finite

CW–complexes such that X is connected. Furthermore, let $\mathfrak{l} \in \mathrm{Lift}(X, Y)$ and ω a homology orientation.

We again write $H = H_1(X; \mathbb{Z})$ and think of H as a multiplicative group. We let $T = \mathrm{Tor}(H)$ be the torsion subgroup. Given a ring R we denote by $Q(R)$ the ring which is given by inverting all non–zero divisors in R. We write $Q(H) = Q(\mathbb{Z}[H])$.

A character $\chi : T \to \mathbb{C}^*$ extends to a ring homomorphism $\chi : \mathbb{Q}[T] \to \mathbb{C}$, its image is a cyclotomic field $\mathbb{F}_\chi$. Two characters χ_1, χ_2 are called equivalent if $\mathbb{F}_{\chi_1} = \mathbb{F}_{\chi_2}$ and if χ_1 is the composition of χ_2 with a Galois automorphism of $\mathbb{F}_{\chi_1}$ over $\mathbb{Q}$. For any complete family of representatives $\chi_1, \dots, \chi_n$ of the set of equivalence classes of characters the homomorphism

$$(\chi_1, \dots, \chi_n) : \mathbb{Q}[T] \to \bigoplus_{i=1}^n \mathbb{F}_{\chi_i}$$

is an isomorphism of rings. We will henceforth identify $\mathbb{Q}[T]$ with $\bigoplus_{i=1}^n \mathbb{F}_{\chi_i}$. Note that under this isomorphism $1 \in \mathbb{Q}[T]$ corresponds to $(1, \dots, 1)$.

Now let F be the free abelian group H/T and pick a splitting $H = F \times T$. We can then identify

$$\mathbb{Q}[H] = \bigoplus_{i=1}^n \mathbb{F}_{\chi_i}[F] \text{ and } Q(H) = \bigoplus_{i=1}^n Q(\mathbb{F}_{\chi_i}[F]).$$

We denote by φ_i the ring homomorphism

$$\mathbb{Z}[H] \to \bigoplus_{i=1}^n \mathbb{F}_{\chi_i}[F] \to \mathbb{F}_{\chi_i}[F] \to Q(\mathbb{F}_{\chi_i}[F]).$$

We now let

$$\tau(X, Y, \mathfrak{l}, \omega) := \sum_{i=1}^n \tau^{\varphi_i}(X, Y, \mathfrak{l}, \omega) \in \bigoplus_{i=1}^n Q(\mathbb{F}_{\chi_i}[F]) = Q(H).$$

Note that $\tau(X, Y, \mathfrak{l}, \omega) \in Q(H)$ is independent of the choices we made (cf. [Tu02, Section K]). Also note that we have

$$\tau(X, Y, h \cdot \mathfrak{l}, \pm\omega) = \pm h \cdot \tau(X, Y, \mathfrak{l}, \omega). \tag{1}$$

In the following we write $\tau(X, Y, \mathfrak{l})$ for the set of torsions corresponding to all possible orientations. Also, if $\chi(X, Y) = 0$ then for $\mathfrak{e} \in \mathrm{Eul}(X, Y)$ we define

$$\tau(X, Y, \mathfrak{e}, \omega) = \tau(X, Y, E^{-1}(\mathfrak{e}), \omega).$$

In the coming sections we will often make use of the following two lemmas.

Lemma 3.4. *Suppose that there are no 0–cells in $X \setminus Y$ and suppose that X is a 2–complex. Then for any $\mathfrak{l} \in$ Lift(X, Y) we have $\tau(X, Y, \mathfrak{l}) \in \mathbb{Z}[H]$.*

Proof. Note that our assumptions say that $C_i(X,Y;\mathbb{Z}[H]) = 0$ for any $i \neq 1,2$. If $\chi(X,Y) \neq 0$, then $H_*(X,Y;\mathbb{F})$ is nontrivial for every $\mathbb{F}$, and the torsion is 0 by definition. If $\chi(X,Y) = 0$, let A be the matrix representing the boundary map

$$C_2(X,Y;\mathbb{Z}[H]) \to C_1(X,Y;\mathbb{Z}[H])$$

with respect to a basis corresponding to $\mathfrak{l} \in \mathrm{Lift}(X,Y)$. Then for any ring homomorphism $\varphi : \mathbb{Z}[H] \to \mathbb{F}$ we have $\tau^{\varphi}(M,\gamma,\mathfrak{l}) = \varphi(\det(A))$. It follows immediately from the definitions that $\tau(X,Y,\mathfrak{l}) = \det(A) \in \mathbb{Z}[H]$. ☐

Lemma 3.5. *Assume that (X',Y') is a CW–pair obtained from (X,Y) by a simple homotopy s, then there exists a natural H–equivariant bijection $b_s\colon Eul(X,Y) \to Eul(X',Y')$ such that for every $\mathfrak{e} \in Eul(X,Y)$*

$$\tau(X,Y,\mathfrak{e}) = \tau(X',Y',b_s(\mathfrak{e})).$$

Proof. It is sufficient to show the result if X' is obtained from X using an elementary expansion. Suppose that we added cells e^i and e^{i+1} to X to get X'. Choose an Euler chain θ representing $\mathfrak{e}$. Let $\delta \subset e^i \cup e^{i+1}$ be a curve such that $\partial\delta = (-1)^i(p_i - p_{i+1})$, where p_i is the center of e^i and p_{i+1} is the center of e^{i+1}. Then define $b_s(\mathfrak{e})$ to be the equivalence class of the Euler chain $\theta + \delta$. From here a standard argument shows that $\tau(X,Y,\mathfrak{e}) = \tau(X',Y',b_s(\mathfrak{e}))$. ☐

3.4. **Torsion for sutured manifolds.** Throughout this section (M,γ) will be a connected weakly balanced sutured manifold.

Definition 3.6. A *sutured handle complex* $\mathcal{A}$ is a pair of spaces $(A, S \times I)$, where S is a compact surface with boundary, together with a decomposition of $A \setminus (S \times I)$ into 3-dimensional handles $e_1 \cup \cdots \cup e_n$. For $0 \leq i \leq n$ we write $A_i = (S \times I) \cup (e_1 \cup \cdots \cup e_i)$ and $S_i = \partial A_i \setminus (S \times \{0\} \cup \partial S \times I)$. Then e_i is smoothly attached to A_{i-1} along S_{i-1} for $1 \leq i \leq n$.

Let $I(r)$ denote the index of the handle e_r. We say that $\mathcal{A}$ is *nice* if I is non-decreasing and $e_i \cap e_j = \emptyset$ whenever $I(e_i) = I(e_j)$ and $i \neq j$.

Definition 3.7. A *handle decomposition* Z of (M,γ) consists of a sutured handle complex $\mathcal{A}$ and a diffeomorphism $d\colon A \to M$ such that $d(S \times \{0\}) = R_-(\gamma)$ and $d(\partial S \times I) = \gamma$. We say that Z is nice if $\mathcal{A}$ is nice.

Notice that if Z is a nice handle decomposition of (M,γ) then we can collapse each handle to its core, starting from e_n and proceeding to e_1, and finally $S \times I$ to $S \times \{0\}$, to obtain a relative CW complex X built upon S. This way it is straightforward to define $\mathrm{Lift}(Z)$, $\mathrm{Eul}(Z)$, and for $\mathfrak{e} \in \mathrm{Eul}(Z)$ the maximal abelian torsion $\tau(Z,\mathfrak{e})$ in a way completely analogous to the case of CW pairs. As in Definition 3.3 we also have a map $E_Z\colon \mathrm{Lift}(Z) \to \mathrm{Eul}(Z)$.

The sets $\mathrm{Eul}(Z)$ and $\mathrm{Spin}^c(M,\gamma)$ are both affine copies of $H_1(M)$. We will show that they are canonically isomorphic. Tothis end, we first define a vector field v_Z on

M with the following properties: v_Z vanishes exactly at the centers of $d(e_1), \dots, d(e_n)$; the index of v_Z at the center of $d(e_i)$ is $(-1)^{I(i)}$; and finally, $v_Z|\partial M = v_0$.

We first construct such a vector field v_A on A, then push it forward using the diffeomorphism d. First, let $v_A|(S \times I) = \partial/\partial t$, where t is the coordinate on I. If $e_i = D^3$ is a 0–handle with coordinates (x, y, z) then let

$$v_Z(x, y, z) = x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z}.$$

On a 3–handle take v_A to be the negative of the previous vector field. If $e_j = D^1 \times D^2$ is a 1–handle with coordinates x on D^1 and (y, z) on D^2 then we define $v_Z(x, y, z)$ to be

$$-x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z}.$$

This vector field points out of e_j along $D^1 \times S^1$ and it lies on the same side of $S^0 \times D^2$ as $\{0\} \times D^2$. We can glue $v_A|(D^1 \times D^2)$ to $v_A|(S \times \{1\}) \cup (0\text{–handles})$ by a smooth handle attachment. Similarly, on a 2–handle $D^2 \times D^1$ with coordinates (x, y, z) we choose v_A to be $-x \cdot \partial/\partial x - y \cdot \partial/\partial y + z \cdot \partial/\partial z$. One can also think of v_Z as the gradient-like vector field for a Morse function compatible with the handle decomposition Z.

Definition 3.8. For $\mathfrak{e} \in \text{Eul}(Z)$ let $s_Z(\mathfrak{e}) \in \text{Spin}^c(M, \gamma)$ be the homology class of the vector field v that is obtained as follows. Pick an Euler chain θ representing $\mathfrak{e}$ that is a union of smoothly embedded arcs and circles inside $\text{Int}(M)$. Choose an open regular neighborhood $N(\theta)$ of θ. Then let $v = v_Z$ on $M \setminus N(\theta)$. If N_0 is a component of $N(\theta)$ diffeomorphic to B^3 then extend v to N_0 as a nowhere zero vector field. This is possible since v_Z has an index 1 and an index -1 singularity inside N_0. The homology class of v is independent of the choice of extension. If N_1 is a component of $N(\theta)$ diffeomorphic to $S^1 \times B^2$ then we get $v|N_1$ from $v_Z|N_1$ using Reeb turbularization, as described in [Tu90, p.639].

Note that we can avoid closed components of θ in the above definition except if (M, γ) is a product and $A = S \times I$.

Lemma 3.9. *For $\mathfrak{e}_1, \mathfrak{e}_2 \in \text{Eul}(Z)$ we have $s_Z(\mathfrak{e}_1) - s_Z(\mathfrak{e}_2) = \mathfrak{e}_1 - \mathfrak{e}_2$.*

Proof. An analogous obstruction theoretic argument as in the proof of [OS04a, Lemma 2.19] works here too. Also see [Tu90]. □

Consequently, s_Z is an H-equivariant bijection between $\text{Eul}(Z)$ and $\text{Spin}^c(M, \gamma)$.

Remark. Note that if the handle decomposition Z arises from a balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ then every $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ defines a unique Euler structure $\mathfrak{e}(\mathbf{x}) \in \text{Eul}(Z)$ as follows. Suppose that $x_i \in \alpha_j \cap \beta_k$ and let a_j be the 1–handle corresponding to α_j and b_k the 2–handle corresponding to β_k. Then let θ_i be a curve that connects the center of a_j to x_i inside a_j and then goes from x_i to the center of b_k inside b_k. The Euler chain $\theta_1 + \dots + \theta_d$ defines $\mathfrak{e}(\mathbf{x})$. Then the Spin^c-structure $\mathfrak{s}(\mathbf{x})$ assigned to $\mathbf{x}$ is exactly $s_Z(\mathfrak{e}(\mathbf{x}))$.

Proposition 3.10. *Suppose that Z and Z' are nice handle decompositions of (M,γ) and let $\mathfrak{s} \in Spin^c(M,\gamma)$. Then*

$$\tau(Z, s_Z^{-1}(\mathfrak{s})) = \tau(Z', s_{Z'}^{-1}(\mathfrak{s})).$$

Proof. By [Ce70] one can get from Z to Z' through a finite sequence of nice handle decompositions, each one obtained from the previous by one of the following operations: a handle isotopy, a handle slide, adding/cancelling a pair of handles, or an isotopy of d. So it suffices to show the claim when $Z = (\mathcal{A}, d)$ and $Z' = (\mathcal{A}', d')$ are related by one of these operations.

Suppose that $\mathcal{A} = \mathcal{A}'$ and d is isotopic to d'. Then there is a diffeotopy $\{\, h_t \colon t \in I \,\}$ of M such that $d' = h_1 \circ d$. If the submanifold $\theta \subset M$ represents $s_Z^{-1}(\mathfrak{s})$ then $\theta' = d' \circ d^{-1}(\theta) = h_1(\theta)$ is isotopic to θ. Moreover, $v_{Z'} = d'_*(v_A) = (h_1)_*(v_Z)$ is isotopic to v_Z. Consequently, θ' represents $s_{Z'}^{-1}(\mathfrak{s})$ and both $\tau(Z, s_Z^{-1}(\mathfrak{s}))$ and $\tau(Z', s_{Z'}^{-1}(\mathfrak{s}))$ are equal to the torsion $\tau(A, S \times I, [d^{-1}(\theta)])$.

Now we consider the case when Z and Z' are related by isotoping a handle. This means the following. We choose a handle e_i of $\mathcal{A}$ and isotope its attaching map f_i to some other map f_i' inside S_{i-1}. Then we extend this isotopy to a diffeotopy $\{\, \varphi_t \colon t \in I \,\}$ of S_{i-1} such that $f_i' = \varphi_1 \circ f_i$. We define the A_j' recursively, together with diffeomorphisms $\nu_j \colon A_j \to A_j'$ for $j \geq i-1$. If $j \leq i-1$ then let $A_j' = A_j$. To define ν_{i-1} choose a collar $S_{i-1} \times I$ of S_{i-1} such that $S_{i-1} \times \{1\}$ is identified with S_{i-1}. Outside this collar let $\nu_{i-1} = \mathrm{id}$, and $\nu_{i-1}(s,t) = (\varphi_t(s), t)$ for $(s,t) \in S_{i-1} \times I$. If A_{j-1}' and ν_{j-1} are already defined then we obtain A_j' by gluing $e_j' = e_j$ to S_{j-1}' along $\nu_j \circ f_j$. Then ν_{j-1} naturally extends to A_j, call this extension ν_j. This defines the handle complex $\mathcal{A}'$, and we set $d' = d \circ (\nu_n)^{-1}$, where n is the number of handles.

Define $\nu = \nu_n$, this is a diffeomorphism from A to A'. For $1 \leq j \leq n$ let p_j be the center of e_j and p_j' the center of e_j'. Then $\nu(p_j) = p_j'$, so $d(p_j) = d'(p_j')$. Hence there is a natural bijection $N \colon \mathrm{Eul}(Z) \to \mathrm{Eul}(Z')$ using the formula $N([\theta]) = [\theta]$. The vector field $\nu_*(v_A)$ agrees with $v_{A'}$ except on $\nu(S_{i-1} \times I)$. But there they both point up (with respect to $\partial/\partial t$), so $\nu_*(v_A)$ and $v_{A'}$ are isotopic on this collar through nowhere zero fields rel boundary. This proves that for every $\mathfrak{e} \in \mathrm{Eul}(Z)$ we have $s_Z(\mathfrak{e}) = s_{Z'}(N(\mathfrak{e}))$.

Let $\hat{M}$ be the maximal abelian cover of M and let $\hat{Z}$ and $\hat{Z}'$ be the induced relative handle structures on $\hat{M}$. We show that $\tau(Z, \mathfrak{e}) = \tau(Z', N(\mathfrak{e}))$ for two special isotopies. Let θ be a submanifold simultaneously representing $\mathfrak{e}$ and $N(\mathfrak{e})$. We can assume that θ has no closed components. Let $\hat{\theta}$ be an arbitrary lift of θ to $\hat{M}$. Then $\partial\hat{\theta}$ defines a lift in both $\hat{Z}$ and $\hat{Z}'$ that represent $E_Z^{-1}(\mathfrak{e})$ and $E_{Z'}^{-1}(N(\mathfrak{e}))$, respectively. Let $\hat{e}_j$ be the lift of $d(e_j)$ and $\hat{e}_j'$ the lift of $d'(e_j')$. Since the centers of the handles of $\hat{Z}$ and $\hat{Z}'$ coincide we have a bijection $B \colon C_*(\hat{Z}) \to C_*(\hat{Z}')$ that takes the generator $\hat{e}_j$ to $\hat{e}_j'$.

First suppose that the isotopy connecting f_i and f_i' avoids every other handle of index $I(i)$, including the attaching map of e_j if $j > i$ and $I(j) = I(i)$. Then B is an isomorphism of based complexes. Indeed, isotoping the attaching map of $\hat{e}_i$ does not change the algebraic intersection number with belt circles of handles of index $I(i)-1$.

The second case is when we handleslide e_i over a handle e_r such that $I(i) = I(r)$. Then we can obtain the based complex $C_*(\hat{Z}')$ from $C_*(\hat{Z})$ by replacing the basis element $\hat{e}_i$ with $\hat{e}_i + \hat{e}_r$. Thus the torsion is again unchanged.

Now suppose that Z' is obtained from Z by adding a canceling pair of handles e and f, and suppose that Z' is also nice (this is not necessarily the case if $I(i) = 1$, but that can be avoided by isotoping the 2-handles beforehand). Suppose that e and f are attached between e_{i-1} and e_i. Similarly to the case of an isotopy, we recursively define $\mathcal{A}'$ together with a sequence of diffeomorphisms $\nu_j \colon A_j \to A'_{j+2}$ for $j \geq i-1$. Let $A'_j = A_j$ for $j \leq i-1$. To define ν_{i-1} choose a collar $S_{i-1} \times I$ of S_{i-1} as before. Outside this collar let $\nu_{i-1} = \text{id}$, and $\nu_{i-1}(S_{i-1} \times I) = (S_{i-1} \times I) \cup e \cup f$. We set $e'_i = e$ and $e'_{i+1} = f$. If $j \geq i$ and the attaching map of e_j in $\mathcal{A}$ is f_j then let $e'_{j+2} = e_j$ attached to A_{j-1} along $\nu_{j-1} \circ f_j$. Finally, set $\nu = \nu_n$ and define $d' = d \circ \nu^{-1}$.

The CW complexes corresponding to $\mathcal{A}$ and $\mathcal{A}'$ are related by an elementary expansion s, so by Lemma 3.5 there is a bijection $b_s \colon \text{Eul}(Z) \to \text{Eul}(Z')$ such that $\tau(Z, \mathfrak{e}) = \tau(Z', b_s(\mathfrak{e}))$ for every $\mathfrak{e} \in \text{Eul}(Z)$. We claim that $s_Z(\mathfrak{e}) = s_{Z'}(b_s(\mathfrak{e}))$. Indeed, let p'_j denote the center of e'_j. Then if θ represents $\mathfrak{e}$ and δ is an arc inside $e'_i \cup e'_{i+1}$ such that $\partial\delta = (-1)^i(p'_i - p'_{i+1})$ then $\theta' = \theta + d'(\delta)$ represents $b_s(\mathfrak{e})$. Let $K = (S'_{i-1} \times I) \cup e'_i \cup e'_{i+1} \subset A'$, then $d'(K) = d(S_{i-1} \times I)$. If v' is a nowhere vanishing vector field on M extending $v_{Z'}|(M \setminus N(\theta'))$ then $v'|d'(K)$ is isotopic to $d_*(\partial/\partial t)$. On the other hand, if v is the nowhere zero vector field obtained from v_Z and θ, then $v|d'(K) = d_*(\partial/\partial t)$ and $v|(M \setminus d'(K)) = v'|(M \setminus d'(K))$. The claim follows. □

Definition 3.11. Let Z be a nice handle decomposition of (M, γ). Given an element $\mathfrak{s} \in \text{Spin}^c(M, \gamma)$ and a homology orientation ω for the pair (M, R_-), we define

$$\tau(M, \gamma, \mathfrak{s}, \omega) = \tau(Z, s_Z^{-1}(\mathfrak{s}), \omega) \in Q(H),$$

where s_Z^{-1} is the canonical identification between $\text{Spin}^c(M, \gamma)$ and $\text{Eul}(Z)$ defined in Definition 3.8. By Proposition 3.10 the torsion $\tau(M, \gamma, \mathfrak{s}, \omega)$ is independent of the choice of Z.

Now suppose that $R_+(\gamma)$ and $R_-(\gamma)$ are both non-empty. This is true for example if (M, γ) is balanced. Then (M, γ) has a handle decomposition with no 0 and 3–handles. From Lemma 3.4 we get that $\tau(M, \gamma, \mathfrak{s}, \omega) \in \mathbb{Z}[H]$. Following Turaev, we define the torsion function as

$$\begin{array}{rcl} T_{(M,\gamma,\omega)} : \text{Spin}^c(M, \gamma) & \to & \mathbb{Z} \\ \mathfrak{s} & \mapsto & \tau(M, \gamma, \mathfrak{s}, \omega)_1 \end{array}$$

where $\tau(M, \gamma, \mathfrak{s}, \omega)_1$ denotes the constant term of $\tau(M, \gamma, \mathfrak{s}, \omega) \in \mathbb{Z}[H]$. Note that in light of (1) we can recover $\tau(M, \gamma, \mathfrak{s}, \omega) \in \mathbb{Z}[H]$ from the above function.

Finally, we extend Definition 3.11 to disconnected weakly balanced sutured manifolds.

Definition 3.12. Suppose that (M, γ) is a weakly balanced sutured manifold whose components are $(M_1, \gamma_1), \dots, (M_n, \gamma_n)$. For $\mathfrak{s} \in \text{Spin}^c(M, \gamma)$ and $1 \leq i \leq n$ let $\mathfrak{s}_i =$

$\mathfrak{s}|M_i$. Then define

$$\tau(M,\gamma,\mathfrak{s}) = \bigotimes_{i=1}^{n} \tau(M_i,\gamma_i,\mathfrak{s}_i) \in \bigotimes_{i=1}^{n} Q(\mathbb{Z}[H_1(M_i)]) = Q(H),$$

where we take the tensor product over $\mathbb{Z}$.

3.5. **Making γ connected.** Let (M,γ) be a connected balanced sutured manifold. In the future, it will often be convenient to assume that γ, and hence $R_\pm(\gamma)$, are connected. This can be arranged by adding product 1–handles to (M,γ) to produce a new sutured manifold (M',γ'). In this section, we describe the effect of this operation on the sutured Floer homology and the torsion. In particular, we show that $SFH(M,\gamma)$ can be recovered from $SFH(M',\gamma')$, and likewise for the torsion.

In terms of Heegaard diagrams, this operation of adding a product 1–handle can be described as follows. Suppose $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a balanced sutured Heegaard diagram for (M,γ), and let Σ' be the result of attaching a 2–dimensional 1–handle h to Σ. Then $(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta})$ is a Heegaard diagram representing a sutured manifold (M',γ'), which is obtained from M by attaching the 3–dimensional 1–handle $h\times[-1,1]$. We can recover (M,γ) from (M',γ') by decomposing along the product disk $c\times[-1,1]$, where c is the cocore of h. If h joins two different components of γ then γ' will have one less component than γ.

From [Ju06, Lemma 9.13] we know that $SFH(M',\gamma')\cong SFH(M,\gamma)$. To be more precise, by [Ju08b, Prop. 5.4] there is an injection $i:\mathrm{Spin}^c(M,\gamma)\to\mathrm{Spin}^c(M',\gamma')$ for which

$$SFH(M,\gamma,\mathfrak{s})\cong SFH(M',\gamma',i(\mathfrak{s})).$$

Moreover, $SFH(M',\gamma',\mathfrak{s}')=0$ if $\mathfrak{s}'$ is not in the image of i. For our purposes, i is most easily described by observing that the generating sets $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $\mathbb{T}'_\alpha\cap\mathbb{T}'_\beta$ are naturally identified, and setting $i(\mathfrak{s}(\mathbf{x}))=\mathfrak{s}(\mathbf{x}')$. The analogous statement for the torsion is proved below.

Lemma 3.13. *Suppose that (M',γ') is obtained from (M,γ) by adding a product 1–handle. Then $T_{(M,\gamma)}(\mathfrak{s}) = T_{(M',\gamma')}(i(\mathfrak{s}))$ for every $\mathfrak{s}\in$ Spin$^c(M,\gamma)$. Moreover, $T_{(M',\gamma')}(\mathfrak{s}')=0$ if $\mathfrak{s}'$ is not in the image of i.*

Proof. Let Z be the handlebody decomposition given by $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ and Z' the decomposition given by $(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta})$. The universal abelian cover $\hat{M}'$ can be constructed as follows. Start with a disjoint union $\sqcup_{i\in\mathbb{Z}}\hat{M}_i$ where each $\hat{M}_i$ is homeomorphic to $\hat{M}$. Now join $\hat{M}_i$ to $\hat{M}_{i+1}$ by 1–handles, one for each element of $H_1(M)$. These 1–handles are all thickenings of 1–handles in $\hat{R}'_-$, so they do not contribute to $C_*(\hat{M}',\hat{R}'_-)$. Choose a basis for $C_*(\hat{M}',\hat{R}'_-)$ all of whose handles are contained in $\hat{M}_0$; let $\mathfrak{l}$ and $\mathfrak{l}'$ be the associated lifts for M and M'. With respect to such a basis,

$$C_*(\hat{M}',\hat{R}'_-)\cong C_*(\hat{M},\hat{R}_-)\otimes_{\mathbb{Z}}\mathbb{Z}[t,t^{-1}].$$

It follows that $\tau(Z', \mathfrak{l}') = i_*(\tau(Z, \mathfrak{l}))$, where $i_* : H_1(M) \to H_1(M')$ is the inclusion. Now the embedding $M \hookrightarrow M'$ naturally induces an affine embedding $i_0 \colon \mathrm{Eul}(Z) \to \mathrm{Eul}(Z')$. Note that $i_0(E_Z(\mathfrak{l})) = E_{Z'}(\mathfrak{l}')$. Then if we define $i = s_{Z'} \circ i_0 \circ s_Z^{-1}$ we get that $\tau(M', \gamma', i(\mathfrak{s})) = i_*(\tau(M, \gamma, \mathfrak{s}))$. Finally, it is easy to see that this definition of i agrees with the one given above. □

4. The torsion function via Fox calculus

In this section, we explain how to compute $T_{(M,\gamma)}$ using Fox calculus. We assume throughout that (M, γ) is balanced and that the subsurfaces $R_\pm(\gamma)$ are connected. In light of Lemma 3.13, this restriction is a very mild one. For brevity, we write $R_\pm$ in place of $R_\pm(\gamma)$ throughout this section.

4.1. **Balanced diagrams and presentations.** We begin by explaining how to find a presentation of $\pi_1(M)$ compatible with a nice handle decomposition Z, or equivalently, with a balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ representing (M, γ). Choose a 2-dimensional handle decomposition of R_- consisting of one 0-handle and l 1-handles; this naturally gives a 3-dimensional handle decomposition of $R_- \times I$, again with one 0-handle and l 1-handles. Without loss of generality, we may assume that the attaching disks of the 1-handles of Z are disjoint from the belt circles of the 1-handles of $R_- \times I$, and thus (after an isotopy) that the 1-handles of Z are attached to the 0-handle of $R_- \times I$.

Fix a basepoint $p \in R_- \times \{0\}$ which is contained in the 0-handle. Then $\pi_1(M, p)$ is generated by loops $\alpha_1^*, \dots, \alpha_d^*, c_1^*, \dots, c_l^*$, where α_i^* runs through the i-th 1-handle of Z, and c_k^* runs through the k-th 1-handle of $R_- \times I$. (Note that α_i^* depends on the choice of handle decomposition for R_- as well as on Z.) Each 2-handle in Z gives rise to a relation as follows. Choose a path b_j from p to the attaching circle β_j of the j-th 2-handle; then $\overline{\beta}_j = b_j \beta_j b_j^{-1}$ is a loop which represents a trivial element of $\pi_1(M, p)$. We have a presentation

$$\pi_1(M, p) = \langle \alpha_1^*, \dots, \alpha_d^*, c_1^*, \dots, c_l^* \,|\, \overline{\beta}_1, \dots, \overline{\beta}_d \rangle.$$

To read off this presentation from a sutured Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ compatible with Z, we proceed as follows. First, surger S along the α-curves to produce a surface homeomorphic to R_- and containing $2l$ marked disks (the traces of the surgery). Next, choose a system of disjoint arcs $c_1, \dots, c_l$ in R_- whose complement is homeomorphic to a disk. (This amounts to choosing a handlebody decomposition of R_-.) Without loss of generality, we may assume that the c_k's are disjoint from the marked disks, so they lift to curves $c_1, \dots, c_l$ in Σ. To write down the word $\overline{\beta}_j$ we simply traverse $\overline{\beta}_j$ and record its intersections with the α_i's and the c_k's as we go.

More precisely, an intersection point x between α_i and $\overline{\beta}_j$ is recorded by α_i^* if the sign of intersection $\alpha_i \cdot \overline{\beta}_j$ at x is positive, and by $(\alpha_i^*)^{-1}$ if the sign of the intersection is negative. Note that in doing this, we have implicitly chosen orientations on the α_i's , the β_j's, and the c_k's.

4.2. **Torsion from a balanced diagram.** Given a presentation of $\pi_1(M,p)$ compatible with a balanced diagram, we can form the Fox derivatives $\dfrac{\partial\overline{\beta}_j}{\partial\alpha_i^*} \in \mathbb{Z}[\pi_1(M)]$.

Write $H = H_1(M)$. The based curves $c_1^*, \dots, c_l^*$ give rise to basis elements $\hat{C}_1^*, \dots, \hat{C}_1^l$ of $C_1(R_-;\mathbb{Z}[H])$. Similarly, the based curves $\alpha_1^*, \dots, \alpha_d^*$ give basis elements $\hat{A}_1^*, \dots, \hat{A}_1^d$ of $C_1(M, R_-;\mathbb{Z}[H])$. Let B_j denote the 2-handle attached along β_j. The basings $b_1, \dots, b_n$ give rise to a choice of lifts $\hat{B}_1, \dots, \hat{B}_d$ of $B_1, \dots, B_d$. These give a basis for $C_2(M;\mathbb{Z}[H])$.

Proposition 4.1. *Let $\mathfrak{l} \in Lift(Z)$ be the element corresponding to the basings. Then up to sign we have*

$$\tau(M,\gamma,\mathfrak{l}) = \det\left(\varphi\left(\frac{\partial\overline{\beta}_j}{\partial\alpha_i^*}\right)\right) \in \mathbb{Z}[H],$$

where $\varphi : \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[H_1(M)]$ is the homomorphism induced by abelianization.

Proof. Consider the following diagram of free $\mathbb{Z}[H]$–modules. We write our choice of basis under the free modules.

$$\begin{array}{ccccccccc}
 & & 0 & \to & C_2(M;\mathbb{Z}[H]) & \to & C_2(M,R;\mathbb{Z}[H]) & \to & 0 \\
 & & & & \hat{B}_1,\dots,\hat{B}_d & & \hat{B}_1,\dots,\hat{B}_d & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_1(R_-;\mathbb{Z}[H]) & \to & C_1(M;\mathbb{Z}[H]) & \to & C_1(M,R_-;\mathbb{Z}[H]) & \to & 0 \\
 & & \hat{C}_1^*,\dots,\hat{C}_l^* & \to & \begin{array}{c}\hat{A}_1^*,\dots,\hat{A}_d^* \\ \hat{C}_1^*,\dots,\hat{C}_l^*\end{array} & & \hat{A}_1^*,\dots,\hat{A}_d^* & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_0(R_-;\mathbb{Z}[H]) & \to & C_0(M;\mathbb{Z}[H]) & \to & 0. & & \\
 & & p & & p & & & &
\end{array}$$

Fox calculus tells us that the boundary map $C_2(M;\mathbb{Z}[H]) \to C_1(M;\mathbb{Z}[H])$ is given by

$$\begin{pmatrix} \varphi\left(\frac{\partial\overline{\beta}_j}{\partial\alpha_i^*}\right) \\ \varphi\left(\frac{\partial\overline{\beta}_j}{\partial c_k^*}\right) \end{pmatrix}.$$

The proposition now follows immediately. □

If we choose a different path b_j' joining p to β_j, the relation $\overline{\beta}_j'$ will be a conjugate of $\overline{\beta}_j$. Write $\overline{\beta}_j' = w\overline{\beta}_j w^{-1}$. Then the effect on the matrix of Fox derivatives is to multiply the j-th column by $\varphi(w)$, so the determinant is multiplied by $\varphi(w)$ as well. This simply expresses the fact that $\hat{B}_j$ and $\hat{B}_j'$ differ by the action of $\varphi(w)$.

In the next section, it will be convenient to choose the path b_j so that its image in Σ does not intersect any of the α_i or c_k. (This is always possible, since the complement of the α's and c's is connected.) In this case, we write the resulting relation as β_j; it is obtained by traversing the curve β_j and recording the intersections with the α's and the c's. Note that the word β_j is not canonical, since we still need to specify our starting point (*i.e.*, the foot of the path b_j).

4.3. **Torsion from a presentation.** In practice, one often wants to compute $\Delta(M,\gamma)$ starting from an arbitary presentation of $\pi_1(M)$, rather than one which comes from a balanced diagram. For example, if M is a handlebody, it is natural to present $\pi_1(M)$ as a free group. This can be done at the cost of losing information associated to the torsion subgroup of H. More precisely, let $T \subset H$ be the torsion subgroup, $F = T/H$, and let $\pi : H \to F$ be the natural projection. Below, we explain how to compute $\pi(\Delta(M,\gamma)) \in \mathbb{Z}[F]$.

Suppose we are given $p \in R_-$ and a presentation

$$\pi_1(M,p) \cong \langle a_1,\dots,a_m \,|\, b_1,\dots,b_n\rangle.$$

Fixing a handlebody decomposition for R_- gives an explicit presentation of $\pi_1(R_-,p)$ as a free group: $\pi_1(R_-,p) \cong \langle c_1,\dots c_l\rangle$. Using the induced map $\pi_1(R_-,p) \to \pi_1(M,p)$, we can view each c_k as a word in the a_i. We form the matrix

$$A = \left(\psi\left(\frac{\partial b_j}{\partial a_i}\right) \;\; \psi\left(\frac{\partial c_k}{\partial a_i}\right)\right)$$

where $\psi = \pi \circ \varphi$ is the natural projection $\pi_1(M,p) \to F$.

Proposition 4.2. *$\pi(\Delta(M,\gamma))$ is the gcd of the $m \times m$ minors of A.*

Note that if our presentation of $\pi_1(M,p)$ comes from a handlebody decomposition of M, the difference $m-n$ is equal to the genus of ∂M. On the other hand, (M,γ) balanced implies that $\chi(R_-) = \chi(\partial M)/2$, so $l = g(\partial M)$ as well. Thus in this case A is an $m \times m$ matrix, and $\pi(\Delta(M,\gamma)) = \det A$.

Proof. We first suppose that our presentation of $\pi_1(M,p)$ comes from a balanced diagram, as described in section 4.1. In this case, $c_k = c_k^*$, and A is of the form

$$A = \phi \begin{pmatrix} \frac{\partial \hat\beta_i}{\partial \hat\alpha_j^*} & 0 \\ \frac{\partial \hat\beta_i}{\partial y_j} & I \end{pmatrix}.$$

Thus the claim follows from Proposition 4.1.

To prove the result in general, recall that any presentation of $\pi_1(M,p)$ can be obtained from any other by a sequence of Tietze moves. Suppose Π and Π' are the presentations before and after such a move, and let A and A' be the associated matrices. We must check that the gcd of the minors of A is the same as the gcd of the minors of A'. This is a more or less standard argument in Fox calculus, *cf.* [CF77]. The only new point is the fact that the c_k are not relations: we must introduce a new, more restrictive move describing how they change.

We consider Tietze moves of four types. The first type consists of adding a new relation $b_{n+1} = b_i^{\pm 1}$ for some i. If the exponent is positive, this has the effect of duplicating one of the columns in A. Minors of the new matrix A' are either minors of A or have two identical columns, in which case their determinant is 0. Thus the gcd is unchanged. If the exponent is negative, the new column is multiplied by a unit in $\mathbb{Z}[F]$, but the argument is otherwise the same.

The second move consists of replacing a relation b_j with $b_j b_{j'}$ for some $j \neq j'$. This has the effect of adding the j'-th column of A to the j-th column. In this case, minors of A' are equal to the corresponding minors of A (if they do not contain the j-th column, or they contain both the j-th and j'-th column), or to a sum of two minors of A (if they contain the j-th column but not the j'-th.) Again, the gcd is unchanged.

The third move consists of replacing b_j by $w b_j w^{-1}$, where w is an arbitrary word in the a_i's. Since b_j is trivial in $\pi_1(M)$, this has the effect of multiplying the i-th column of A by w. Thus the gcd is the same up to multiplication by a unit in $\mathbb{Z}[F]$.

The fourth move consists of adding a new generator a_{m+1} and a new relation $b_{n+1} = a_{m+1} w$, where w is an arbitrary word. The $(m+1) \times (m+1)$ minors vanish unless they contain the term $\partial b_{n+1} / \partial a_{m+1}$, in which case they reduce to the $m \times m$ minors of A. Thus the gcd is unchanged in this case as well.

The final, new move which we must consider consists of replacing c_k with an equivalent word with respect to a fixed presentation for $\pi_1(M)$. To be precise, we replace $c_k = xy$ with $c_k' = x b_j y$ for some j. Then we have an equation of column vectors

$$\left\langle \psi\left(\frac{\partial c_k'}{\partial a_i}\right)\right\rangle = \left\langle \psi\left(\frac{\partial c_k}{\partial a_j}\right)\right\rangle + \psi(x)\left\langle \psi\left(\frac{\partial b_i}{\partial a_j}\right)\right\rangle,$$

so each $m \times m$ minor of A' is either equal to a minor of A or a sum of the form $M_1 + \psi(x) M_2$, where the M_i are minors of A. Again, it is easy to see that the gcd is unchanged. □

Corollary 4.3. *$\pi(\Delta(M,\gamma))$ is determined by the map $\pi_1(R_-(\gamma), p) \to \pi_1(M, p)$.*

Proof. A classical theorem of Nielsen on automorphisms of free groups [Ni24] implies that if $\langle c_1, \dots, c_l \rangle$ and $\langle d_1, \dots, d_l \rangle$ are two bases of a free group, then they are related by a sequence of the following moves. We either 1) replace c_k by c_k^{-1} or 2) replace c_k by $c_k c_{k'}$ for some $k \neq k'$. It is easy to see that (up to multiplication by a unit in $\mathbb{Z}[F]$) the gcd is unchanged by these moves. □

When $H_1(M)$ is free, $\pi(\Delta(M,\gamma)) = \Delta(M,\gamma)$, so Proposition 2 of the introduction follows immediately.

5. The proof of Theorem 1

We are now in a position to prove that the Euler characteristic of $SFH(M,\gamma,\mathfrak{s})$ coincides with the torsion $T_{(M,\gamma)}(\mathfrak{s})$. First, observe that it is enough to prove the equality in the case where $R_-(\gamma)$ is connected. Indeed, if $R_-(\gamma)$ is not connected, we can add product one-handles to obtain a new sutured manifold (M',γ') with $R_-(\gamma')$ connected. Lemma 3.13 and the discussion preceding it show that if $\chi(SFH(M',\gamma',i(\mathfrak{s}))) = T_{(M',\gamma')}(i(\mathfrak{s}))$ then $\chi(SFH(M,\gamma,\mathfrak{s})) = T_{(M,\gamma)}(\mathfrak{s})$.

5.1. **Equality in $\mathbb{Z}[H]/\pm H$.** For the rest of this section, we assume $R_-(\gamma)$ is connected. Our next step is to show that the torsion and the Euler characteristic agree up to multiplication by $\pm[h]$ for some $h \in H$. In light of Proposition 4.1, it suffices to prove

Proposition 5.1.
$$\chi(SFH(M,\gamma)) \sim \det \varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right),$$
where $\sim$ indicates equality up to multiplication by $\pm[h]$.

Proof. We argue along the lines of chapter 3 in [Ra03]. First, observe that there are natural bijections

$$\{\text{elements of } \alpha_i \cap \beta_j\} \longleftrightarrow \{\text{appearances of } \alpha_i^* \text{ in } \beta_j\} \longleftrightarrow \left\{\text{monomials in } \varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right)\right\},$$

where the free derivative $\partial\beta_j/\partial\alpha_i^*$ has been expanded *without canceling any terms.* Equivalently,
$$\varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right) = \sum_{x\in\alpha_i\cap\beta_j} m(x)[E(x)],$$
where each $m(x) = \pm 1$ and each $E(x)$ is an element of $H_1(M)$. The sign $m(x)$ is given by the exponent of the corresponding appearance of α_i^* in β_j, or equivalently, by the sign of intersection $\alpha_i \cdot \beta_j$ at x.

Recall that the chain complex computing SFH is generated by d-tuples of intersection points $\mathbf{x} = \{x_1, \dots x_d\}$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma \in \Sigma_d$. On the other hand, the determinant of an $d \times d$ matrix B_{ij} can be expanded as
$$\det B_{ij} = \sum_{\sigma\in S_d} \operatorname{sign}(\sigma) B_{1\sigma(1)} \cdots B_{d\sigma(d)}.$$
Thus we get a bijection
$$\mathbb{T}_\alpha \cap \mathbb{T}_\beta \longleftrightarrow \left\{\text{monomials in } \det\varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right)\right\}.$$
Again, all terms in the determinant are to be expanded *without cancellation.*

It follows that
$$\tau(M,\gamma) \sim \det\varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right) = \sum_{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} m(\mathbf{x})[E(\mathbf{x})].$$

Here $E(\mathbf{x}) = \sum_{i=1}^d E(x_i)$ and $m(\mathbf{x})$ is the sign of the intersection $\mathbb{T}_\alpha \cdot \mathbb{T}_\beta$ at $\mathbf{x}$. The orientations on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are induced by the orderings $\langle\alpha_1, \dots, \alpha_d\rangle$ and $\langle\beta_1, \dots, \beta_d\rangle$. On the other hand, we know from the equation at the end of section 2.5 that
$$\chi(SFH(M,\gamma)) \sim \sum_{\{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\}} m(\mathbf{x})[\iota(\mathbf{x})].$$

where $\iota : \mathrm{Spin}^c(M,\gamma) \to H_1(M)$ is an affine isomorphism. Comparing, we see that the two expressions will agree up to multiplication by $\pm[h]$ if we can show

Lemma 5.2. $E(\mathbf{x}) - E(\mathbf{y}) = \mathbf{x} - \mathbf{y}$.

Proof. The coefficients $E(x)$ appearing in the Fox derivative $\partial\beta_j/\partial\alpha_i^*$ can be interpreted geometrically as follows. In the universal abelian cover $\hat{M}$, we have fixed lifts $\hat{\alpha}_i$ of α_i and $\hat{\beta}_j$ of β_j coming from the basings. An intersection point $x \in \alpha_i \cap \beta_j$ lifts to a unique $\hat{x} \in \hat{\beta}_j$, and this point $\hat{x}$ is contained in $(E(x)\cdot\hat{\alpha}_i) \cap \hat{\beta}_j$.

Suppose $\mathbf{x} = \{x_1, \dots, x_d\}$ and $\mathbf{y} = \{y_1, \dots, y_d\}$, and that the ordering of the x's and y's is arranged so that $x_j, y_j \in \beta_j$. Then there are i, i' so that $\hat{x}_j \in (E(x_j)\cdot\hat{\alpha}_i)\cap\hat{\beta}_j$ and $\hat{y}_j \in (E(y_j)\cdot\hat{\alpha}_{i'}) \cap \hat{\beta}_j$. If $\tilde{\beta}_j$ is an arbitrary lift of β_j, and $\tilde{x}_j, \tilde{y}_j$ are the lifts of x_j, y_j which lie on it, then we can write $\tilde{x}_j \in (\tilde{E}(x_j)\cdot\hat{\alpha}_i)\cap\tilde{\beta}_j$ and $\tilde{y}_j \in (\tilde{E}(y_j)\cdot\hat{\alpha}_{i'})\cap\tilde{\beta}_j$. Then we have

$$\tilde{E}(x_j) - \tilde{E}(y_j) = E(x_j) - E(y_j).$$

Now suppose we are given a 1–cycle θ which runs from $\mathbf{x}$ to $\mathbf{y}$ along the α curves and a 1–cycle η which runs from $\mathbf{y}$ to $\mathbf{x}$ along the β curves. The closed 1–cycle $\theta - \eta$ may be divided into components. Let δ be one such component. If we start at some y_{j_1} on δ, and traverse δ, we will successively encounter points labelled $x_{j_1}, y_{j_2}, x_{j_2}, \dots, y_{j_r}, x_{j_r}$ before returning to y_{j_1}.

Let $\hat{\delta}$ be the lift of δ to $\hat{M}$ which starts at $\hat{y}_{j_1}$ and ends at $[\delta]\cdot\hat{y}_{j_1}$. As we traverse $\hat{\delta}$, we denote the α curve containing the lift of y_{j_k} by $\tilde{E}(y_{j_k})\cdot\hat{\alpha}_{i_k}$. Similarly, we denote the α curve containing the lift of x_{j_k} by $\tilde{E}(x_{j_k})\cdot\hat{\alpha}_{i_k}$. Since x_{j_k} is joined to $y_{j_{k+1}}$ along an α curve, $\tilde{E}(x_{j_k}) = \tilde{E}(y_{j_{k+1}})$. On the other hand, since y_{j_k} is joined to x_{j_k} by a β curve, we have

$$\tilde{E}(x_{j_k}) - \tilde{E}(y_{j_k}) = E(x_{j_k}) - E(y_{j_k}).$$

We now compute

$$\begin{aligned}[\delta] &= \tilde{E}(x_{j_r}) - \tilde{E}(y_{j_1}) \\ &= \sum_{k=1}^{r}\left(\tilde{E}(x_{j_k}) - \tilde{E}(y_{j_k})\right) \\ &= \sum_{k=1}^{r}\left(E(x_{j_k}) - E(y_{j_k})\right).\end{aligned}$$

A similar relation holds for each component of $\theta - \eta$. Adding them all together, we find that

$$\mathbf{x} - \mathbf{y} = [\theta - \eta] = \sum_{i=1}^{n}\left(E(x_i) - E(y_i)\right) = E(\mathbf{x}) - E(\mathbf{y}).$$

□

This completes the proof of Proposition 5.1 □

5.2. **Spin^c structures.** Our next step is to show that $\chi(SFH(M,\gamma,\mathfrak{s})) = \pm T_{M,\gamma}(\mathfrak{s})$. To see this, fix a single generator $\mathbf{x} = \{x_1,\dots,x_d\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and consider the associated lift $\mathfrak{l}(\mathbf{x}) = E^{-1}(\mathfrak{e}(\mathbf{x})) \in \mathrm{Lift}(M,R_-)$. We will show that the term in

$$\det \varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right)$$

corresponding to $\mathbf{x}$ contributes to $T_{M,\gamma}(\mathfrak{s})$, or equivalently, that it contributes ± 1 to the torsion $\tau(M,\gamma,\mathfrak{l}(\mathbf{x}))$. It will then follow from Proposition 5.1 that the same relation holds for every generator $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Without loss of generality, we may assume $x_i \in \alpha_i \cap \beta_i$. Consider the matrix

$$F_{ij} = \varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right)$$

of Fox derivatives. Fix lifts $\tilde{A}_i^*$ of the 1-handles to the universal abelian cover. Then after an appropriate normalization (multiplying each column in the matrix by a unit in $\mathbb{Z}[H_1(M)]$), the column vector

$$\varphi\left(\frac{\partial \beta_j}{\partial \alpha_1^*},\dots,\frac{\partial \beta_j}{\partial \alpha_d^*}\right)$$

expresses the boundary of the lift $\tilde{B}_j$ of the two-handle B_j in terms of the lifts $\tilde{A}_i^*$. It is well-defined up to multiplication by a unit in $\mathbb{Z}[H_1(M)]$, corresponding to changing the lift $\tilde{B}_j$. Let us choose the $\tilde{B}_j$ so that the monomial $E(x_i)$ in F_{ij} is ± 1. This choice of basis is clearly compatible with the Euler structure $\mathfrak{e}(\mathbf{x})$, so the corresponding element of $\mathrm{Lift}(Z)$ is $\mathfrak{l}(\mathbf{x})$. Then

$$\tau(M,\gamma,\mathfrak{l}(\mathbf{x})) = \det \varphi\left(\frac{\partial \beta_j}{\partial \alpha_i^*}\right),$$

and the monomial $E(\mathbf{x})$ contributes to $\tau(M,\gamma,\mathfrak{l}(\mathbf{x}))$ with coefficient ± 1. In other words, it is assigned to the Spin^c structure $\mathfrak{s}(\mathbf{x})$.

5.3. **Homology orientations.** To complete the proof of Theorem 1, it remains to check that if we fix a homology orientation ω for $H_1(M,R_-)$, then we have $\chi(SFH(M,\gamma,\mathfrak{s},\omega)) = T_{(M,\gamma,\omega)}(\mathfrak{s})$. This follows easily from the definitions. To compute the torsion, we choose bases $A_1^*,\dots,A_d^*$ for $C_1(Z)$ and $B_1,\dots,B_d$ for $C_2(Z)$ which induce ω; we lift these bases to bases of $C_*(\tilde{Z})$, and the torsion is given by the determinant of the boundary map $\partial : C_2(\tilde{Z}) \to C_1(\tilde{Z})$, or equivalently, of F.

To fix an absolute $\mathbb{Z}/2$ grading on $SFH(M,\gamma)$, we must orient the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. By Lemma 2.6, this choice determines a homology orientation ω for $H_*(M,R_-)$. With respect to ω, the sign with which a generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ contributes to the Euler characteristic is the intersection sign $\mathbb{T}_\alpha \cdot \mathbb{T}_\beta$ at $\mathbf{x}$. But this sign is precisely the sign of the corresponding term in $\det F$. $\square$

6. Algebraic properties of the torsion

In this section, we collect some algebraic properties of the torsion function and their consequences. We begin by describing some known examples which appear as special cases of the torsion for sutured manifolds. We then turn our attention to the "evaluation homomorphism" $H_1(M) \to H_1(M, R_-)$ and prove Proposition 5 from the introduction. Finally, we discuss sutured L-spaces and give the proof of Corollary 6.

6.1. **Special Cases of the Torsion.** In this section, we summarize some useful special cases of the torsion. These are all "decategorifications" of known facts about sutured Floer homology, although in many cases, they admit more elementary proofs as well.

Lemma 6.1. *Let (M, γ) be a product sutured manifold. Denote by $\mathfrak{s}_0$ the canonical horizontal Spin^c–structure of (M, γ) and take ω to be the positive orientation on $H_*(M, R_-(\gamma)) = 0$. Then*

$$T_{(M,\gamma,\omega)}(\mathfrak{s}) = \begin{cases} 1, & \text{if } \mathfrak{s} = \mathfrak{s}_0, \\ 0, & \text{if } \mathfrak{s} \neq \mathfrak{s}_0. \end{cases}$$

This decategorifies the fact that $SFH(M, \gamma, \mathfrak{s})$ is isomorphic to $\mathbb{Z}$ if $\mathfrak{s} = \mathfrak{s}_0$ and is trivial otherwise [Ju06].

Proof. Let Z be the handle decomposition of (M, γ) with underlying sutured handle complex $(A, S \times I)$ such that $S = R_-(\gamma)$ and $A = S \times I$. Then $s_Z^{-1}(\mathfrak{s}_0) \in \mathrm{Eul}(Z)$ can be represented by the 1-cycle $\theta = \emptyset$. Since there are no handles, we have a canonical identification between $\mathrm{Eul}(Z)$ and $H_1(M)$, and by the definitions of subsection 3.2 in this case $\tau(Z, h, \omega) = h$ for every $h \in \mathrm{Eul}(Z) \cong H_1(M)$. The result follows. □

Lemma 6.2. *Let Y be a closed 3–manifold and let $Y(1)$ be the balanced sutured manifold defined in Example 2.2. Then for any $\mathfrak{s} \in$ Spin$^c(Y(1))$ we have*

$$T_{Y(1)}(\mathfrak{s}) = \begin{cases} 1 & \text{if } b_1(Y) = 0, \\ 0 & \text{if } b_1(Y) > 0. \end{cases}$$

Proof. Since $R_- = D^2$ the map $p_* \colon H_1(M) \to H_1(M, R_-)$ is an isomorphism. So the result follows immediately from Proposition 5. □

This also follows from the isomorphism $SFH(Y(1)) \cong \widehat{HF}(Y))$ [Ju06], together with the corresponding calculation of $\chi(\widehat{HF}(Y, \mathfrak{s}))$ [OS04a].

Let $L \subset S^3$ be an ordered oriented k–component link. Let $S^3(L)$ be the corresponding balanced sutured manifold as defined in Example 2.3. Then $H = H_1(S^3 \setminus N(L))$ is the free abelian multiplicative group generated by $t_1, \ldots, t_k$, where t_k is represented by the meridian of the k-th component.

Lemma 6.3. *Let $L \subset S^3$ be as above. Then*

$$\sum_{h\in H} T_{S^3(L)}(h)\cdot h \sim \Delta_L(t_1,\dots,t_k)\cdot \prod_{i=1}^{k}(t_i-1).$$

Proof. $SFH(S^3(L))$ is equal to the link Floer homology $\widehat{HFL}(L)$, so this follows from [OS05, Theorem 1.3] □

The following lemma can be viewed as the decategorification of [Ju08, Theorem 1.5]. It is a straightforward consequence of Proposition 5 and Lemma 6.3.

Lemma 6.4. *Let $K \subset S^3$ be a null–homologous knot and let R be a genus minimizing Seifert surface for K. Then*

$$\sum_{\mathfrak{s}\in Spin^c(S^3(R))} T_{Y(R)}(\mathfrak{s})$$

is equal to the $t^{g(R)}$–coefficient of the symmetrized Alexander polynomial of K.

6.2. **Evaluation.** In this section, we study the behavior of the torsion under the natural map $p : H_1(M) \to H_1(M, R_-)$. We prove the following statement, which is clearly equivalent to Proposition 5 of the introduction.

Proposition 6.5. *If (M,γ) is a balanced sutured manifold, $p_*(\tau(M,\gamma)) = \pm I_{H_1(M,R_-)}$, where $I_G \in \mathbb{Z}[G]$ is the sum of all elements in G if G is finite, and is 0 otherwise.*

Equivalently, if $K = PD[\ker p_*] \subset H^2(M,\partial M)$ and $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$, then

$$\sum_{h\in K} T_{(M,\gamma)}(\mathfrak{s}+h) = \begin{cases} 1 & \text{if } H_1(M,R_-) \text{ is finite,}\\ 0 & \text{otherwise.}\end{cases}$$

This result generalizes the fact that $\Delta_K(1) = 1$ whenever $K \subset Y$ is a knot in a homology sphere. Indeed, if we take (M,γ) to be the manifold $Y(K)$ from Example 2.3, then $H_1(M,R_-) = 0$, and p_* induces the map $\mathbb{Z}[t^{\pm1}] \to \mathbb{Z}$ which is evaluation at $t = 1$.

Proof. By adding product 1-handles, we reduce to the case where $R_-(\gamma)$ is connected. Indeed, if (M',γ') is obtained from (M,γ) by adding a product 1-handle, it is easy to see that $H_1(M',R_-(\gamma')) \cong H_1(M,R_-(\gamma))$, and that the diagram

$$\begin{array}{ccc} H_1(M) & \xrightarrow{p_*} & H_1(M,R_-(\gamma)) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle\cong} \\ H_1(M') & \xrightarrow{p'_*} & H_1(M',R_-(\gamma')) \end{array}$$

commutes. Applying Lemma 3.13, we see that if we know $p_*(\tau(M',\gamma')) = 0$, then $p_*(\tau(M,\gamma)) = 0$ as well.

From now on, we assume that $R_- = R_-(\gamma)$ is connected. Denote the group $H_1(M, R_-)$ by G. Let $\psi : \pi_1(M) \to G$ be the composition of p_* with the abelianization map, and consider the connected covering map $\pi : \widetilde{M} \to M$ corresponding to the kernel of ψ. Let $\overline{\tau}$ be the maximal abelian torsion of $C_*(\widetilde{M}, \widetilde{R}_-)$, viewed as a module over $\mathbb{Z}[G]$. Then

$$C_*(\widetilde{M}, \widetilde{R}_-) \cong C_*(\hat{M}, \hat{R}_-) \otimes_{\mathbb{Z}[H_1(M)]} \mathbb{Z}[G].$$

It now follows immediately from the proof of Lemma 3.4 that $\overline{\tau} = p_*(\tau(M, \gamma))$. Thus it suffices to show that $\overline{\tau} = I_G$.

As in subsection 3.3, let $T \subset G$ be the torsion subgroup, and pick a splitting $G = F \times T$, where F is a free abelian group. Under the isomorphism

$$\mathbb{Q}[T] \cong \bigoplus_i \mathbb{F}_{\chi_i}$$

I_T maps to the element whose $\mathbb{F}_{\chi_i}$ component is 0 for all non–trivial χ_i and whose component in $\mathbb{F}_{\chi_{\text{id}}} \cong \mathbb{Q}$ is $|G|$. (Here χ_{id} denotes the trivial character.) To show that $\overline{\tau}$ is a multiple of I_G, it suffices to show that the torsion $\tau^{\phi_i}(\widetilde{M}, \widetilde{R}_-)$ vanishes whenever either the group F or the character χ_i is nontrivial. Equivalently, we must prove that that the complex $C_*(\widetilde{M}, \widetilde{R}_-) \otimes Q[\mathbb{F}_{\chi_i}(F)]$ has nontrivial homology.

To this end, we consider the groups $H_0(\widetilde{M})$ and $H_0(\widetilde{R}_-)$. By construction $H_0(\widetilde{M}) = \mathbb{Z}$. Applying the universal coefficient theorem, we see that $H_0(\widetilde{M}, Q(\mathbb{F}_{\chi_i}[F])) = 0$ unless $F = 0$ and χ_i is the trivial character. On the other hand, if $i_* : \pi_1(R_-) \to \pi_1(M)$ is the inclusion, $\psi \circ i_*$ factors as a composition

$$\pi_1(R_-) \to H_1(R_-) \to H_1(M) \to H_1(M, R_-)$$

and is therefore the zero map. It follows that $\pi : \widetilde{R}_- \to R_-$ is a trivial covering. We assumed R_- is connected, so p_* is a surjection and the deck group of $\widetilde{M}$ is isomorphic to G. Thus $H_0(\widetilde{R}_-) \cong \mathbb{Z}[G]$, from which it follows that $H_0(\widetilde{R}_-, Q(\mathbb{F}_{\chi_i}[F])) \cong Q(\mathbb{F}_{\chi_i}[F])$ for any character χ_i.

We now consider the long exact sequence of the pair $(\widetilde{M}, \widetilde{R}_-)$:

$$\to H_1(\widetilde{M}, \widetilde{R}_-; Q(\mathbb{F}_{\chi_i}[F])) \to H_0(\widetilde{R}_-; Q(\mathbb{F}_{\chi_i}[F])) \to H_0(\widetilde{M}; Q(\mathbb{F}_{\chi_i}[F])) \to$$

The middle group in this sequence has rank 1, but the last group is trivial unless $F = 0$ and $\chi_i = 1$. It follows that $H_1(\widetilde{M}, \widetilde{R}_-; Q(\mathbb{F}_{\chi_i}[F]))$ is nontrivial unless $F = 0$ and χ_i is trivial.

To finish the proof, we need only compute the torsion $\tau^{\phi_{\text{id}}}(\tilde{M}, \tilde{R}_-)$ when $F = 0$ and ϕ_{id} is the homomorphism induced by the trivial character. But in this case $C_*(\widetilde{M}) \otimes Q[\mathbb{F}_{\chi_{\text{id}}}(F)]$ reduces to the ordinary chain complex $C_*(M, R_-; \mathbb{Q})$. This complex is trivial for $i \neq 1, 2$, so the torsion is $\det(d)$, where $d : C_2(M, R_-) \to C_1(M, R_-)$ is the boundary map. In other words, $\tau^{\phi_{\text{id}}}(\tilde{M}, \tilde{R}_-) = \pm|H_1(M, R_-)| = \pm|G|$ as desired. □

6.3. **Sutured L-spaces.** From the introduction, we recall the following

Definition. We say that (M,γ) is a *sutured L-space* if $SFH(M,\gamma)$ is torsion free and supported in a single $\mathbb{Z}/2$ homological grading.

Lemma 6.6. *Suppose (M,γ) is a balanced sutured manifold and that $S \subset M$ is a nice decomposing surface. Let (M',γ') be the result of decomposing (M,γ) along S. Let O_S be the set of outer Spin^c structures for S, as defined in* [Ju08]*. If*

$$\bigoplus_{s\in O_S} SFH(M,\gamma,\mathfrak{s})$$

is torsion free and supported in a single $\mathbb{Z}/2$ grading, then (M',γ') is a sutured L-space.

Proof. By the decomposition formula of [Ju08],

$$SFH(M',\gamma') \cong \bigoplus_{s\in O_S} SFH(M,\gamma,\mathfrak{s}).$$

□

Corollary 6.7. *If (M,γ) is a sutured L-space and (M',γ') is obtained by decomposing (M,γ) along a nice surface, then (M',γ') is a sutured L-space.*

Corollary 6.8. *Suppose $L \subset S^3$ is an oriented link and Σ is a minimal genus Seifert surface of L. If $\widehat{HFK}(L,g(\Sigma))$ is torsion free and supported in one $\mathbb{Z}/2$ grading, then $S^3(\Sigma)$ is a sutured L-space.*

Proof. $S^3(\Sigma)$ is obtained by decomposing $S^3(L)$ along Σ. There is an isomorphism $SFH(S^3(L)) \cong \widehat{HFK}(L)$. The part coming from the outer Spin^c structures with respect to Σ is exactly $\widehat{HFK}(L,g(\Sigma))$. □

Corollary 6.9. *If $L \subset S^3$ is an alternating link and Σ is a Seifert surface of L, then $S^3(\Sigma)$ is a sutured L-space.*

Proof. The main theorem of [OS03] implies that $\widehat{HFK}(L,g(\Sigma))$ is torsion free and supported in a single homological grading. □

We remark that there are many non-alternating links which also satisfy the hypothesis of Corollary 6.8. For example, it is satisfied by all knots of ten or fewer crossings.

The sutured Floer homology of a sutured L-space has an especially simple form:

Corollary 6.10. *If (M,γ) is a sutured L-space, then for each $\mathfrak{s} \in$ Spin$^c(M,\gamma)$, the group $SFH(M,\gamma,\mathfrak{s})$ is either trivial or isomorphic to $\mathbb{Z}$.*

Proof. Since SFH is supported in a single homological grading, $T_{(M,\gamma)}(\mathfrak{s})$ is equal to the rank of $SFH(M,\gamma,\mathfrak{s})$. Then Proposition 5 shows that

$$\sum_{h\in K} \text{rank } SFH(M,\gamma,\mathfrak{s}+h) \leq 1.$$

This clearly implies the statement above. □

7. The Thurston norm for sutured manifolds

Let (M, γ) be a sutured manifold. In [Sc89], Scharlemann introduced a natural seminorm on $H_2(M, \partial M)$ which generalizes the usual Thurston norm of [Th86]. In this section, we investigate the relation between this norm and the sutured Floer homology.

Definition 7.1. Let (M, γ) be a sutured manifold. Given a properly embedded, compact, connected surface $S \subset M$ let

$$x^s(S) = \max\{0, \chi(S \cap R_+(\gamma)) - \chi(S)\},$$

and extend this definition to disconnected surfaces by taking the sum over the components. Note that $S \cap R_+(\gamma)$ is necessarily a one–dimensional manifold and $\chi(S \cap R_+(\gamma))$ equals the number of components of $S \cap R_+(\gamma)$ which are not closed. Equivalently, we have

$$x^s(S) = \max\left\{0, -\chi(S) + \frac{1}{2}|S \cap s(\gamma)|\right\}.$$

For $\alpha \in H_2(M, \partial M; \mathbb{Z})$ let

$$x^s(\alpha) = \min\{x^s(S) : S \subset M \text{ is properly embedded and } [S, \partial S] = \alpha\}.$$

Theorem 7.2. *(*[Sc89]*) Let (M, γ) be a sutured manifold. Then the function*

$$x^s : H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}_{\geq 0}$$

defined above has the following two properties:

(1) *$x^s(n\alpha) = |n| \cdot x^s(\alpha)$ for all $n \in \mathbb{Z}$ and $\alpha \in H_2(M, \partial M; \mathbb{Z})$*
(2) *$x^s(\alpha + \beta) \leq x^s(\alpha) + x^s(\beta)$ for all $\alpha, \beta \in H_2(M, \partial M; \mathbb{Z})$.*

It follows that x^s extends to a continuous map $x^s : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}_{\geq 0}$ which is convex and linear on rays from the origin. Put differently, x^s is a seminorm on $H_2(M, \partial M; \mathbb{R})$. It is called the *sutured Thurston norm.*

Example. Let Y be a closed 3–manifold and let $(M, \gamma) = Y(1)$ be Y with an open ball removed and having a connected suture. Then we can identify $H_2(Y; \mathbb{R})$ with $H_2(Y(1), \partial Y(1); \mathbb{R})$. It is straightforward to see that under this identification the Thurston norms of Y and $Y(1)$ agree.

Example. Let $K \subset S^3$ a knot and let $(M, \gamma) = S^3(K)$ be the associated sutured manifold with two meridional sutures. If $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is a generator then $x^s(\alpha) = 2g(K)$. Note that this differs from the usual Thurston norm x of M, which satisfies $x(\alpha) = 2g(K) - 1$ for a non–trivial knot.

The following proposition should be compared with [Ju08b, Theorem 5.1] and [Ju08b, Proposition 7.5].

Proposition 7.3. *Suppose that (M,γ) is taut, balanced, reduced, and horizontally prime, and that $H_2(M;\mathbb{Z})=0$. Then x^s is a norm.*

Proof. Assume there exists an $\alpha \neq 0$ in $H_2(M,\partial M;\mathbb{Z})$ with $x^s(\alpha)=0$. This implies that there exists a connected homologically non–trivial orientable surface $(S,\partial S)\subset (M,\partial M)$ with $x^s(S)=0$. Hence

$$-\chi(S)\leq -\chi(S)+\chi(S\cap R_+(\gamma))\leq 0.$$

So $\chi(S)\geq 0$ and S is either S^2, T^2, D^2, or $S^1\times I$. Since $[S,\partial S]\neq 0$ and $H_2(M;\mathbb{Z})=0$ the surface S is not S^2 or T^2. Furthermore, we can assume that $S\cap\gamma$ consists of arcs connecting $R_-(\gamma)$ and $R_+(\gamma)$.

Now suppose that $S=D^2$. Then $\chi(S\cap R_+(\gamma))$ is 0 or 1. In the latter case S would be a homologically non-trivial product disc, contradicting the assumption that (M,γ) is reduced. In the former case S is a compressing disk for $R(\gamma)$. Since (M,γ) is taut $R(\gamma)$ is incompressible, so ∂S bounds a disk S' in $R(\gamma)$. Now $S\cup(-S')$ is a sphere, which has to bound a D^3 since M is irreducible. But then $[S,\partial S]=0$, a contradiction.

Finally, assume that S is an annulus. Then $\chi(S\cap R_+(\gamma))=0$. Since (M,γ) is reduced we know that S can not be a product annulus. So suppose that $\partial S\subset R$, where R is either $R_-(\gamma)$ or $R_+(\gamma)$. Pick a product neighborhood $S\times[0,1]$ and let

$$R'=R\setminus(\partial S\times(0,1))\cup(S\times 0)\cup(S\times 1).$$

Then R' is homologous to R, $\partial R'=\partial R$ and $\chi(R')=\chi(R)$. Hence R' is a horizontal surface. Note that R' is not parallel to R. If R' were parallel to $R(\gamma)\setminus R$ then $\partial S\times[0,1]$ would give rise to a non-trivial product annulus. So the existence of R' would contradict our assumption that (M,γ) is horizontally prime. □

Definition 7.4. Let $\mathcal{S}(M,\gamma)=\{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)\,|\,SFH(M,\gamma,\mathfrak{s})\neq 0\}$ be the support of $SFH(M,\gamma)$. If $\alpha\in H_2(M,\partial M;\mathbb{R})$, we define

$$z(\alpha)=\max\{\langle\,\mathfrak{s}-\mathfrak{t},\alpha\,\rangle\,|\,\mathfrak{s},\mathfrak{t}\in\mathcal{S}(M,\gamma)\}.$$

Remark. In [Ju08b, Section 8] another "seminorm" y on $H_2(M,\partial M;\mathbb{R})$ was constructed using sutured Floer homology. Hence y satisfies all properties of a seminorm except that $y(\alpha)\neq y(-\alpha)$ can happen. It is straightforward to see that

$$z(\alpha)=\frac{1}{2}(y(\alpha)+y(-\alpha)).$$

To understand the relationship between x^s and z, we consider the double of the sutured manifold (M,γ) along $R(\gamma)$. More precisely, the double DM of (M,γ) is obtained from the disjoint union of M and $-M$ by identifying the two copies of $R(\gamma)$ via the identity map. The boundary of DM is a union of tori; each torus is the double of a component of γ. This operation was first used by Gabai [Ga83].

A theorem of Cantwell and Conlon relates the sutured Thurston norm on (M,γ) to the Thurston norm of the double. To be precise, suppose (M,g) is a sutured manifold, and let $X = DM$ be the double of M along $R(\gamma)$. There is a natural "doubling map" $D_* : H_2(M, \partial M) \to H_2(X, \partial X)$, and we have

Theorem 7.5. [CC06, Theorem 2.3] $x(D_*(\alpha)) = 2x^s(\alpha)$

Here x denotes the usual Thurston norm on $H_2(X, \partial X)$ and x^s is the sutured Thurston norm on $H_2(M, \partial M)$.

Definition 7.6. We make X into a sutured manifold (X, γ_X) in the following canonical way. Let the components of γ be $\gamma_1, \dots, \gamma_l$. For each component γ_i of γ choose two parallel, oppositely oriented arcs m_i and m_i' that connect $R_+(\gamma)$ and $R_-(\gamma)$. Then on the torus $D\gamma_i \subset \partial X$ the sutures are $\mu_i = m_i \cup (-m_i)$ and $\mu_i' = m_i' \cup (-m_i')$. These sutures are well defined up to isotopy. Let γ_X be a regular neighborhood of $\bigcup_{i=1}^{l} (\mu_i \cup \mu_i')$ inside ∂X.

Lemma 7.7. *Let (M, γ) be a taut sutured manifold and let (X, γ_X) be its double. Then for all $\alpha \in H_2(M, \partial M)$ we have*

$$2x^s(\alpha) = \max\{\langle\, \mathfrak{s} - \mathfrak{t}, D_*(\alpha)\,\rangle \,|\, \mathfrak{s}, \mathfrak{t} \in \mathcal{S}(X, \gamma_X)\}.$$

Proof. Observe that we can assign to (X, γ_X) a link L in a 3-manifold Y which is obtained by Dehn filling ∂X such that the μ_i become meridians of the filling tori, see [Ju06, Example 2.4]. Then by [Ju08, Remark 8.5] we can assign to each $\mathfrak{s} \in \mathrm{Spin}^c(X, \gamma_X)$ a relative first chern class $c_1(\mathfrak{s}) \in H^2(X, \partial X)$ in such a way that the set $\{c_1(\mathfrak{s}) \,|\, \mathfrak{s} \in \mathcal{S}(X, \gamma_X)\}$ is symmetric about the origin. Then for every $h \in H_2(X, \partial X)$

$$\max\{\langle\, c_1(\mathfrak{s}, t), h\,\rangle \,|\, \mathfrak{s} \in S(X, \gamma_X)\,\} = x(h) + \sum_{i=1}^{l} |\langle\, h, \mu_i\,\rangle|.$$

Since the image of $\mathcal{S}(X, \gamma_X)$ is centrally symmetric, this is equivalent to saying

$$\max\{\langle\, \mathfrak{s} - \mathfrak{t}, h\,\rangle \,|\, \mathfrak{s} \in S(X, \gamma_X)\,\} = x(h) + \sum_{i=1}^{l} |\langle\, h, \mu_i\,\rangle|.$$

Note that the right hand side is exactly $x^s(h)$.

We claim that if $h = D_*(\alpha)$ for some $\alpha \in H_2(M, \partial M)$ then $\langle\, h, \mu_i\,\rangle = 0$ for every $1 \le i \le l$. To see this, choose a surface S representing α. We may assume that $S \cap \gamma_i$ consists of a collection of parallel arcs. If we take m_i and m_i' parallel to these arcs, then $\partial S \cap \mu_i$ and $\partial S \cap \mu_i'$ are empty. Combining this with the fact that $x(h) = 2x^s(\alpha)$, we obtain the statement of the lemma. □

The surface $R(\gamma)$ defines an oriented surface $R \subset X$. Note that R has the orientation coming from $R(\gamma)$, *not* the induced orientation coming from ∂M. In particular, the homology class represented by R is twice the class of $R_+(\gamma)$. It is easy to see that R is a nice decomposing surface for (X, γ_X) in the sense of [Ju08b]. Let $O_R \subset \mathrm{Spin}^c(X, \partial X)$ be the set of outer Spin^c structures for R.

Lemma 7.8. *Let* (M,γ) *be a taut sutured manifold. Then for all* $\alpha \in H_2(M,\partial M)$ *we have*

$$2z(\alpha) = \max\{\, \langle\, \mathfrak{s}-\mathfrak{t}, D_*(\alpha)\,\rangle \mid \mathfrak{s},\mathfrak{t} \in O_R \cap \mathcal{S}(X,\gamma_X)\,\}\,.$$

Proof. If we decompose (X,γ_X) along R we get $(M',\gamma') = (M,\gamma) \sqcup (-M,-\gamma)$. $\mathrm{Spin}^c(M',\gamma')$ is naturally identified with $\mathrm{Spin}^c(M,\gamma)\times\mathrm{Spin}^c(-M,-\gamma)$. If $i : M' \to X$ is the inclusion, there is a natural map $i_R : \mathrm{Spin}^c(M',\gamma') \to O_R$ with the property that $i_R(\mathfrak{s}) - i_R(\mathfrak{t}) = i_*(\mathfrak{s}-\mathfrak{t})$. (Here we view $\mathfrak{s}-\mathfrak{t}$ as an element of $H_1(X)$, or equivalently, define $i_* = PD \circ i^* \circ PD$.) The decomposition theorem of [Ju08] implies that

$$i_R(\mathcal{S}(M',\gamma')) = \mathcal{S}(X,\gamma_X) \cap O_R.$$

Clearly

$$\mathcal{S}(M',\gamma') = \mathcal{S}(M,\gamma) \times \mathcal{S}(-M,-\gamma)$$

Recall that there is a bijection $\mathrm{Spin}^c(M,\gamma) \to \mathrm{Spin}^c(M,\gamma)$ which sends a nonvanishing vector field v to $-v$. By Proposition 2.8, $\mathcal{S}(-M,-\gamma) = -\mathcal{S}(M,\gamma)$. Thus an element of $\mathcal{S}(X,\gamma_X) \cap O_R$ can be written as $\mathfrak{s} = i_R(\mathfrak{s}_1,-\mathfrak{s}_2)$, where $\mathfrak{s}_1,\mathfrak{s}_2 \in \mathrm{Spin}^c(M,\gamma)$. We compute

$$\begin{aligned}\langle \mathfrak{s}-\mathfrak{t}, D_*(\alpha)\rangle &= \langle i_R(\mathfrak{s}_1,-\mathfrak{s}_2) - i_R(\mathfrak{t}_1-\mathfrak{t}_2), D_*(\alpha)\rangle \\ &= \langle i_*(\mathfrak{s}_1-\mathfrak{t}_1, \mathfrak{t}_2-\mathfrak{s}_2), D_*(\alpha)\rangle \\ &= \langle \mathfrak{s}_1-\mathfrak{t}_1, \alpha\rangle + \langle \mathfrak{s}_2-\mathfrak{t}_2, \alpha\rangle\end{aligned}$$

In particular, we see that

$$\max\{\, \langle\, \mathfrak{s}-\mathfrak{t}, D_*(\alpha)\,\rangle \mid \mathfrak{s},\mathfrak{t} \in O_R \cap \mathcal{S}(X,\gamma_X)\,\} = 2\cdot\max\{\langle\, \mathfrak{s}-\mathfrak{t}, \alpha\,\rangle \mid \mathfrak{s},\mathfrak{t} \in \mathcal{S}(M,\gamma)\}$$

□

Proof. (of Theorem 3). If (M,γ) is taut, this is an immediate consequence of Lemmas 7.7 and 7.8. If (M,γ) is not taut, then $SFH(M,\gamma) = 0$, so $z = 0$ and the inequality is obviously true. □

Remark. We introduce the notation B_{x^s} for the unit norm ball of x^s and B_z for the unit norm ball of z. Let $P \subset H_2(X,\partial X;\mathbb{R})$ for the sutured Floer polytope of (X,γ_X), and for any $h \in H_2(X,\partial X)$ let P_h be the corresponding face of P. Similarly, let $Q \subset H_2(M,\partial M)$ be the sutured Floer polytope of (M,γ). Then Lemmas 7.7 and 7.8 tell us that

$$B_{x^s} = P^* \cap \mathrm{Im}(D_*)$$

and

$$\begin{aligned}B_z &= P_\rho^* \cap \mathrm{Im}(D_*) \\ &= (Q + (-Q))^*.\end{aligned}$$

where $\rho = [R]$.

When (M,γ) is a rational homology product, the relation between $P^* \cap \mathrm{Im}(D_*)$ and $P_\rho^* \cap \mathrm{Im}(D_*)$ is easy to understand. Let $i : R_-(\gamma) \to X$ be the inclusion, and let $A \subset H_1(X, \partial X)$ be the image of i_*. Let $B \subset H_1(X, \partial X)$ be the one dimensional space spanned by the double of an arc joining $R_-(\gamma)$ to $R_+(\gamma)$. Then $H_1(X, \partial X) = A \oplus B$.

There is a natural "reflection" $r : X \to X$ which acts as the identity on A and by multiplication by -1 on B. The image of D_* is fixed by r_*, so B is orthogonal to Im D_*. The action of r_* exchanges the faces P_ρ and $P_{-\rho}$. Composing the central symmetry of P with r_* gives an involution on P_ρ. (This realizes the central symmetry of B_z.) It follows that $P_{-\rho}$ is a translate of P_ρ in the direction of B.

Since B is orthogonal to Im D_*, restricting our attention to the image of D_* has the effect of projecting P in the direction of B. The norms x^s and z will agree if and only if the image of P under this projection is equal to P_ρ. Somewhat surprisingly, this is not always the case.

Proposition 7.9. *For the sutured manifold* (M,γ) *given in* [CC06, Example 2] *we have* $x^s \neq z$.

Proof. In [CC06] the unit ball B_{x^s} is computed explicitly. To get B_z one proceeds by computing $T_{(M,\gamma)}$, which is straightforward since M is a genus three handlebody. Now to get $P(M,\gamma,t)$ one uses the adjunction inequality of [Ju08b] for the decomposing discs given in [CC06], each of which intersects $s(\gamma)$ in four points. After symmetrizing one can just observe that $B_z \neq B_{x^s}$. □

Remark. In the above example the polytope P is four-dimensional and is composed of three layers in the ρ direction. The opposite faces P_ρ and $P_{-\rho}$ are in the two outer layers, and they are are both smaller than the middle layer, which is exactly $B^*_{x^s}$. This picture shows that this is in some sense the smallest possible counterexample where M is a handlebody, since on a genus two handlebody we would have $P = P_\rho \cup P_{-\rho}$.

8. Examples and Applications

We conclude with some sample computations of the torsion and/or the sutured Floer homology, with emphasis on the case where (M,γ) is the complement of a Seifert surface $\Sigma \subset S^3$.

Example 8.1. Suppose that $\Sigma \subset S^3$ is an embedded annulus. Then $\partial\Sigma$ consists of two parallel copies of a knot K with some linking number n corresponding to the framing of the annulus. The complementary sutured manifold $S^3(\Sigma)$ is homeomorphic to $S^3 \setminus N(K)$. Its boundary is a torus with two sutures, each representing the homology class $\ell + nm$ with respect to the canonical basis on $H_1(\partial(S^3 \setminus N(K))$. Let K_n be the manifold obtained by filling this homology class (*i.e.*, by performing $n/1$ Dehn surgery on K), and let $K(n) \subset K_n$ be the core circle of the filling. Then $SFH(S^3(\Sigma))$ is isomorphic to $\widehat{HFK}(K(n))$, from which it follows that

$$\tau(S^3(\Sigma)) = \Delta_K(t) \cdot \frac{t^n - 1}{t - 1}.$$

Note that when $n = 0$, the torsion vanishes, regardless of what K is. The group $\widehat{HFK}(K(n))$ has been studied by Eftekhary [Ef05] (in the case $n = 0$) and by Hedden [He05], who gives a complete calculation in terms of the groups $HFK^-(K)$. In particular, $\widehat{HFK}(K(0))$ is nontrivial unless K is the unknot.

Example 8.2. Suppose M is a solid torus, and that γ consists of $2n$ parallel curves on ∂M, each of which represents p times the generator of $H_1(M)$. $SFH(M, \gamma)$ was computed by the second author in [Ju08b]; its Euler characteristic is given by

$$\tau(M, \gamma) \sim \frac{(t^p - 1)^n}{t - 1}.$$

The homology in each Spin^c structure is a free module of rank equal to the Euler characteristic. An important special case is when $n = 1$. In this case $(M, \gamma) = S^3(\Sigma)$, where Σ is a twisted band with p full twists (in other words, an unknotted annulus in S^3 with framing p.) $SFH(M, \gamma)$ is supported in p consecutive Spin^c structures, each containing a single copy of $\mathbb{Z}$.

Example 8.3. Suppose $K = K(p, q) \subset S^3$ is the two-bridge knot or link corresponding to the fraction p/q. The set of Seifert surfaces for K has been classified up to isotopy by Hatcher and Thurston [HT85]. Any such surface is obtained as a Murasugi sum of twisted bands. By [Ju08, Cor. 8.8], the sutured Floer polytope is a rectangular prism, the length of whose sides is determined by the number of twists in the corresponding band. These, in turn, are given by the coefficients of the unique continued fraction expansion of p/q all of whose terms are even [HT85]. For example, the knot $K(56, 15)$ has continued fraction expansion

$$\frac{56}{15} = 4 - \frac{1}{4 - \frac{1}{4}}.$$

Any Seifert surface Σ of K is a Murasugi sum of three twisted bands, each with two full twists. $SFH(S^3(\Sigma)) \cong \mathbb{Z}^8$ is supported at the vertices of a $2 \times 2 \times 2$ cube.

Whenever two of the bands have more than one twist, K will have more than one Seifert surface. The calculation above shows that these surfaces cannot be distinguished by their sutured Floer polytope alone. In contrast, we have

Theorem. *(*[HJS08]*) There exist two minimal genus Seifert surfaces R_1 and R_2 for $K(17, 4)$ which can be distinguished by combining sutured Floer homology with the Seifert form. More precisely, there does not exist an orientation–preserving diffeomorphism of the pairs (S^3, R_1) and (S^3, R_2).*

For $i = 1, 2$ the groups $SFH(S^3(R_i), \mathfrak{s})$ for $\mathfrak{s} \in \text{Spin}^c(S^3(R_i))$ are determined by $T_{S^3(R_i)}(\mathfrak{s})$. Hence it is straightforward to modify the proof of the theorem to show that the Seifert surfaces can also be distinguished by using torsion and the Seifert form.

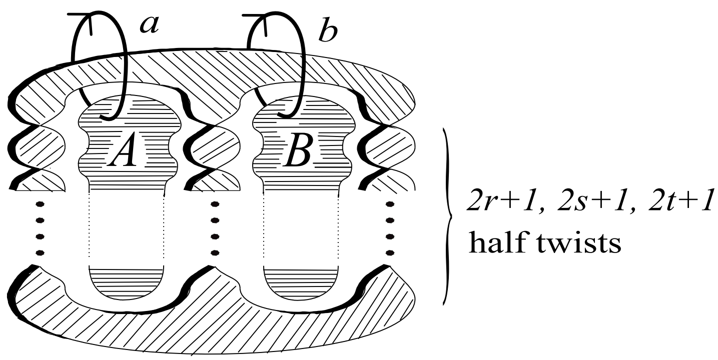

FIGURE 1. The pretzel knot $P(2r+1, 2s+1, 2t+1)$ with canonical Seifert surface.

Example 8.4. The pretzel knot $P(2r+1, 2s+1, 2t+1)$ has an obvious Seifert surface Σ, as shown in Figure 1. The complement $M = S^3 \setminus N(\Sigma)$ is a handlebody of genus 2. An obvious pair of compressing disks A and B is shown in the figure. Cutting M along these disks and using Seifert-Van Kampen gives an isomorphism between $\pi_1(M)$ and the free group generated by a and b. If α is a curve on ∂M, we can read off the word it represents in $\pi_1(M)$ by traversing α and recording its intersections with ∂A and ∂B.

Suppose r, s and t are all positive and that Σ is oriented so that the uppermost region in the figure belongs to R_-. Put p in this region, and let α be a loop which runs from p down the left-hand strip, and back up via the middle strip. Similarly, let β be a loop which runs down the right-hand strip and back up the middle, so that $\pi_1(R_-, p)$ is generated by α and β. The reader can easily verify that

$$\alpha = a^{r+1}(b^{-1}a)^s b^{-1} \qquad \beta = b^{-t}(b^{-1}a)^s b^{-1}$$

Evaluating the 2×2 determinant

$$\begin{bmatrix} \partial\alpha/\partial a & \partial\beta/\partial a \\ \partial\alpha/\partial b & \partial\beta/\partial b \end{bmatrix}$$

we find that $\tau(S^3(\Sigma))$ is supported on a hexagon, as illustrated in Figure 2. With respect to the natural basis given by a and b, the sides of the hexagon have slope $0, 1$ and ∞. Parallel sides have the same length, and the sides are of length $r+1, t+1$, and $s+1$. The coefficient of the torsion at each lattice point in the hexagon is 1, and the sutured Floer homology consists of a single copy of $\mathbb{Z}$ at each lattice point.

The case where $r, s > 0$ and $t < 0$ can be treated similarly. We distinguish two subcases, depending on whether $|2t-1|$ is less than $\min(2r+1, 2s+1)$, or greater. In the first, the coefficients of $\tau(S^3(\Sigma))$ take on both positive and negative signs. The torsion is supported on a "bowtie", as shown in Figure 3. The coefficient of the torsion is 1 at each lattice point in the rectangle, and -1 at each lattice point

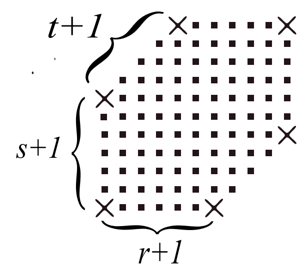

FIGURE 2. Support of $\tau(S^3(\Sigma))$ for $r, s, t > 0$.

in the two triangles. In the second case, the support is a nonconvex hexagon, as illustrated in Figure 4. The coefficient of the torsion is 1 at each lattice point in the hexagon. To determine the sutured Floer homology, we compare with the calculation of $\widehat{HFK}(P(2r+1, 2s+1, 2t+1)$ given in [OS04c]. In both cases, the top group in the knot Floer homology is torsion free and its rank is equal to the number of vertices in the support of $\tau(S^3(\Sigma))$. It follows that $SFH(S^3(\Sigma))$ has rank one at each vertex in the support and is trivial elsewhere.

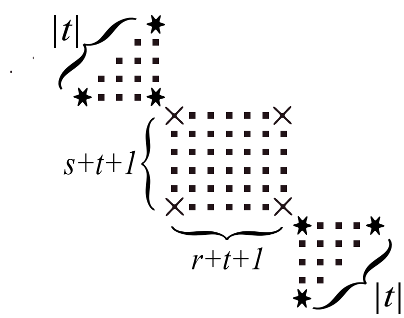

FIGURE 3. Support of $\tau(S^3(\Sigma))$ for $r, s > 0$ and $-\min(r, s) \leq t \leq 0$.

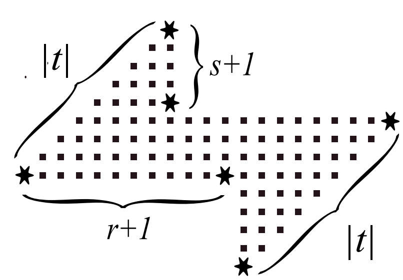

FIGURE 4. Support of $\tau(S^3(\Sigma))$ for $r, s > 0, t < 0$ and $t \leq -\min(r, s)$.

Example 8.5. The three-component pretzel link $P(2r, 2s, 2t)$ has a Seifert surface Σ similar to that shown in Figure 1, and $\tau(S^3(\Sigma))$ can be computed as in the previous example. When r, s, t are all positive, the torsion is again supported on a hexagon with sides of slopes $0, 1$ and ∞. However in this case parallel sides of the hexagon do not have the same lengths. Instead, the sides have lengths $r+1, t, s+1, r, t+1, s$ as we go around the hexagon. This gives a simple family of examples for which the torsion does not exhibit any symmetry. The phenomenon is already evident for $P(2, 2, 2)$. In this case, the hexagon degenerates to a triangle supported at three vertices in the plane. With respect to the standard basis a, b, these vertices can be taken to be $(0, 0)$, $(1, 0)$ and $(1, 1)$.

Example 8.6. Seifert surfaces of small knots. Let K be a knot in S^3 and suppose Σ is a Seifert surface for K. Among knots with nine crossings or fewer, most are either two-bridge or fibred. (See *e.g.* the tables in [Ka96] or [CL09].) If K is fibred, $SFH(S^3(\Sigma)) \cong \mathbb{Z}$; if it is two-bridge, $SFH(S^3(\Sigma))$ was determined in Example 8.3. We briefly describe $SFH(S^3(\Sigma))$ for the remaining knots here. They fall into two broad classes, as well as a few knots with more interesting homology.

- The knots $9_{16}, 9_{37}$, and 9_{46} have $SFH(S^3(\Sigma)) \cong SFH(A_2)$, where A_2 is an unknotted annulus with two full twists.
- The knots $8_{15}, 9_{25}, 9_{39}, 9_{41}$, and 9_{49} have $SFH(S^3(\Sigma)) \cong SFH(S^3(\Sigma_{2,2,2}))$, where $\Sigma_{2,2,2}$ is the Seifert surface of the $(2, 2, 2)$ pretzel link.
- The knot 9_{35} is $P(3, 3, 3)$. Its sutured Floer polytope is a hexagon with sides of length 2.
- The knot 9_{38} is the only knot with fewer than 10 crossings whose sutured Floer polytope is 3-dimensional. The polytope is contained in a $2 \times 2 \times 2$ cube, with $\mathbb{Z}$ summands at five of the vertices: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1), (1, 1, 0)$, and $(1, 0, 1)$.

For all of these knots, the top group in knot Floer homology is torsion free and supported in a single homological grading, so SFH is determined by the torsion.

Example 8.7. The four-strand pretzel link $L = P(n, -n, n, -n)$ has a genus one Seifert surface analogous to the one shown in Figure 1. The multivariable Alexander polynomial of this link is 0, but a calculation similar to the one in Example 4 shows that the torsion polytope is a "pinwheel" consisting of four square pyramids, each with side length n. It follows that the rank of $HFK(L)$ in the top Alexander grading is at least

$$4 \sum_{I=1}^{n} n^2 = \frac{2n(n+1)(2n+1)}{3}.$$

Example 8.8. We conclude by using the torsion to give an example of a phenomenon first observed by Goda [Go94]. Namely, there exist sutured manifolds whose total space is a genus two handlebody, but which are not disk-decomposable. Consider the

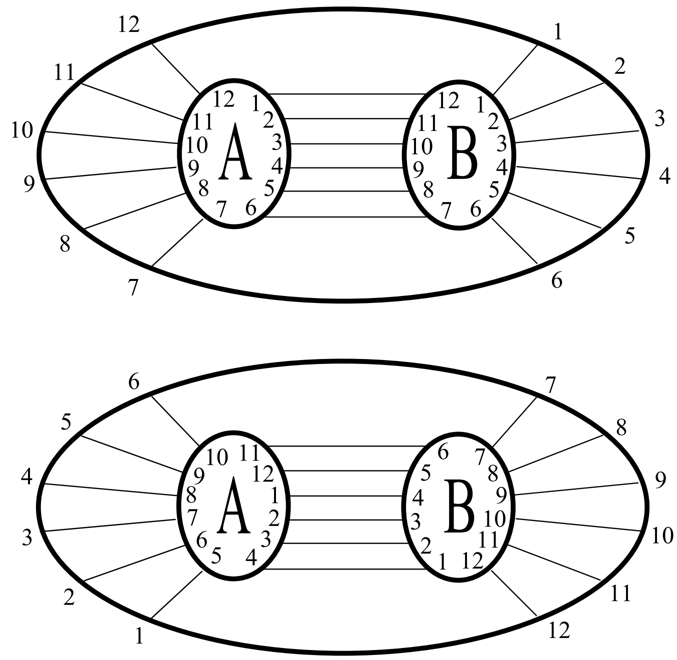

FIGURE 5. Sutured genus two handlebody with no disk decomposition.

genus two surface obtained by gluing two pairs of pants as illustrated in Figure 8. Let M be the handlebody in which the curves labeled A and B bound compressing disks, and let g be the curve shown in the figure. Then we easily compute

$$\tau(M,\gamma) \sim 2a - 3 + 2a^{-1}.$$

Proposition 8.1. *(M,γ) is not disk-decomposable.*

Proof. Suppose we decompose (M,γ) along a disk D to obtain a sutured manifold (M',γ'). If ∂D is a non-separating curve in ∂M, then M' is homeomorphic to $S^1 \times D^2$. If (M',γ') were taut, then by [Ju08b], $SFH(M',\gamma')$ would be isomorphic to the restriction of $SFH(M,\gamma)$ to those Spin^c structures which are extremal with respect to evaluation on $[D]$. It follows that either $\tau(M',\gamma') = 0$, $\tau(M',\gamma') \sim 2$, or $\tau(M',\gamma') \sim 2t - 3 + 2t^{-1}$. Comparing with Example 2, we see that none of these are the torsion of a taut sutured manifold whose total space is the solid torus.

Similarly, if ∂D is a separating curve, then M' is homeomorphic to the disjoint union of two solid tori M_1 and M_2, and

$$SFH(M,\gamma) \cong SFH(M',\gamma') \cong SFH(M_1,\gamma_1) \otimes SFH(M_2,\gamma_2).$$

Again, comparing $\tau(M,\gamma)$ with Example 2 shows that this is not possible. $\square$

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